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An analytical approximate method for solving unsteady state two-dimensional convection-diffusion equations

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Abstract:

In this paper, an analytic approximate method for solving the unsteady two-dimensional convection-diffusion equations is introduced. Also, the convergence of the approximate methods is analyzed. Three test examples are presented, two have exact and one has not exacted solutions. The results obtained show that these methods are powerful mathematical tools for solving linear and nonlinear partial differential equations, moreover, new analytic Taylor method (NATM), reduced differential transform method (RDTM), and homotopy perturbation method (HPM), are more accurate and have less CPU time than the other methods.

Keywords: unsteady, convection-diffusion, RDTM, HPM, NATM, RK-4, accuracy, convergence.

1-Introduction

This work is interested in two unsteady state two-dimensional initial-boundary value problems, which were formulated as follows;

Problem-I: Linear transport (convection-diffusion) equation.

$$\frac{\partial u}{\partial t}+\beta_x\frac{\partial u}{\partial x}+\beta_y\frac{\partial u}{\partial y}-\alpha_x\frac{\partial^2 u}{\partial x^2}-\alpha_y\frac{\partial^2 u}{\partial y^2}=0,\ \ (x,y,t)\in[0,L]\times[0,L]\times[0,T] \tag{1}$$

with initial condition

$$u(x,y,0)=\emptyset_o(x,y), 0\le x,y\le L, \tag{2}$$

and the boundary conditions

$$\left.\begin{matrix} u(x,0,t)=f_o(x,t), u(x,L,t)=f_1(x,t), 0\le x\le L, t\ge 0 \\ u(0,y,t)=g_o(y,t), u(L,y,t)=g_1(y,t), 0\le y\le L, t\ge 0\end{matrix}\right\}, \tag{3}$$

where $u(x,y,t)$ is a transported variable, β_x and β_y, are arbitrary constants witch show the speed of convection, α_x and α_y are positive constants of diffusion coefficients, and $f_{0,}f_{1,}g_{0,}g_1$ and $\emptyset_o$ are known functions.

Problem-II: Nonlinear Burgers equation:

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\alpha v\frac{\partial u}{\partial y}-\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0,\ \ (x,y,t)\in[0,L]\times[0,L]\times[0,T] \tag{4}$$

$$\frac{\partial v}{\partial t}+\alpha u\frac{\partial v}{\partial x}+\alpha v\frac{\partial v}{\partial y}-\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0, \tag{5}$$

with initial conditions

$$\left.\begin{matrix} u(x,y,0)=a_1(x,y) \\ v(x,y,0)=a_2(x,y)\end{matrix}\right\}(x,y)\in\Omega, \tag{6}$$

and boundary conditions

$$\left.\begin{matrix} u(x,y,t)=b_1(x,y,t), \\ v(x,y,t)=b_2(x,y,t),\end{matrix}\right\}(x,y)\in\Omega, t>0, \tag{7}$$

where $\Omega=\{(x,y): a\le x\le b, c\le y\le d\}$ is the computational domain, $\partial\Omega$ is its boundary, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined, a_1, a_2, b_1 and b_2 are the known functions, and Re is the Reynolds number.

These problems are essential examples of partial differential equations that are represent a wide range of phenomena such as heat transfer, mass transfer, petroleum reservoir modeling, subsurface pollution remediation, continuum mechanics, shock waves, acoustic waves, gas dynamics, elasticity, and so on [1, 2, 3, 4]. There is a wide body of literature on many forms of transport equations that are solved using various numerical and analytical approaches, for example: Tanaka and Chen [5] studied coupling dual reciprocity boundary element method and differential quadrature method for time-dependent diffusion problems. Bahadir [6] applied fully implicit finite-difference scheme for solving two-dimensional Burgers equations .Al-Saif and Al-Kanani [2] suggested alternative direction implicit formulation of the differential quadrature method for solving the unsteady state two-dimensional convection-diffusion equations. Abdou and Soliman [7] used variational iteration method for solving Burger's and coupled Burger's equations. Djidjeli et al. [8] studied global and compact meshless schemes for the unsteady convection-diffusion equation. Sharma and Methi [9] presented homotopy perturbation method approach for the solution of equation to unsteady flow of a polytropic gas. You [10] proposed a high-order pade' ADI method for solving unsteady convection-diffusion equations. Ali [11] studied radial basis function based meshless methods for the solution of Burger's equations.We will solve the two problems specifically utilizing these methods. The first is HPM that is created by Ji-Huan He for the first time in 1999. This method was further developed and improved by He, and applied it to nonlinear oscillators with discontinuities[12], nonlinear wave equation [13] and boundary value problem[14]. The second is RDTM, this method suggested by the Turkish mathematician Keskin [15-17] for the first time in 2009. It has received much attention since it has applied to solve a wide variety of problem by many author[7,18,19]. The third method is NATM this method suggested by Sabah [20] in 2018 , based on Taylors expansion in calculating non-linear terms, the new technique provided analytical solutions (approximate and exact). Comparison these methods with fourth-order Range-Kutta (RK-4) method. The aim of this study is to comparison between these methods in terms of accuracy of the solution and speed of the convergence. The results obtained indicate that the RDTM,HPM and NATM have high accuracy and less CPU time than the other methods.

2- The solution methods

In what follows, we will highlight briefly the main points of each of HPM, RDTM, and NATM to know each one how to works when apply to handle the equations of the problem.

2.1 The basic ideas of new analytic Taylor method:

The new analytic Taylor method is analytical method introduced in 2018, by Sabah [20], this technique based on Taylor's expansion in calculating non-linear terms in two-dimensional non-linear initial value problems.
To illustration the methodology of the proposed method, consider an equation of two-dimensional space and first order in time as the follows:

$$u_t(x,y,t) = F[u] + g(x,y), \tag{8}$$

with the initial condition

$$u(x,y,t_0) = f(x,y), \tag{9}$$

where, u unknown function , $F[u]$ is linear and non-linear operator and $g(x,y)$ is the known function.
Firstly, let us define Taylor's formula that we will use in the next theorem.

Definition 2.1.1 (Taylors Formula)[21]

Suppose $f:(a,b) \to R$ has $n+1$ derivatives on (a,b), and let $a < c < b$. For every $a < x < b$, there exists ω between c and x such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where, $R_n(x) = \frac{f^{(n+1)(\omega)}}{(n+1)!}(x-c)^{n+1}$.

Theorem 2.1.1[21]

Let u be the analytic solution of Eqs. (8)-(9) and $F(u)$ is the analytic of his arguments. Then we can find the Taylor series about t_0 for the solution $u(x,y,t)$ by using the following formula:

$$u(x,y,t) = a_0 + a_1\Delta t + a_2\frac{(\Delta t)^2}{2!} + a_3\frac{(\Delta t)^3}{3!} + \cdots + a_n\frac{(\Delta t)^n}{n!} + \cdots, \tag{10}$$

$$a_0 = u(x,y,t_0), a_1 = g(x,y) + F[u]\,|_{t=t_0}, a_2 = \frac{\partial F[u]}{\partial t}\,|_{t=t_0}, a_3 = \frac{\partial^2 F[u]}{\partial t^2}\,|_{t=t_0}$$

$$a_n = \frac{\partial^{n-1}F[u]}{\partial t^{n-1}}\,|_{t=t_0}. \tag{11}$$

To illustrate the above theorem, we need to integrate Eq.(8) from t_0 to t, to obtain :

$$u(x,y,t) = u(x,y,t_0) + g(x,y)\Delta t + \int_{t0}^{t} F[u]\,dt, \tag{12}$$

Where, $\Delta t = t - t_0$ and $F[u]$ can be expressed by the expand Taylor's series about t_0 as:

$$F[u] = [F[u]]_{t_0} + [F'[u]]_{t_0}\frac{\Delta t}{1!} + [F''[u]]_{t_0}\frac{(\Delta t)^2}{2!} + [[F'''[u]]_{t_0}\frac{(\Delta t)^3}{3!} + [F^{(n)}[u]]_{t_0}\frac{(\Delta t)^n}{n!} + \ldots, \tag{13}$$

Substituting Eq. (13) into Eq. (12) and integrating resulting equation to obtain the series solution (10). Eq. (10) is corresponding with the expand Taylors series of u about t_0 that means $a_i, i = 1,2 \ldots$ is having another represent, which is

$$a_1 = \frac{\partial u}{\partial t}\Big|_{t=t_0}, \; a_2 = \frac{\partial^2 u}{\partial t^2}\Big|_{t=t_0}, a_3 = \frac{\partial^3 u}{\partial t^3}\Big|_{t=t_0}, \ldots, \; a_n = \frac{\partial^n u}{\partial t^n}\Big|_{t=t_0},$$

That is very important in the process a solutions, because it is the key of the present method to find more terms of series solutions. To computing, the derivatives of $F[u]$, we will use the chain rule

$$F'[u] = \sum_{i=0}^{n}\sum_{j=0}^{i} F_{u_{x^{i-j}y^j}}[u]u_{x^{i-j}y^j t}, \tag{14}$$

$$F''[u] = \sum_{i=0}^{n}\sum_{j=0}^{i}\left(\sum_{k=0}^{n}\sum_{r=0}^{k} F_{u_{x^{i-j}y^j}, u_{x^{k-r}y^r}}[u]u_{x^{i-j}y^j t}u_{x^{k-r}y^r t} + F_{u_{x^{i-j}y^j}}[u]u_{x^{i-j}y^j tt}\right), \tag{15}$$

where, n is the highest partial derivative of u , $u_{x^{i-j}y^j} = \frac{\partial^i u}{\partial x^{i-j}\partial y^j}$, $u_{x^{i-j}y^j} = u_{y^j x^{i-j}}$, $F_{u_{x^{i-j}y^j}}[u] = \frac{\partial F[u]}{\partial u_{x^{i-j}y^j}}$, $F_{u_{x^{i-j}y^j}, u_{x^{k-r}y^r}}[u] = \frac{\partial^2 F[u]}{\partial u_{x^{i-j}y^j}\partial u_{x^{k-r}y^r}}$.

The series solution (10) at initial time $(t_0 = 0)$ is

$$u(x,y,t) = a_0 + a_1 t + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + \cdots + a_n\frac{t^n}{n!} + \cdots. \tag{16}$$

2.2 The basic ideas of reduced differential transform method:

The reduced differential transform method is an analytical-numerical technique introduced for the first time by Keskin [15,16] to study the analytical solutions of linear and nonlinear wave equations. It has received much attention since it has been applied to solve a wide variety of problems by many authors [22]. This suggested technique is highly efficient and powerful in obtaining the exact solutions as well as approximate solutions of mathematical modeling of many problems in technology, finance, engineering disciplines, natural sciences such as biology, physics, chemistry, and earth science, gives the solution in the form of rapidly convergent successive approximations, and is capable of handling Linear and nonlinear equations in a similar manner. The basic definitions and operations of Two-dimensional reduced differential transform method [23-27] are introduced as follows.

Definition 2.2.1

If function $u(x,y,t)$ is analytic and differentiated continuously with respect to time and space in the domain of interest, then let

$$U_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}, \tag{17}$$

Where, the t-dimensional spectrum function $U_k\,(x,y)$ is the transformed function. In this paper, the lowercase $u(x,y,t)$ represents the original function, while the uppercase $U_k\,(x,y)$stand for the transformed function.

Definition 2.2.2

The differential inverse transform of $U_k\,(x,y)$ is defined as;

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)(t-t_0)^k\,, \tag{18}$$

then by combining Eqs. (17), and (18) we obtain

$$u(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0} t^k. \tag{19}$$

Note that the function $u(x,y,t)$ can be written in a finite series as follows:

$$u_n(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)(t-t_0)^k + R_n(x,y,t). \tag{20}$$

Where the tail function $R_n(x,y,t)$, is negligibly small. Therefore the exact solution of problem is given

$$u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t).$$

Table 1: The fundamental operations of RDTM

Functional Form	Transformed Form
$u(x,y,t)$	$U_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}$
$w(x,y,t) = \alpha u(x,y,t) \pm \beta v(x,y,t)$	$W_k(x,y) = \alpha U_k(x,y) \pm \beta V_k(x,y)$
$w(x,y,t) = u(x,y,t)v(x,y,t)$	$W_k(x,y) = \sum_{r=0}^{\infty} V_r(x,y)U_{k-r}(x,y) = \sum_{r=0}^{\infty} U_r(x,y)V_{k-r}(x,y)$
$w(x,y,t) = \frac{\partial^r}{\partial t^r}u(x,y,t)$	$W_k(x,y) = \frac{(k+r)!}{k!}U_{k+r}(x,y)$
$w(x,y,t) = \frac{\partial^2}{\partial y^2}u(x,y,t)$	$W_k(x,y) = \frac{\partial^2}{\partial y^2}U_k(x,y)$
$w(x,y,t) = \frac{\partial^2}{\partial x^2}u(x,y,t)$	$W_k(x,y) = \frac{\partial^2}{\partial x^2}U_k(x,y)$

To illustration the methodology of the proposed method, consider the following nonlinear partial differential equation written in an operator form

$$L\big(u(x,y,t)\big) + R\big(u(x,y,t)\big) + N\big(u(x,y,t)\big) = g(x,y,t), \tag{21}$$

with initial condition

$$u(x,y,0) = f(x,y). \tag{22}$$

Where, L and R are a linear differential operators, Nu is a nonlinear operator and $g(x,y,t)$ is an inhomogeneous term.
According to the RDTM and Table 1, we can construct the following iteration formula:
$$(k+1)U_{k+1}(x,y,t) = G_k(x,y) - RU_k(x,y) - NU_K(x,y), \quad (23)$$
where, $U_k(x,y)$,and $G_k(x,y)$ are the transformations of the functions $u(x,y,t)$,and $g(x,y,t)$ respectively.
From the initial condition (22), we have
$$U_0(x,y) = f(x,y). \quad (24)$$
Substituting Eq.(24) into Eq.(23) and by straightforward iterative calculations, we get $U_k(x,y)$, values.Then the inverse transformation of the set of values $\{U_k(x,y)\}_{k=0}^{n}$ gives the n-terms approximation solution as,
$$\breve{u}(x,y,t) = \sum_{k=0}^{n} U_k(x,y)t^k, \quad (25)$$
Where n is order of approximation solution, Therefore, the exact solution of the problem is given by
$$u(x,y,t) = \lim_{n\to\infty} \tilde{u}_n(x,y,t). \quad (26)$$

2.3 The basic ideas of homotopy perturbation method:

The homotopy perturbation method was found for the first time by Chinese Mathematician, He [12]. The method is a powerful and efficient method to find the solutions to nonlinear equations. The coupling of the perturbation and homotopy methods is called the homotopy perturbation method. In this method, the solution is considered as the summation of an infinite series$u = \sum_{n=0}^{\infty} u_n$, which usually converges rapidly to the exact solutions. By the homotopy technique in topology, a homotopy is constructed with an embedding parameter$p \in [0,1]$, which is considered as a "small parameter", to illustrate the basic idea of Homotopy technique [28]. We consider the following nonlinear differential equation in operator form:
$$A(u) - f(r) = 0, r \in \Omega, \quad (27)$$
with boundary conditions
$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma, \quad (28)$$
where, A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω . The operator A can generally be divided into two parts L and N, where L a linear operator is and N is a nonlinear operator as;
$$A(u) = L(u) + N(u). \quad (29)$$
Thus, Eq. (27) can be rewritten as follows:
$$L(u) + N(u) - f(r) = 0. \quad (30)$$
By the homotopy technique, one constructs a homotopy $v(r,p)\colon \Omega \times [0,1] \to R$, which satisfies
$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0,1], r \in \Omega. \quad (31)$$
Where $p \in [0,1]$ is embedding parameter, and u_0 is an initial approximation of Eq.(27), which satisfies the boundary conditions. Clearly, from Eq. (31) we have
$$H(v,0) = L(v) - L(u_0) = 0, \quad (32)$$
$$H(v,1) = A(v) - f(r) = 0. \quad (33)$$
In addition, the process of changing p from zero to unity is just that of changing $v(r,p)$ from $u_0(r)$ to $u(r)$ in topology, this is called deformation, and $L(v) - L(u_0), A(v) - f(r)$ are called homotopic. We consider v as follows:
$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots = \sum_{n=0}^{\infty} p^n v_n. \quad (34)$$
According to the homotopy perturbation method, the best approximation solution of Eq.(30) can be explained as a following series of powers of p:
$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots = \sum_{n=0}^{\infty} v_n . \quad (35)$$

In the next, we will test these analytical –approximate methods by its application on three convection- diffusion problems.

3- Applications:

In this section, we solve the issues stated in (1) and (4,5) using analytical techniques to demonstrate the accuracy, efficiency, and convergence of the recommended methods NATM, RDTM, and HPM identified in the previous.

Problem-I (The Unsteady state two-dimensional convection - diffusion equation) [29]

Consider Eq.(1) with $\beta_x = \beta_y = -1\,,\ and\ L = 1$.Then Eq.(1) can be written as:

$$\frac{\partial u}{\partial t}(x,y,t) - \frac{\partial u}{\partial x}(x,y,t) - \frac{\partial u}{\partial y}(x,y,t) - \alpha_x \frac{\partial^2 u}{\partial x^2}(x,y,t) - \alpha_y \frac{\partial^2 u}{\partial y^2}(x,y,t) = 0, \quad (36)$$

with initial condition, $u(x,y,0) = a(e^{-c_x \cdot x} + e^{-c_y \cdot y}), 0 \le x, y \le 1, t > 0\,,$ (37)

where, $c_x = \frac{1 \pm \sqrt{1+4b\alpha_x}}{2\alpha_x} > 0, c_y = \frac{1 \pm \sqrt{1+4b\alpha_y}}{2\alpha_y} > 0.$

To solve this problem by NATM, we note that the highest derivative $of\ u$ is $n = 2$ and $t_0 = 0$, then according to relations in Eq. (11), we get

$$a_0 = u(x,y,0) = a(e^{-c_x \cdot x} + e^{-c_y \cdot y}), \quad (38)$$

$$a_1 = [F[u]]_0 = a\left((c_x^2\alpha_x - c_x) - e^{c_x x} + (c_y^2\alpha_y - c_y)e^{-c_y,y}\right), \quad (39)$$

$$\begin{aligned} a_2 &= [F'[u]]_0 = \textstyle\sum_{i=0}^{2}\sum_{j=0}^{i} F u_x^{j-j} y^j [a_0](a_1) x^{i-j} y^j, \\ &= a\left(c_x^2(c_x\alpha_x - 1)^2 e^{-c_x x} + c_y^2\left(c_y\alpha_y - 1\right)^2 e^{-c_y,y}\right), \end{aligned} \quad (40)$$

$$\begin{aligned} a_3 &= [F''[u]]_0 = \textstyle\sum_{i=0}^{2}\ \sum_{j=0}^{i} \Big(F_{u_{x^{i-j}y^j}}[a_0](a_2)_{x^{i-j}y^j} \\ &\quad + \textstyle\sum_{k=0}^{2}\ \sum_{r=0}^{k}\ F_{\rho_{x^{i-j}y^j},\rho_{x^{k-r}y}}[a_0](a_1)_{x^{i-j}y^j}(a_1)_{x^{k-r}y^r}\Big) \\ &= a\left(c_x^3(c_x\alpha_x - 1)^3 e^{-c_x x} + c_y^3\left(c_y\alpha_y - 1\right)^3 e^{-c_y,y}\right), \end{aligned} \quad (41)$$

$\vdots$

From Eq.(16), we get the exact solution

$$\begin{aligned} u(x,y,t) &= \textstyle\sum_{k=0}^{\infty} U_k(x,y)t^k = a(e^{c_x.x} + e^{-c_y.y}) + at\left((c_x^2\,\alpha_x - c_x)e^{c_x x} + \ (c_y^2\,\alpha_y - \ c_y)e^{-c_y,y}\right) + a\frac{t^2}{2}\Big(c_x^2(c_x\alpha_x - \\ &\quad 1)^2 e^{-c_x \cdot x} + c_y^2\left(c_y\alpha_y - 1\right)^2\Big)e^{-c_y \cdot y}\Big) + \cdots \\ &= a(e^{-c_x \cdot x} + e^{-c_y \cdot y})\left[1 + bt + \frac{(bt)^2}{2!} + \cdots\right] = a(e^{-c_x \cdot x} + e^{-c_y \cdot y})e^{bt}. \end{aligned} \quad (42)$$

This is the same exact solution of Eq.(**I**) [29].

Also, the analytical-approximate solution of the convection-diffusion equation can be found by using RDTM, as follows;

By taking the differential transform technique for Eq.(36), with the initial condition (37), we get the recurrence relation of the approximate solutions in the form

$$(k+1)U_{k+1}(x,y) = \frac{\partial}{\partial x}U_k(x,y) + \frac{\partial}{\partial y}U_k(x,y) + \ \alpha_x \frac{\partial}{\partial x^2}U_k(x,y) + \alpha_y \frac{\partial}{\partial y^2}U_k(x,y) \quad (43)$$

From the initial condition (37), we have

$$U_0(x,y) = a(e^{-c_x.x} + e^{-c_y.y}). \quad (44)$$

Now, substituting Eq.(44) into Eq.(43), we obtain

$$U_1 = \ a\left((c_x^2\alpha_x - c_x)e^{-c_x \cdot x} + (c_y^2\alpha_y - c_y)e^{-c_y \cdot y}\right), \quad (45)$$

$$U_2 = \frac{1}{2}a\left(c_x^2(c_x\alpha_x - 1)^2 e^{-c_x \cdot x} + c_y^2\left(c_y\alpha_y - 1\right)^2\right)e^{-c_y \cdot y}\Big), \quad (46)$$

$$U_3 = \frac{1}{6}a\left(c_x^3\left((c_x\alpha_x - 1)^3 e^{-c_x \cdot x} + c_y^3\left(\alpha_y c_y - 1\right)^3 e^{-c_y \cdot y}\right)\right), \quad (47)$$

$U_4 = \frac{1}{24} a\left(c_x^4\left((c_x\alpha_x - 1)^4 e^{-c_x \cdot x} + c_y^4\left(\alpha_y c_y - 1\right)^4 e^{-c_y \cdot y}\right)\right),$ (48)

$\vdots$

Finally, the differential inverse transform of $U_k(x, y)$ gives

$u(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y) t^k$ (49)

$= a(e^{c_x \cdot x} + e^{-c_y \cdot y}) + at\left((c_x^2\,\alpha_x - c_x)e^{c_x x} + \left(c_y^2\,\alpha_y - c_y\right)e^{-c_y \cdot y}\right) + a\frac{t^2}{2}(c_x^2(c_x\,\alpha_x - 1)^2 e^{-c_x x}$
$+ c_y^2\left(c_y\,\alpha_y - 1\right)^2 e^{-c_y \cdot y})) + \cdots$

$= a(e^{-c_x \cdot x} + e^{-c_y \cdot y})\left[1 + bt + \frac{(bt)^2}{2!} + \cdots\right] = a(e^{-c_x \cdot x} + e^{-c_y \cdot y})e^{bt}.$ (50)

This is the same exact solution of Eq.(1) [29].

Now, for solving Eq.(36) by HPM, we construct the following homotopies:

$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = P\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \alpha_x \frac{\partial^2 u}{\partial x_2} + \alpha_y \frac{\partial^2 u}{\partial y^2} - \frac{\partial u_0}{\partial t}\right],$ (51)

Suppose that the solution of the problem (36) is in the form

$u = p^{\circ} u_0 + p^1 u_{1+} p^2 u_2 + \cdots$ (52)

Substitute Eq.(52) into Eq.(51) and equating the coefficients of like power p; we will have the following set of differential equations:

$$\left.\begin{aligned} &p^0: \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \\ &p^1: \frac{\partial u_1}{\partial t} = \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y} + \alpha_x \frac{\partial u_0}{\partial x^2} + \alpha_y \frac{\partial u_0}{\partial y^2} \\ &p^2: \frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y} + \alpha_x \frac{\partial^2 u_1}{\partial x^2} + \alpha_y \frac{\partial^2 u_1}{\partial y^2} \\ &p^3: \frac{\partial u_3}{\partial t} = \frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial y} + \alpha_x \frac{\partial^2 u_2}{\partial x^2} + \alpha_y \frac{\partial u_2}{\partial y^2} \end{aligned}\right\} \quad (53)$$

$\vdots$

Solve equations in (53) to get the solutions:

$u_0 = a\left(e^{-c_x \cdot x} + e^{e^{-c_y \cdot y}}\right),$ (54)

$u_1 = at\left((c_x^2\alpha_x - c_x)e^{-c_x \cdot x} + (c_y^2\alpha_y - c_y)e^{-c_y \cdot y}\right),$

$u_2 = \frac{1}{2} at^2\left(c_x^2\left(c_x\alpha_x - 1\right)^2 e^{-c_x \cdot x} + c_y^2\left(c_y\alpha_y - 1\right)^2 e^{-c_y \cdot y}\right),$ (55)

$u_3 = \frac{1}{6} at^3\left(c_x^3(c_x\alpha_x - 1)^3 e^{-c_x \cdot x} + c_y^3\left(c_y\alpha_y - 1\right)^3 e^{-c_y \cdot y}\right),$ (56)

$\vdots$

Therefore, the approximate solution will be

$$u(x, y, t) = u_0 + u_1 + u_2 + u_3 + \cdots = a(e^{-Cxx} + e^{-Cy \cdot y})\left(1 + bt + \frac{(bt)^2}{2} + \cdots\right)$$
$$= a(e^{-c_x \cdot x} + e^{-c_y \cdot y}).e^{bt} \quad (57)$$

This is the same exact solution of Eq. (1)[29].

Problem-II (System of two-dimensional Burgers' equations)[30]

Consider the Eqs. (4) and (5) with $\alpha = 1$. Then these equations becomes

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0, \end{aligned} \quad (58)$$

with initial conditions

PII-1: $u(x, y, 0) = \frac{3}{4} - \frac{1}{4[1+\exp(\omega(x,y))]}, \quad v(x, y, 0) = \frac{3}{4} + \frac{1}{4[1+\exp(\omega(x,y))]}$ (59)

where, $\omega(x, y) = \frac{Re(y-x)}{8}$.

To solve this problem by NATM, we comparing these equations with Eq.(8) to get:

$g_1(x, y) = 0, g_2(x, y) = 0$ (60)

$$F[u,v] = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right),$$
$$G[u,v] = \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right), \quad (61)$$

we note that the highest derivative of u is $n = 2$ and $t_0 = 0$, then according to(11), we get

$$a_0 = u(x,y,0) = \frac{3}{4} - \frac{1}{4[1+\exp(\omega(x,y))]}, \quad (62)$$

$$b_0 = v(x,y,0) = \frac{3}{4} + \frac{1}{4[1+\exp(\omega(x,y))]}, \quad (63)$$

$$a_1 = [F[u,v]]_0 = \frac{-1}{128}\frac{Re\ \exp(\omega(x,y))}{[1+\exp(\omega(x,y))]^2}, \quad (64)$$
$$b_1 = [G[u,v]]_0 = \frac{1}{128}\frac{Re\ \exp(\omega(x,y))}{[1+\exp(\omega(x,y))]^2},$$

$$a_2 = [F'[u,v]]_0 = \sum_{i=0}^{2}\sum_{j=0}^{i} F_{u_{x^{i-j}y^j}}[a_0,b_0](a_1)_{x^{i-j}y^j}$$
$$= -\frac{1}{4096}\frac{[-1+\exp(\omega(x,y))]Re^2\exp(\omega(x,y))}{[1+\exp(\omega(x,y))]^3}, \quad (65)$$

$$b_2 = [G'[u,v]]_0 = \sum_{i=0}^{2}\sum_{j=0}^{i} G_{u_{x^{i-j}y^j}}[a_0,b_0](b_1)_{x^{i-j}y^j}$$
$$= \frac{1}{4096}\frac{[-1+\exp(\omega(x,y))]Re^2\exp(\omega(x,y))}{[1+\exp(\omega(x,y))]^3}, \quad (66)$$

From Eq.(16), we get the analytical approximate solution

$$S_2 = \sum_{i=0}^{2} a_i\frac{(t)^i}{(i)!}\ and\ K_2 = \sum_{i=0}^{2} b_i\frac{(t)^i}{(i)!}. \quad (67)$$

Also, the analytical-approximate solution can be found by using RDTM for the two-dimensional Burgers' equations. By taking the differential transform technique for equations in (58), with the initial conditions in (59), we get the recurrence relation of the approximate solutions in the form

$$(k+1)U_{k+1}(x,y) = \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}U_k(x,y) + \frac{\partial^2}{\partial y^2}U_k(x,y)\right) - \sum_{r=0}^{k} U_r\frac{\partial}{\partial x}U_{k-r} - \sum_{r=0}^{k} V_r\frac{\partial}{\partial y}U_{k-r}, \quad (68)$$

$$(k+1)V_{k+1}(x,y) = \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}V_k(x,y) + \frac{\partial^2}{\partial y^2}V_k(x,y)\right) - \sum_{r=0}^{k} U_r\frac{\partial}{\partial x}V_{k-r} - \sum_{r=0}^{k} V_r\frac{\partial}{\partial y}V_{k-r}, \quad (69)$$

where the t-dimensional spectrum functions $U_k(x,y)$ and $V_k(x,y)$ are transform functions . From equations in (59), we have

$$U_0(x,y) = u(x,y,0) = \frac{3}{4} - \frac{1}{4\omega^+}, \quad (70)$$
$$V_0(x,y) = v(x,y,0) = \frac{3}{4} + \frac{1}{4\omega^+}, \quad (71)$$

Embed the initial conditions (70) and (71) in the recurrence relations (68) and (69), give us the values of $U_k(x,y)$ and $V_k(x,y)$ as follows;

$$U_1(x,y) = -Re\ e^{\omega(x,y)}/128(\omega^+)^2, \quad (72)$$
$$U_2(x,y) = -Re^2 e^{\omega(x,y)}(-\omega^-)/8192(\omega^+)^3, \quad (73)$$
$$U_3(x,y) = -\left(e^{2\omega(x,y)Re} - 4e^{\omega(x,y)Re} + 1\right)Re^3 e^{\omega(x,y)Re}/786432(\omega^+)^4, \quad (74)$$
$$\vdots$$

And

$$V_1(x,y) = Re\ e^{\omega(x,y)}/128(\omega^+)^2, \quad (75)$$
$$V_2(x,y) = Re^2 e^{\omega(x,y)}(-\omega^-)/8192(\omega^+)^3, \quad (76)$$
$$V_3(x,y) = \left(e^{2\omega(x,y)} - 4e^{\omega(x,y)} + 1\right)Re^3 e^{\omega(x,y)}/786432(\omega^+)^4, \quad (77)$$
$$\vdots$$

After taking the inverse differential transformations of the set of values, we obtain

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^k = \frac{3}{4} - \frac{1}{4\omega^+} + \frac{-Re^{\omega(x,y)}}{128(\omega^+)^2}t - \frac{Re^2 e^{\omega(x,y)}(\omega^-)}{8192(\omega^+)^3}t^2 + \ldots$$
$$= \frac{3}{4} - 1/4[1 + \exp(\omega(x,y) - (Re/32)t)]\,. \quad (78)$$

$$v(x,y,t) = \sum_{k=0}^{\infty} V_k(x,y)t^k = \frac{3}{4} + \frac{1}{4\omega^+} + \frac{Re^{\omega(x,y)}}{128(\omega^+)^2}t + \frac{Re^2 e^{\omega(x,y)}(\omega^-)}{8192(\omega^+)^3}t^2 + \cdots$$

$$= \frac{3}{4} + 1/4[1 + \exp(\omega(x, y) - (Re/32)t)], \tag{79}$$

Where, $\omega^- = 1 - \exp(\omega(x, y))$,and $\omega^+ = 1 + \exp(\omega(x, y))$.

These are the exact solutions of Burger equations in (58) with initial conditions (59) [30].

Now, for solving equations in (58) by HPM, we construct the following homotopies:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} &= P\left[-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial u_0}{\partial t}\right] \\ \frac{\partial v}{\partial t} - \frac{\partial v_0}{\partial t} &= P\left[-u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial v_0}{\partial t}\right] \end{aligned}\right. , \tag{80}$$

We assume that the solutions for the system (58) as power series in P, of the form:

$$\left.\begin{aligned} u &= p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots \\ v &= p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots \end{aligned}\right\}, \tag{81}$$

Substitute these solutions(81) in system(80) and equating the coefficients of like power P, we will have the sets of differential equations :-

$$\left.\begin{aligned} p^0 &: \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \\ &\quad \frac{\partial v_0}{\partial t} - \frac{\partial v_0}{\partial t} = 0 \\ p^1 &: \frac{\partial u_1}{\partial t} = -u_0\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_0}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right] \\ &\quad \frac{\partial v_1}{\partial t} = -u_0\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_0}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right] \\ p^2 &: \frac{\partial u_2}{\partial t} = -u_0\frac{\partial u_1}{\partial x} - u_1\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_1}{\partial y} - v_1\frac{\partial u_0}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right] \\ &\quad \frac{\partial v_2}{\partial t} = -u_0\frac{\partial v_1}{\partial x} - u_1\frac{\partial v_0}{\partial x} - v_0\frac{\partial v_1}{\partial y} - v_1\frac{\partial v_0}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2}\right] \\ p^3 &: \frac{\partial u_3}{\partial t} = -u_0\frac{\partial u_2}{\partial x} - u_1\frac{\partial u_1}{\partial x} - u_2\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_2}{\partial y} - v_1\frac{\partial u_1}{\partial y} - v_2\frac{\partial u_0}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right] \\ &\quad \frac{\partial v_3}{\partial t} = -u_0\frac{\partial v_2}{\partial x} - u_1\frac{\partial v_1}{\partial x} - u_2\frac{\partial v_0}{\partial x} - v_0\frac{\partial v_2}{\partial y} - v_1\frac{\partial v_1}{\partial y} - v_2\frac{\partial v_0}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2}\right] \\ &\vdots \end{aligned}\right\} \tag{82}$$

The solutions of the above equations give the same result as solutions in equations (70-77), and the same analytical solutions (78) and (79).

Problem-2(II) (System of two-dimensional Burgers' equations) [31]

Consider the same system that is given in (58) with initial conditions;

PII-2: $u(x, y, 0) = S^x S^y, v(x, y, 0) = (S^x + 2S^x C^x)(S^y + 2S^y C^y),$ (89)

where, $S^x = \sin(\pi x)$, $S^y = \sin(\pi y)$, $C^x = \cos(\pi x), C^y = \cos(\pi y)$.

To solve this problem by NATM, we have been used the same definitions that are given in Eqs. (60) and (61).

We note that the highest derivative of u is $n = 2$ and$t_0 = 0$, then according to (11), we get

$$a_0 = u(x, y, 0) = S^x S^y, \tag{90}$$

$$b_0 = u(x, y, 0) = (S^x + 2S^x C^x)(S^y + 2S^y C^y), \tag{91}$$

$$a_1 = [F[u, v]]_0 = -\pi S^x[2ReC^y S^x C^x(S^y + 2S^y C^y) + 2ReS^x S^y (C^y)^2 - ReS^y (C^y)^2 + ReS^x S^y C^y + ReC^y + 2\pi S^y]/Re \tag{92}$$

$$\begin{aligned} b_1 = [G[u, v]]_0 = -\frac{1}{Re}\Big[4\pi\Big(\Big(Re\Big(2(S^x)^2 C^x + 1 - \left(\tfrac{1}{4}\right)(C^x)^2((C^x)^2 - (S^x)^2)^2\Big)((C^y)^2 - (S^y)^2) \\ -\left(\frac{1}{4}\right)ReC^y((C^x)^2 - (S^x)^2)^2 + \left(\frac{1}{2}\right)ReS^x S^y((C^x)^2 - (S^x)^2) + \left(\left(\frac{1}{2}\right)ReS^x C^y + 2\pi\right)2S^x C^x \\ +\left(\left(\frac{1}{4}\right)ReS^y C^x + \left(\frac{5}{4}\right)\pi\right)C^x - \left(\frac{1}{4}\right)ReC^y (C^x)^2 - 2\Big)2S^y C^y \\ +ReS^y\left(2(S^x)^2 C^x + 1 - \left(\frac{1}{2}\right)(C^x)^2 - \left(\frac{1}{2}\right)((C^x)^2 - (S^x)^2)^2\right)((C^y)^2 - (S^y)^2) \end{aligned}$$

$$-\left(\frac{1}{4}\right)ReS^yC^y((C^x)^2-(S^x)^2)-\left(\frac{1}{2}\right)ReS^x(C^y-1)(C^y+1)((C^x)^2-(S^x)^2)$$

$$+2\left(\frac{1}{2}\left(ReS^xC^y+\left(\frac{5}{2}\right)\pi\right)\right)(S^x)^2C^x+\left(\left(\frac{1}{2}\right)\pi S^y-\left(\frac{1}{4}\right)ReC^x(C^y-1)(C^y+1)\right)S^x$$
$$-\left(\tfrac{1}{4}\right)ReS^yC^y((C^x)^2-2)\Big)\Big], \tag{93}$$

from equation (16), we get the analytical approximate solution

$$s_1=\sum_{i=0}^{1}a_i\frac{(t)^i}{(i)}and k_1=\sum_{i=0}^{1}b_i\frac{(t)^i}{(i)!}. \tag{94}$$

To solve this problem by using RDTM, we have been used the same definitions that are given in equations (68-71) and obtain on the same analytical solutions (90-94) .Also, for solving this problem by using HPM, we have been used the same definitions that are given in equations (80-82) and obtain on the same analytical solutions (90-94), such that

$$p^0:\begin{cases}u_0=a_0\\ v_0=b_0\end{cases},\ p^1:\begin{cases}u_1=a_1.t\\ v_1=b_1.t\end{cases},\ \cdots \tag{95}$$

therefore, the approximate solution given as:

$$\begin{aligned}u(x,y,t)&=u_0+u_1+\cdots\\ v(x,y,t)&=v_0+v_1+\cdots\end{aligned} \tag{96}$$

4- Convergence Analysis

In this subject, we study the analysis of convergence depending on the theorem, which we will mention it in the next paper.

Theorem 4.1]32]

Let M be an operator from a Hilbert space H into H and u be the exact solution .The approximate solution $\sum_{i=0}^{\infty}u_i=\sum_{i=0}^{\infty}a_i\frac{(\Delta t)^i}{(i)!}$ is convergence to exact solution u when $\exists 0\le\alpha<1\ \|u_{i+1}\|\le\alpha\|u_i\|,\forall i\in\mathbb{N}\cup\{0\}$

Proof: We want to show that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence,

$$\|S_{n+1}-S_n\|=\|u_{n+1}\|\le\alpha\|u_n\|\le\alpha^2\|u_{n-1}\|\le\cdots\le\alpha^n\|u_1\|\le\alpha^{n+1}\|u_0\|$$

Now for $n,m\in\mathbb{N},n\ge m$

$$\begin{aligned}\|S_n-S_m\|&=\|(S_n-S_{n-1})+(S_{n-1}-S_{n-2})+\cdots+(S_{m+1}-S_m)\|\\ &\le\|S_n-S_{n-1}\|+\|S_{n-1}-S_{n-2}\|+\cdots+\|S_{m+1}-S_m\|\\ &\le\alpha^n\|u_0\|+\alpha^{n-1}\|u_0\|+\cdots+\alpha^{m+1}\|u_0\|\\ &\le(\alpha^{m+1}+\alpha^{m+2}+\cdots+\alpha^n)\|u_0\|=\alpha^{m+1}\frac{1-\alpha^{n-m}}{1-\alpha}\|u_0\|\end{aligned}$$

Hence, $\lim_{n,m\to\infty}\|S_n-S_m\|=0$ that is mean $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space H then there exist $S\in H$ such that$\lim_{n\to\infty}S_n=S$, where$S=u$.

Corollary 4.1.1

From theorem (4.1) $\sum_{i=0}^{\infty}u_i=\sum_{i=0}^{\infty}a_i\frac{(\Delta t)^i}{(i)!}$ convergence to exact solution u when $0\le\alpha_i<1,i=0,1,2,\ldots$

Now, to illustrate the convergence of analytical approximate solutions for the two problems of equation, we applied Corollary (4.1.1) as in the table on (4).

5- Discussions

In this section, we show the three-dimensional figures obtained by using HPM, RDTN, NATM and RK-4, we give data for the errors between the numerical solution and analytical solutions of two-dimensional convection diffusion equation .Figs. (1-3) show that the exact solution, approximate solution and RK-4 solution for problems I and (II), figs. (4-5) show that the approximate solution for problem 2(II) obtained by applying the suggested methods NATM, HPM, RDTM and RK-4. L_∞ For the series approximation solutions of the two-test problem computed by:

$$\| E \|_{L_\infty} = \max_{i,j=0,1,\dots,N} \left(\left|\rho_{\text{exact}}\,(x,y) - \rho_{\text{approx}}\,(x,y)\right|\right)$$

Where t is fixed. Table (1) shows comparison of errors and CPU time between RDTM, HPM, NATM, RK4, LDQM and BDQM for different values of h and at$t = 0.1$. The result confirms that the NATM, RDTM, and HPM more accurate and less CPU time compared to other methods. Tables (2, 3) are clarify errors of u and v obtained in solving problem (II) for different values of h ,at $t = 0.01$ and $R = 100$. We can say that NATM, RDTM and HPM are effective and good approaches to find the solution of non-linear system of two-dimensional Burgers equation. To test the convergence of the proposed methods, that is, the convergence of the approximate solution to the exact solution, we have successfully applied Corollary$(4.1.1)$; this is explained in the tables (4).With exception of the RK-4, because the solution is not in a form$K_n = \sum_{i=0}^{n} v_i$.

6- Conclusion and remarks

In this study, as a conclusion, we have four methods that have been successfully applied to find the solutions of the convection-diffusion equations. We use maple 16 to calculate the functions obtained from the suggested methods. Results show that RDTM, HPM, and NATM are powerful mathematical tools for solving systems of nonlinear partial differential equations. , also NATM, RDTM, and HPM are more accurate and have less CPU time compared to other methods.

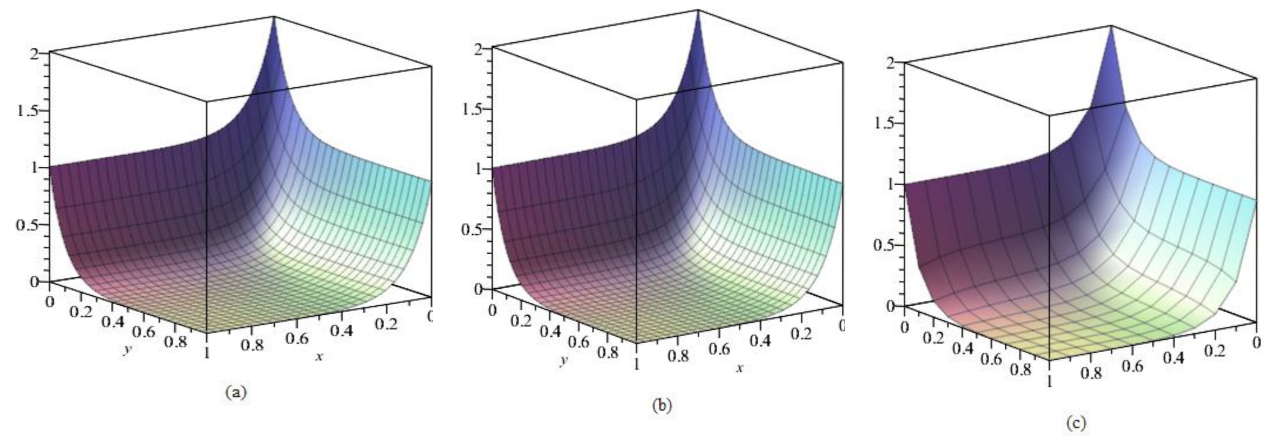


Fig.1. Graphics of $u(x, y, t)$for problem(**I)** (a) Exact (b) Approximate solution (S_5) , (c) RK-4.At $t = 0.1, \alpha_x = \alpha_y = 0.1$.

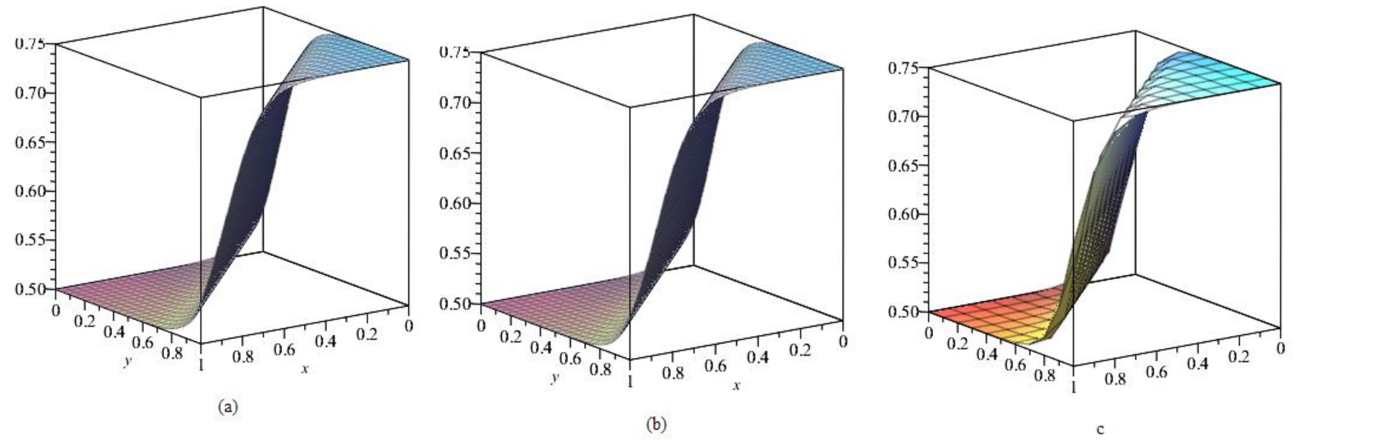


Fig.2. Graphics of $u(x, y, t)$for problem $(\mathbf{II})$(a) Exact, (b) Approximate solution (S_5), (c) RK-4. $Re = 100, t = 0.01$.

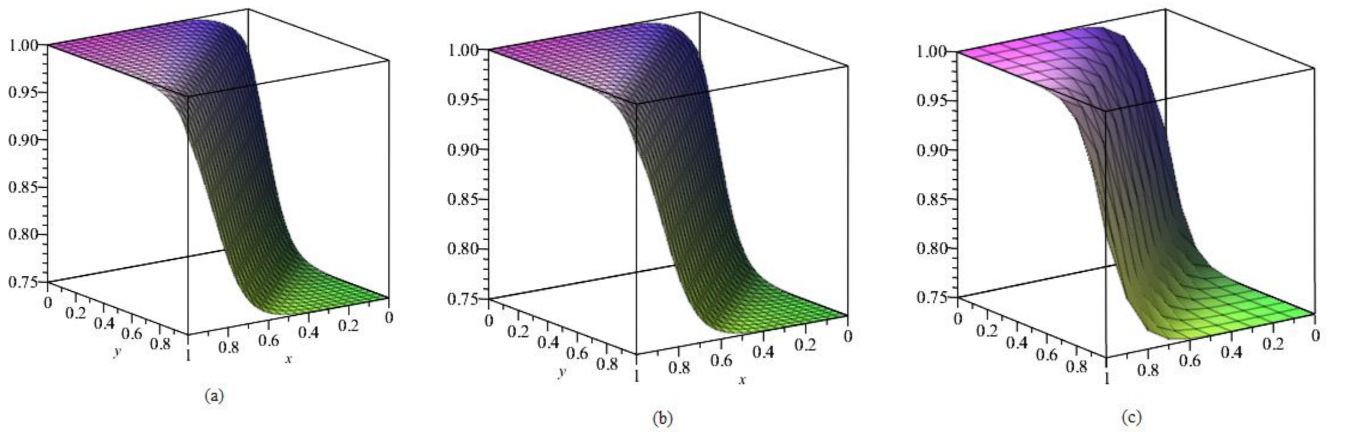
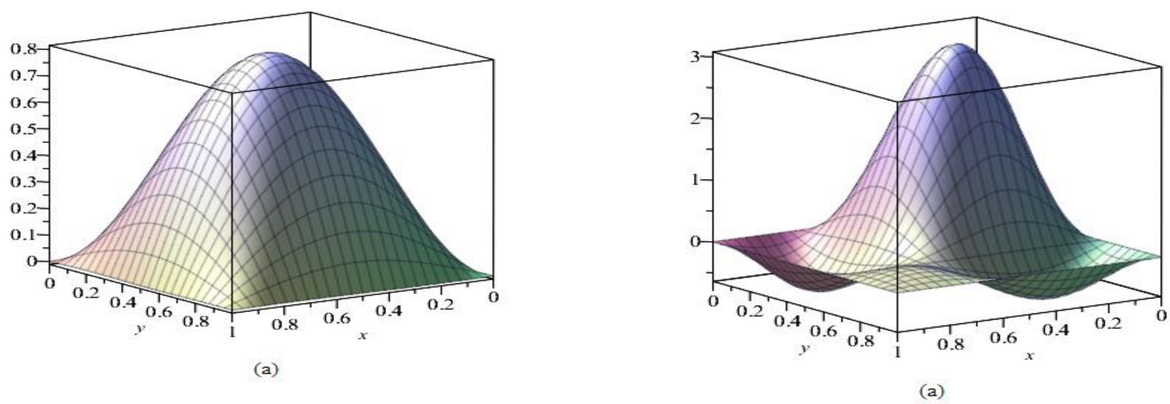


Fig.3. Graphics of $v(x, y, t)$for problem $(\mathbf{II})$ (a) Exact, (b) Approximate solution (k_5), (c) RK-4. $Re = 100, t = 0.01$.



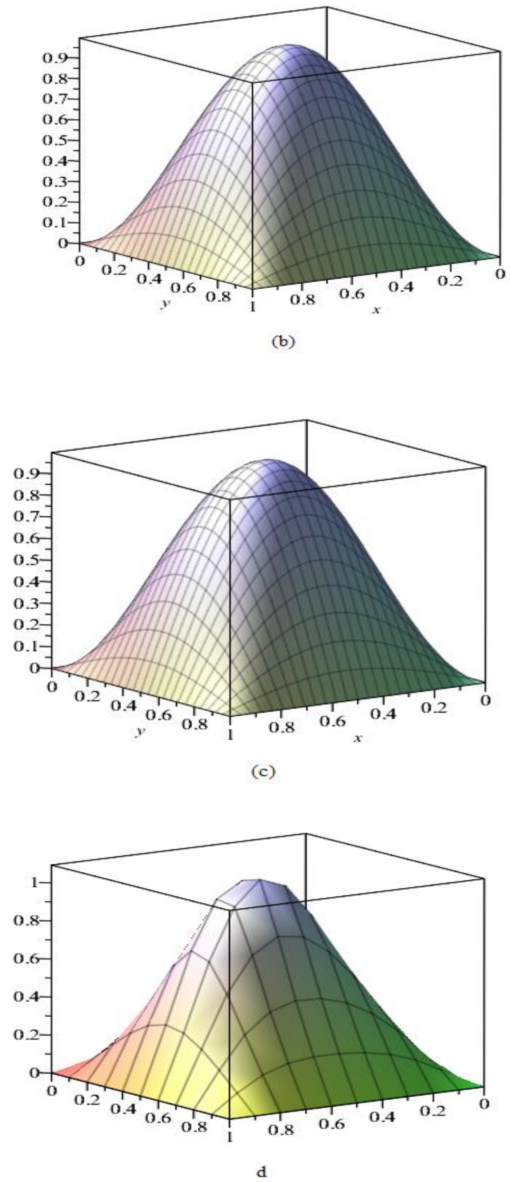


Fig.4. Graphics $u(x,y,t)(S_3)$ of problem 2(**II**) for *(a) NATM,* $(b)HPM, (c)RDTM, (d)\ RK4at\ R = 100, t = 0.01$

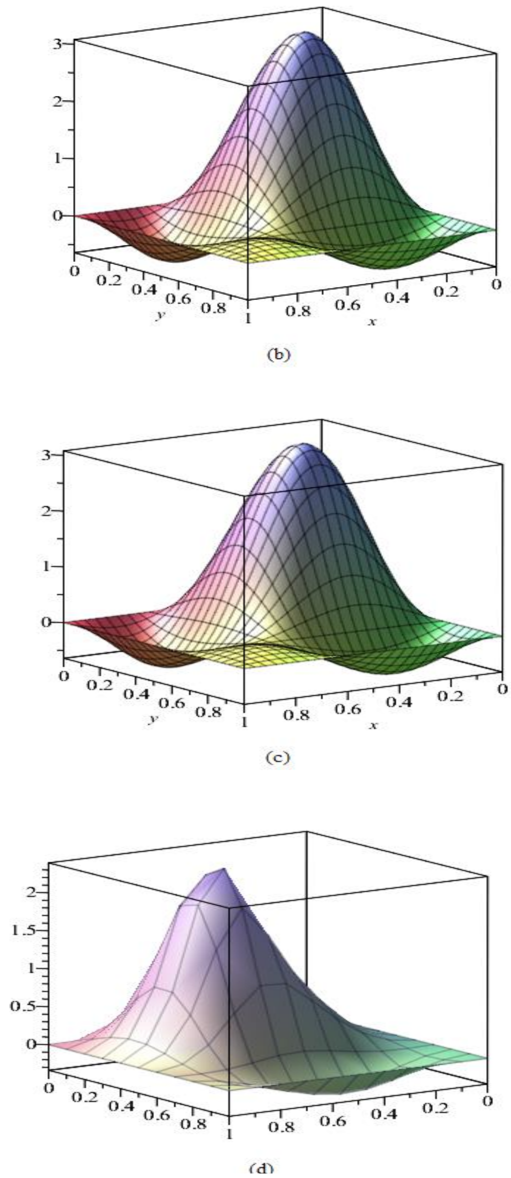


Fig.5. Graphics $v(x,y,t)(K_3)$ of problem 2(**II**) for (a)NAM,(b) HPM, (c) RDTM, (d) RK $at\ R = 100, t = 0.01$

Table 1: Comparison **L∞** and CPU for problem **I** between RDTM, HPM, NATM (S_5), and RK4.$at\ \alpha_y = \alpha_x = 0.1$,

h	**Error**	**Method with** $t = 0.1$					
		RDTM	**HPM**	**NATM**	**RK-4**	**LDQM**	**BDQM**
0.25	**L∞** **CPU**	$1.67E-12$ 0.031	$1.67E-12$ 0.031	$1.67E-12$ 0.031	0.2849 2.2820	$2.57E-06$ 0.509	$4.18E-07$ 0.503
0.17	**L∞** **CPU**	$1.67E-12$ 0.047	$1.67E-12$ 0.047	$1.67E-12$ 0.047	0.2691 1.3120	$1.38E-05$ 0.509	$5.64E-06$ 0.503
0.125	**L∞** **CPU**	$1.67E-12$ 0.031	$1.67E-12$ 0.031	$1.67E-12$ 0.031	0.2130 1.5310	$3.22E-05$ 0.772	$7.95E-06$ 0.757
0.1	**L∞** **CPU**	$1.67E-12$ 0.031	$1.67E-12$ 0.031	$1.67E-12$ 0.031	0.1609 2.4530	$5.35E-05$ 1.047	$9.18E-06$ 1.015

Table2: Comparison **L∞** and CPU for problem (**II**) between RDTM, NATM, HPM (S_5)and RK-4.$at\ Re = 100$.for u.

h	Error	Method with $t = 0.01$			
		RDTM	**HPM**	**NATM**	**RK-4**
$\mathbf{0.25}$	**L∞** **CPU**	$1.55\ E-11$ 0.031	$1.55\ E-11$ 0.031	$1.55\ E-11$ 0.031	$1.219\ E-01$ 1.310
$\mathbf{0.17}$	**L∞** **CPU**	$1.55\ E-11$ 0.016	$1.55\ E-11$ 0.016	$1.55\ E-11$ 0.016	$7.83\ E-02$ 1.328
$\mathbf{0.125}$	**L∞** **CPU**	$1.55\ E-11$ 0.032	$1.55\ E-11$ 0.032	$1.55\ E-11$ 0.032	$3.70\ E-02$ 1.406
$\mathbf{0.1}$	**L∞** **CPU**	$1.55\ E-11$ 0.047	$1.55\ E-11$ 0.047	$1.55\ E-11$ 0.047	$1.96\ E-02$ 1.641

Table3: Comparison **L∞** and CPU for problem (**II**) between RDTM, NATM, HPM(k_5)and RK-4.$at\ Re = 100$.for v.

h	Error	Method with $t = 0.01$			
		RDTM	**HPM**	**NATM**	**RK-4**
$\mathbf{0.25}$	**L∞** **CPU**	$1.55\ E-11$ 0.031	$1.55\ E-11$ 0.031	$1.55\ E-11$ 0.031	$1.219\ E-01$ 1.310
$\mathbf{0.17}$	**L∞** **CPU**	$1.55\ E-11$ 0.016	$1.55\ E-11$ 0.016	$1.55\ E-11$ 0.016	$7.83\ E-02$ 1.328
$\mathbf{0.125}$	**L∞** **CPU**	$1.55\ E-11$ 0.032	$1.55\ E-11$ 0.032	$1.55\ E-11$ 0.032	$3.70\ E-02$ 1.406
$\mathbf{0.1}$	**L∞** **CPU**	$1.55\ E-11$ 0.047	$1.55\ E-11$ 0.047	$1.55\ E-11$ 0.047	$1.96\ E-02$ 1.641

Table4: Convergence solutions problems **I** with $t = 0.1$ and **(II)&2(II)** with $t = 0.01,\ and\ Re = 100,$

solutions		u_3			v_3		
Problem/method		**NATM**	**HPM**	**RDTM**	**NATM**	**HPM**	**RDTM**
I	α_0	0.1000001	0.1000001	0.1000001	-----------	------------	-----------
	α_1	0.0050000	0.0050000	0.0050000	-----------	------------	-----------
	α_2	0.0033333	0.0033333	0.0033333	-----------	------------	-----------
(II)	α_0	0.0013711	0.0013711	0.0013711	0.0009861	0.0009861	0.0009861
	α_1	0.0067586	0.0067586	0.0067586	0.0067586	0.0067586	0.0067586
	α_2	0.0091022	0.0091022	0.0091022	0.0091022	0.0091022	0.0091022
2(II)	α_0	0.0373419	0.0373419	0.0373419	0.0713448	0.0713448	0.0713448
	α_1	0.0731304	0.0731304	0.0731304	0.1132454	0.1132454	0.1132454
	α_2	0.1222978	0.1222978	0.1222978	0.1428962	0.1428962	0.1428962

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