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Backward doubly stochastic differential equations (BDSDEs): Existence and Uniqueness

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Abstract: In this paper, we present a class of stochastic differential equations with terminal condition, called backward doubly stochastic differential equations (BDSDEs). Precisely, we will prove the existence and uniqueness of the solutions of FBDSDEs but under weaker conditions.

Keywords: Itô Formula, Brownian motion/Wiener process, Backward doubly stochastic differential equations, stochastic partial differential-integral equations.

1 Introduction

Linear backward stochastic differential equations were first studied by Bismut [4] in 1973. The general nonlinear backward stochastic differential equations were first studied by Pardoux and Peng, [12], in 1990.

Forward-backward stochastic differential equations (FBSDEs in short) were first studied by Antonelli (see [3]), where the system of such equations is driven by Brownian motion on a small time interval. The proof there relies on the fixed point theorem. Since then FBSDEs are encountered in stochastic optimal control problem and mathematical finance. There are also many other methods to study FBSDEs on an arbitrarily given time interval. For example, the four-step scheme approach of Ma et al. [9], in which the authors proved the existence and uniqueness of solutions for fully coupled FBSDEs on an arbitrarily given time interval, where the diffusion coefficients were assumed to be nondegenerate and deterministic. Their work is based on continuation method. See also Hu and Peng [7], Pardoux and Tang [14], Peng and Wu [15], and Yong [18]. There is also a numerical approach for handling some linear FBSDEs as e.g. in Delarue and Menozzi [5] and Ma et al. [11]; see also Ma and Yong [10], Hu et al. [6] and [8].

A new class of stochastic differential equations with terminal condition, called backward doubly stochastic differential equations (BDSDEs) was introduced in 1994 by Pardoux and Peng in [13]. Precisely, they proved there the existence and uniqueness of the solutions of this kind of systems and produced also a probabilistic representation of certain quasi-linear stochastic partial differential equations (SPDEs) extending a Feynman Kac formula for linear SPDEs.

Peng and Shi [16], introduced fully coupled FBDSDEs and showed the existence and uniqueness of their solutions with arbitrarily fixed time duration and under some monotone conditions.

In this paper, we will prove the existence and uniqueness of the solutions of FBDSDEs but under weaker conditions than [16]. We establish in particular the existence and uniqueness of the solutions of the following Markovian fully coupled FBDThe Existence and Uniqueness Theorem of FBDSDEs

2 Existence and Uniqueness Theorem of FBDSDEsSDEs

Let

$$\begin{aligned}
&b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \to \mathbb{R}^n,\\
&\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \to \mathbb{R}^{n\times d},\\
&f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \to \mathbb{R}^m,\\
&g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n\times l} \times \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times l},\\
&\Psi : \Omega \times \mathbb{R}^m \to \mathbb{R}^n,\\
&h : \Omega \times \mathbb{R}^n \to \mathbb{R}^m,
\end{aligned}$$

be a given mappings satisfying assumptions to be given shortly.

Consider the following system of equations, which we call *forward-backward doubly stochastic differential equations* (FBDSDEs):

$$\begin{cases}
dy_t = b\left(t, y_t, Y_t, z_t, Z_t\right) dt + \sigma\left(t, y_t, Y_t, z_t, Z_t\right) dW_t - z_t d\overleftarrow{B}_t\\
dY_t = f\left(t, y_t, Y_t, z_t, Z_t\right) dt + g\left(t, y_t, Y_t, z_t, Z_t\right) d\overleftarrow{B}_t + Z_t dW_t,\\
y_0 = \Psi\left(Y_0\right), Y_T = h\left(y_T\right).
\end{cases} \tag{1}$$

A *solution* of (1) is a stochastic process (y, Y, z, Z) such that for each $t \in [0,T]$ we have $a.s$:

$$\begin{cases}
y_t = \Psi\left(Y_0\right) + \int_0^t b\left(s, y_s, Y_s, z_s, Z_s\right) ds + \int_0^t \sigma\left(s, y_s, Y_s, z_s, Z_s\right) dW_s - \int_0^t z_t d\overleftarrow{B}_s,\\
Y_t = h\left(y_T\right) - \int_t^T f\left(s, y_s, Y_s, z_s, Z_s\right) ds - \int_t^T g\left(s, y_s, Y_s, z_s, Z_s\right) d\overleftarrow{B}_s - \int_t^T Z_t dW_s.
\end{cases} \tag{2}$$

We remark that this system is studied only if either we are given $y_0 = \Psi\left(Y_0\right)$ or $Y_T = h\left(y_T\right)$. Usually, we assume $y_0 = x$, a given element x of $\mathbb{R}^n$, in which case $\Psi : \mathbb{R}^m \to \mathbb{R}^n$ is the construct mapping $\Psi\left(Y\right) = x \ \ \forall Y \in \mathbb{R}^m$. If this is the case, however, the proof we shall introduce below can be reduced, and in particular, in the proof of Lemma 6 below, we would not have the cases I) below because $\beta_2 = 0$ in this case.

We want now to set our assumptions for the system (2). Given an $m \times n$ full-rank matrix R let us introduce the following notation:

$$\upsilon = \left(y, Y, z, Z\right),$$

$$A\left(t, \upsilon\right) := \left(R^* f, Rb, R^* g, R\sigma\right)\left(t, \upsilon\right),$$

$$\left\langle A, \upsilon\right\rangle := \left\langle y, R^* f\right\rangle + \left\langle Y, Rb\right\rangle + \left\langle z, R^* g\right\rangle + \left\langle Z, R\sigma\right\rangle,$$

where $(^*)$ denotes matrix transpose, and

$$R^* g = \left(R^* g_1, \cdots, R^* g_l\right), R\sigma = \left(R\sigma_1, \cdots, R\sigma_d\right).$$

We set the following assumptions.

(A1) (Monotonicity condition): $\forall\, \upsilon = (y, Y, z, Z), \overline{\upsilon} = \left(\overline{y}, \overline{Y}, \overline{z}, \overline{Z}\right) \in \mathbb{R}^{n+m+n\times l+m\times d},\ \forall\, t \in [0,T]$,

$$\langle A(t,\upsilon) - A(t,\overline{\upsilon}), \upsilon - \overline{\upsilon}\rangle \leq -\theta_1 \left(|R(y-\overline{y})|^2 + \|R(z-\overline{z})\|^2\right) - \theta_2 \left(\left|R^*\left(Y-\overline{Y}\right)\right|^2 + \left\|R^*\left(Z-\overline{Z}\right)\right\|^2\right).$$

(A2) We have

$$\left\langle \Psi(Y) - \Psi\left(\overline{Y}\right), R^*\left(Y-\overline{Y}\right)\right\rangle \leq -\beta_2 \left|R^*\left(Y-\overline{Y}\right)\right|^2,\ \forall\, Y, \overline{Y} \in \mathbb{R}^m,$$
$$\langle h(y) - h(\overline{y}), R(y-\overline{y})\rangle \geq \beta_1 |R(y-\overline{y})|^2,\ \forall\, y, \overline{y} \in \mathbb{R}^n.$$

Here $\theta_1, \theta_2, \beta_1$, and β_2 are given nonnegative constants with $\theta_1 + \theta_2 > 0$, $\beta_1 + \beta_2 > 0$, $\theta_1 + \beta_2 > 0$, $\theta_2 + \beta_1 > 0$. Moreover, we have $\theta_1 > 0, \beta_1 > 0$ (resp. $\theta_2 > 0, \beta_2 > 0$) when $m > n$ (resp. $n > m$).

(A3) For each $\upsilon \in \mathbb{H}^2$, $A(t,\upsilon)$ is an $\mathcal{F}_t$-measurable vector process defined on $[0,T]$ with $\widetilde{A}(.,0) \in \mathbb{H}^2$ and $\forall\, y \in \mathbb{R}^n, h(y)$ is an $\mathcal{F}_T$-measurable vector process with $h(0) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m)$, and for each $Y \in \mathbb{R}^m, \Psi(Y)$ is an $\mathcal{F}_0$-measurable vector process with $\Psi(0) \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n)$.

(A4) (Lipschitz condition): $\exists\, c > 0,\ 0 < \gamma < 1$ and $0 < \gamma' < 1$ such that

$$\left|b(t,y,Y,z,Z) - b\left(t,\overline{y},\overline{Y},\overline{z},\overline{Z}\right)\right|^2 \leq c\left(|y-\overline{y}|^2 + \left|Y-\overline{Y}\right|^2 + \|z-\overline{z}\|^2 + \left\|Z-\overline{Z}\right\|^2\right),$$
$$\left|f(t,y,Y,z,Z) - f\left(t,\overline{y},\overline{Y},\overline{z},\overline{Z}\right)\right|^2 \leq c\left(|y-\overline{y}|^2 + \left|Y-\overline{Y}\right|^2 + \|z-\overline{z}\|^2 + \left\|Z-\overline{Z}\right\|^2\right),$$
$$\left|\sigma(t,y,Y,z,Z) - \sigma\left(t,\overline{y},\overline{Y},\overline{z},\overline{Z}\right)\right|^2 \leq c\left(|y-\overline{y}|^2 + \left|Y-\overline{Y}\right|^2 + \left\|Z-\overline{Z}\right\|^2\right) + \gamma' \|z-\overline{z}\|^2,$$
$$\left|g(t,y,Y,z,Z) - g\left(t,\overline{y},\overline{Y},\overline{z},\overline{Z}\right)\right|^2 \leq c\left(|y-\overline{y}|^2 + \left|Y-\overline{Y}\right|^2 + \|z-\overline{z}\|^2\right) + \gamma \left\|Z-\overline{Z}\right\|^2,$$
$$\left|\Psi(Y) - \Psi\left(\overline{Y}\right)\right| \leq c\left|Y-\overline{Y}\right|, |h(y) - h(\overline{y})| \leq c\,|y-\overline{y}|,$$

for all argument written in the right hand side of each inequality and for all $t \in [0,T]$ whenever they appear.

(1) **Remark.** Assumptions (A1) and (A2) can be replaced by the following ones with essentially the same proofs of the solution theorem and its lemmas.

(A1)′ $\forall\, \upsilon = (y, Y, z, Z), \overline{\upsilon} = \left(\overline{y}, \overline{Y}, \overline{z}, \overline{Z}\right) \in \mathbb{R}^{n+m+n\times l+m\times d},\ \forall\, t \in [0,T]$

$$\langle A(t,\upsilon) - A(t,\overline{\upsilon}), \upsilon - \overline{\upsilon}\rangle \geq \theta_1 \left(|R(y-\overline{y})|^2 + \|R(z-\overline{z})\|^2\right) + \theta_2 \left(\left|R^*\left(Y-\overline{Y}\right)\right|^2 + \left\|R^*\left(Z-\overline{Z}\right)\right\|^2\right),$$

and

(A2)′

$$\left\langle \Psi(Y) - \Psi\left(\overline{Y}\right), R^*\left(Y-\overline{Y}\right)\right\rangle \geq \beta_2 \left|R^*\left(Y-\overline{Y}\right)\right|^2,\ \forall\, Y, \overline{Y} \in \mathbb{R}^m,$$
$$\langle h(y) - h(\overline{y}), R(y-\overline{y})\rangle \leq -\beta_1 |R(y-\overline{y})|^2,\ \forall\, y, \overline{y} \in \mathbb{R}^n.$$

We shall consider only assumptions (A1)–(A4).

(2) **Remark.** Note that, since $R : \mathbb{R}^n \to \mathbb{R}^m$ (a matrix $m \times n$) has a full-rank, if $m > n$, then R is injective and there exists $C_1, C_2 > 0$ such that

$$C_1 \left|y\right|_{\mathbb{R}^n} \leq \left|Ry\right|_{\mathbb{R}^m} \leq C_2 \left|y\right|_{\mathbb{R}^n}, \forall\, y \in \mathbb{R}^n.$$

In fact $C_1 = \min\{|Rx|_{\mathbb{R}^m}, x \in S^{n-1}\}$, which is positive since R is injective, and $C_2 = \max\{|Rx|_{\mathbb{R}^m}, x \in S^{n-1}\} = \|R\|$.

In the case $m < n$, we have a similar inequality for the transpose matrix R^*, which is injective as a mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$, namely:

$$C_3 \left|Y\right|_{\mathbb{R}^m} \leq \left|R^*Y\right|_{\mathbb{R}^n} \leq C_4 \left|Y\right|_{\mathbb{R}^m}, \forall\, Y \in \mathbb{R}^m,$$

where $C_3 = \min\{|R^*Y|_{\mathbb{R}^n}; Y \in S^{m-1}\} > 0$, and $C_4 = \max\{|R^*Y|_{\mathbb{R}^n}; Y \in S^{m-1}\} = \|R^*\|$.

If $m = n$, then $\text{rank}(R) = m = n$, and so R is invertible. Therefore we get the following result:

$$C_5 \left|y\right|_{\mathbb{R}^n} \leq \left|Ry\right|_{\mathbb{R}^n} \leq C_6 \left|y\right|_{\mathbb{R}^n}, \forall\, y \in \mathbb{R}^n,$$

where $C_5 = \|R^{-1}\|^{-1}$ and $C_6 = \|R\|$.

Let us state on Itô's formula (see e.g. [17]), that will be used throughout the paper.

(3) **Proposition.** *Let* $(\alpha, \widehat{\alpha}) \in \left[\mathcal{M}^2(0,T;\mathbb{R}^n)\right]^2, (\beta, \widehat{\beta}) \in \left[\mathcal{M}^2(0,T;\mathbb{R}^n)\right]^2, (\gamma, \widehat{\gamma}) \in \left[\mathcal{M}^2(0,T;\mathbb{R}^{n\times l})\right]^2$, *and* $(\delta, \widehat{\delta}) \in \left[\mathcal{M}^2(0,T;\mathbb{R}^{n\times d})\right]^2$ *Assume that*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s \overleftarrow{dB}_s + \int_0^t \delta_s dW_s,$$

and

$$\widehat{\alpha}_t = \widehat{\alpha}_0 + \int_0^t \widehat{\beta}_s ds + \int_0^t \widehat{\gamma}_s \overleftarrow{dB}_s + \int_0^t \widehat{\delta}_s dW_s,$$

for $t \in [0,T]$. *Then, for each* $t \in [0,T]$,

$$\langle \alpha_t, \widehat{\alpha}_t \rangle = \langle \alpha_0, \widehat{\alpha}_0 \rangle + \int_0^t \langle \alpha_s, d\widehat{\alpha}_s \rangle + \int_0^t \langle d\alpha_s, \widehat{\alpha}_s \rangle + \int_0^t d \left\langle \alpha, \widehat{\alpha} \right\rangle_s \quad \mathbb{P} - a.s.,$$

and

$$\begin{aligned} \mathbb{E}\left[\langle \alpha_t, \widehat{\alpha}_t \rangle\right] = \mathbb{E}\left[\langle \alpha_0, \widehat{\alpha}_0 \rangle\right] + \mathbb{E}\Big[\int_0^t \langle \alpha_s, d\widehat{\alpha}_s \rangle\Big] + \mathbb{E}\Big[\int_0^T \langle d\alpha_s, \widehat{\alpha}_s \rangle\Big] \\ - \mathbb{E}\Big[\int_0^t \langle \gamma_s, \widehat{\gamma}_s \rangle ds\Big] + \mathbb{E}\Big[\int_0^t \langle \delta_s, \widehat{\delta}_s \rangle ds\Big]. \end{aligned}$$

In the following, our main theorems will be given.

(4) **Theorem.** *[Uniqueness] Under assumptions (A1)–(A4) (or (A1)′, (A2)′, (A3)–(A4); cf. Remark 1) FBDSDEs (1) has at most one solution* (y, Y, z, Z) *in* $\mathbb{H}^2$.

Proof. Let $\upsilon := (y, Y, z, Z)$ and $\upsilon' := (y', Y', z', Z')$ be two solutions of (1). Denote

$$\widehat{\upsilon} = \left(\widehat{y}, \widehat{Y}, \widehat{z}, \widehat{Z}\right) = (y - y', Y - Y', z - z', Z - Z').$$

Applying integration of parts (Proposition 3) to $\left\langle R\widehat{y}, \widehat{Y}\right\rangle$ on $[0,T]$ we get

$$\begin{aligned} 0 &\leq \mathbb{E}\left[\langle R\widehat{y}_T, h(y_T) - h(y'_T)\rangle\right] - \mathbb{E}\left[\left\langle R\left(\Psi(Y_0) - \Psi(Y'_0)\right), \widehat{Y}_0\right\rangle\right] \\ &= \mathbb{E}\left[\int_0^T \langle A(t,\upsilon_t) - A(t,\upsilon'_t), \widehat{\upsilon}_t\rangle \, dt\right] \\ &\leq -\theta_1 \mathbb{E}\left[\int_0^T \left(|R\widehat{y}_t|^2 + \|R\widehat{z}_t\|^2\right) dt\right] \\ &\quad - \theta_2 \mathbb{E}\left[\int_0^T \left(\left|R^*\widehat{Y}_t\right|^2 + \left\|R^*\widehat{Z}_t\right\|^2\right) dt\right]. \end{aligned}$$

Thus by using (A2)

$$\theta_1 \mathbb{E}\left[\int_0^T \left(|R\widehat{y}_t|^2 + \|R\widehat{z}_t\|^2\right) dt\right] + \theta_2 \mathbb{E}\left[\int_0^T \left(\left|R^*\widehat{Y}_t\right|^2 + \left\|R^*\widehat{Z}_t\right\|^2\right) dt\right] \leq 0.$$

If $m > n$ then $\theta_1 > 0$, and so we have $|R\widehat{y}_t|^2 \equiv 0$ and $\|R\widehat{z}_t\|^2 \equiv 0$ a.s. $\forall t \in [0,T]$. Thus $y_t = y'_t$ and $z_t = z'_t$ a.s. $\forall t \in [0,T]$. In particular, $h(y_T) = h(y'_T)$. The system (2) implies

$$\begin{cases} 0 = \Psi(Y_0) - \Psi(Y'_0) + \int_0^t \widehat{b}(s,Y_s,Z_s)\, ds + \int_0^t \widehat{\sigma}(s,Y_s,Z_s)\, dW_s, \\ \widehat{Y}_t = -\int_t^T \widehat{f}(s,Y_s,Z_s)\, ds - \int_t^T \widehat{g}(s,Y_s,Z_s)\, d\overleftarrow{B}_s - \int_t^T \widehat{Z}_t dW_s, \end{cases}$$

where

$$\widehat{\pi}(t,Y_t,Z_t) := \pi(t,y_t,Y_t,z_t,Z_t) - \pi(t,y_t,Y_t,z_t,Z_t)$$

for $\pi := b,\sigma,f,g$. Consequently, from the uniqueness of the solution of BDSDEs and forward SDEs by Itô's formula (Proposition 3) to $\left|\hat{Y}_t\right|$, it follows that $Y_t = Y'_t$, and $Z_t = Z'_t$ a.s. $\forall t \in [0,T]$.

If $m < n$ then $\theta_2 > 0$, and so we have $Y_t = Y'_t$ and $Z_t = Z'_t$. In particular, $\Psi(Y_0) = \Psi(Y'_0)$, and so system (2) yields

$$\begin{cases} \widehat{y}_t = \int_0^t \widehat{b}(s,y_s,z_s)\, ds + \int_0^t \widehat{\sigma}(s,y_s,z_s)\, dW_s - \int_0^t \widehat{z}_t d\overleftarrow{B}_s \\ 0 = -\int_t^T \widehat{f}(s,y_s,z_s)\, ds - \int_t^T \widehat{g}(s,y_s,z_s)\, d\overleftarrow{B}_s. \end{cases}$$

As done earlier, we deduce that $y_t = y'_t$ and $z_t = z'$.

By arguments similar to the above cases, the desired uniqueness property of solutions can be obtained easily in the case $m = n$ ∎

(5) **Theorem.** *[Existeness] Assume (A1)–(A4) or ((A1)′, (A2)′, (A3)–(A4)) with $0 < \gamma' \leq \gamma/2$ when $m \leq n$. Then FBDSDEs (1) has a solution (y, Y, z, Z) in $\mathbb{H}^2$.*

We divide the proof into three case: $m > n, m < n$ and $m = n$.

Case 1: $m > n$ $(\theta_1 > 0)$. Consider the following family of FBDSDEs parameterized by $\alpha \in [0,1]$:

$$\begin{cases} dy_t = \left[\alpha b(t,\upsilon_t) + \widetilde{b}_0(t)\right] dt + \left[\alpha\sigma(t,\upsilon_t) + \widetilde{\sigma}_0(t)\right] dW_t - z_t d\overleftarrow{B}_t \\ dY_t = \left[\alpha f(t,\upsilon_t) - (1-\alpha)\theta_1 R y_t + \widetilde{f}_0(t)\right] dt + Z_t dW_t \\ \qquad\qquad + \left[\alpha g(t,\upsilon_t) - (1-\alpha)\theta_1 R z_t + \widetilde{g}_0(t)\right] d\overleftarrow{B}_t, \\ y_0 = \alpha\Psi(Y_0) + \psi, Y_T = \alpha h(y_T) + (1-\alpha)\theta_1 R y_T + \phi, \end{cases} \tag{3}$$

where $\upsilon_t := (y_t, Y_t, z_t, Z_t)$, $\left(\widetilde{b}_0, \widetilde{f}_0, \widetilde{\sigma}_0, \widetilde{g}_0\right) \in \mathbb{H}^2$, $\psi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n)$ and $\phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m)$ are given arbitrarily.

Note that when $\alpha = 1$ the existence of the solution of (3) implies clearly that of (2) simply by letting $(\widetilde{b}_0, \widetilde{f}_0, \widetilde{\sigma}_0, \widetilde{g}_0) = (0,0,0,0)$, while when $\alpha = 0$, (3) becomes a decoupled FBDSDEs of the form:

$$\begin{cases} dy_t = \widetilde{b}_0(t)\, dt + \widetilde{\sigma}_0(t)\, dW_t - z_t d\overleftarrow{B}_t \\ dY_t = -\theta_1 R y_t dt + Z_t dW_t + \left[-\theta_1 R z_t + \widetilde{g}_0(t)\right] d\overleftarrow{B}_t, \\ y_0 = \psi, Y_T = \theta_1 R y_T + \phi, 0 \le t \le T. \end{cases} \tag{4}$$

We can rewrite

$$\int_0^t \widetilde{\sigma}_0(s) dW_s = -\int_{T-t}^T \breve{\widetilde{\sigma}}_0(s) d\overleftarrow{\breve{W}}_s,$$

where

$$\breve{\widetilde{\sigma}}_0(s) = \widetilde{\sigma}_0(T-s),$$

i.e., as a backward Itô integral, and

$$\int_0^t Z_s d\overleftarrow{B}_s = -\int_{T-t}^T \breve{Z}_s d\breve{B}_s, \breve{Z}_s = Z_{T-s},$$

as a forward Itô integral, and similarly, for Lebesgue integrals (with respect to dt) in (2), which then enable us to rewrite the first (forward) equation as a BDSDE as follows:

$$\breve{y}_t = \breve{y}_T + \int_t^T \breve{\widetilde{b}}_0(s) ds - \int_t^T \breve{\widetilde{\sigma}}_0(s) d\overleftarrow{W}_s - \int_t^T \breve{z}_s d\breve{B}_s, 0 \le t \le T, \tag{5}$$

where $\breve{y}_t = y_{T-t}$, (hence $\breve{y}_T = y_0$), $\breve{\widetilde{b}}_0(s) = \widetilde{b}(T-s)$. So under our assumptions (A3) we deduce that (4) has a unique solution $(\breve{y}, \breve{z})$ in $\mathcal{M}^2(0,T;\mathbb{R}^n) \times \mathcal{M}^2\left(0,T;\mathbb{R}^{n\times l}\right)$. Alternatively, one can simply apply a generalized martingale representation theorem (as in Al-Hussein and Gherbal [1]) to get an explicit formula for this unique solution $(\breve{y}, \breve{z})$ since all integrals here do not depend on $\breve{y}$ or $\breve{z}$. Consequently, $\{(y_s, z_s) := (\breve{y}_{T-s}, \breve{z}_{T-s}), 0 \le t \le T\}$ is the unique solution of the forward equation of (4). By substituting (y,z) in the second equation of (3), it becomes a BDSDE of type (4) discussed earlier, and so it admits a unique solution (Y,Z). Therefore, we derive a unique solution $\upsilon = (y,Y,z,Z)$ of (4) in $\mathbb{H}^2$.

The following apriori lemma is a key step in the proof of the method of continuation. It shows that for a fixed $\alpha = \alpha_0 \in [0,1)$, if (3) is uniquely solvable, then it is also uniquely solvable for any $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$, for some positive constant δ_0 independent of α_0.

(6) **Lemma.** *We assume $m > n$. Under assumption (A1)–(A4), there exists a positive constant δ_0 such that if, apriori, for each $\psi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n), \phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m)$ and $\left(\widetilde{b}_0, f_0, \widetilde{\sigma}_0, \widetilde{g}_0\right) \in \mathbb{H}^2$, (3) is uniquely solvable for some $\alpha_0 \in [0,1)$, then for each $\alpha \in [\alpha_0, \alpha_0 + \delta_0], \psi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n), \phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m)$ and $\left(\widetilde{b}_0, \widetilde{f}_0, \widetilde{\sigma}_0, \widetilde{g}_0\right) \in \mathbb{H}^2$, (3) is uniquely solvable.*

The proof of this lemma is very long and it can be found in [?]

Case 3. $m = n$. We assume (A1)-(A4) with $0 < \gamma' \leq \gamma/2$. From (A1) and (A2) we only need to consider two cases:
1) If $\theta_1 > 0, \theta_2 \geq 0, \beta_1 > 0$, and $\beta_2 \geq 0$, we can have the same result.
2) If $\theta_1 \geq 0, \theta_2 > 0, \beta_1 \geq 0$, and $\beta_2 > 0$, we can have the same result.

We are now ready to complete the proof of Theorem 5.

Proof completion of Theorem 5. For **Case 1**, we know that, for each

$$\psi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n), \phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m), \left(\widetilde{b}_0, \widetilde{f}_0, \widetilde{\sigma}_0, \widetilde{g}_0\right) \in \mathbb{H}^2,$$

FBDSDEs has a unique solution as $\alpha = 0$. It follows that there exists a positive constant $\delta_0 = \delta_0(c, \gamma, \beta_1, \theta_1, R, T)$ such that for any $\delta \in [0, \delta_0]$ and $\psi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n), \phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m), \left(\widetilde{b}_0, \widetilde{f}_0, \widetilde{\sigma}_0, \widetilde{g}_0\right) \in \mathbb{H}^2$, has a unique solution for $\alpha = \delta$. Since δ_0 depends only on $c, \gamma, \beta_1, \theta_1, R$ and T. In particular, for $\alpha = 1$ with $\left(\widetilde{b}_0, \widetilde{f}_0, \widetilde{\sigma}_0, \widetilde{g}_0\right) \equiv 0, \phi \equiv 0, \psi \equiv 0$, FBDSDEs has a unique solution in $\mathbb{H}^2$.

For **Case 2**, we know that, for each

$$\psi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n), \phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m), \left(\tilde{b}_0, \widetilde{f}_0, \widetilde{\sigma}_0, \widetilde{g}_0\right) \in \mathbb{H}^2,$$

FBDSDEs has a unique solution as $\alpha = 0$. It follows from that there exists a positive constant $\delta_0 = \delta_0(c, \gamma, \beta_2, \theta_2, R, T)$ such that for any $\delta \in [0, \delta_0]$ and $\psi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n), \phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m), \left(\widetilde{b}_0, \widetilde{f}_0, \widetilde{\sigma}_0, \widetilde{g}_0\right) \in \mathbb{H}^2$, has a unique solution for $\alpha = \delta$. Since δ_0 depends only on $c, \gamma, \beta_2, \theta_2, R$ and T, and then deduce as in the preceding case that FBDSDEs has a unique solution in $\mathbb{H}^2$.

Similar to these cases, the desired result can be obtained in **Case 3**. ∎

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