

DOI: <https://doi.org/10.24297/jam.v21i.9207>**Hypergraphs: Application in Food networks**

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Abstract—

A hypergraph is a generalization of a graph since, in a graph an edge relates only a pair of points, but the edges of a hypergraph known as hyperedges can relate groups of more than two points. The representation of complex systems as graphs is appropriate for the study of certain problems. We give several examples of social, biological, ecological and technological systems where the use of graphs gives very limited information about the structure of the system. We propose to use hypergraphs to represent these systems.

Keywords—Hypergraphs, relation algebra, complex systems

INTRODUCTION

The calculus of relations has been an important component of the development of logic and algebra since the middle of the nineteenth century. George Boole, in his. Mathematical Analysis of Logic [1], initiated the treatment of logic as part of mathematics, specifically as part of algebra.

A graph is a representation of a set of objects where some pairs of the objects are connected by links. The interconnected objects are represented by mathematical abstractions called vertices, and the links that connect some pairs of vertices are called edges. The edges may be directed or undirected [1,4].

Hypergraphs generalization of graphs, have been widely and deeply studied in Berge (1973, 1984, 1989), and quite often have proved to be a successful tool to represent and model concepts and structures in various areas of Computer Science and Discrete Mathematics. In a graph an edge relates only a pair of points, but the edges of the hypergraphs known as hyperedges can relate groups of more than two points [1, 5].

The study of graphs represents an important area of multidisciplinary research involving physics, mathematics, chemistry, biology, social sciences, and information sciences. In some cases the use of simple or directed graphs to represent complex systems does not provide a complete description of the real-world systems under investigation. For instance, in a collaboration network represented as a simple graph we only know whether scientists have collaborated or not, but we cannot know whether three or more authors linked together in the network were coauthors of the same paper or not. collaboration network as a hypergraph in which points represent authors and hyperedges represent the groups of authors that have published papers together. We will show some applications of complex systems for which hypergraph representation is necessary. Specially, we will more investigate the food web case [1, 4, 5].

HOMOGENEOUS RELATION

Relations between elements of the same set are called homogeneous; they are the easier ones to investigate. The homogeneous case already presents many of the essential features and is notationally simpler.

Definitions 1.

- 1) A homogeneous relation R on V is a subset of the Cartesian product $V \times V$.
- 2) If R is a relation between the finite indexed sets X and Y then R can be represented by the Boolean matrix B whose row and column indices index the elements of X and Y , respectively, such that the entries of B are defined by:

$$B_{ij} = \begin{cases} 1 & (x_i, y_j) \in R \\ 0 & (x_i, y_j) \notin R \end{cases}$$

GRAPHS

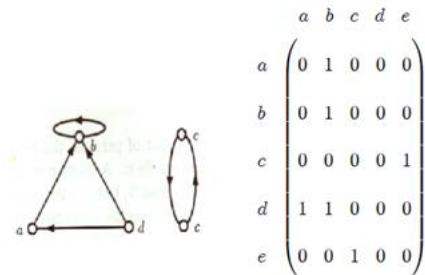
Relations and graphs are closely related. Any homogeneous relation can be interpreted as the transition relation of a graph and vice versa.

Definition 2.

A graph $G = (V,B)$ consists of a set V of points (also vertices) and a relation $B \subset V \times V$ the associated relation. A loop is an arrow whose head and tail coincide, in the example 1, a loop (b, b) is attached to point b .

Example 1. We present $V = \{a, b, c, d, e, f\}$ and

$$B = \{(a, b), (b, b), (c, e), (d, a), (d, b), (e, c)\}$$



which is also shown as a graph and a matrix in Fig. 1.

Fig. 1: A graph and its associated relation as a Boolean matrix.

Particular relations on V are the identity and nonidentity of elements

$$I = \{(x, y) | x = y\} \subset V \times V$$

$$\bar{I} = \{(x, y) | x \neq y\} \subset V \times V$$

The identity relation is helpful for defining the following simple properties of relations.

Definition 3.

Let R be a homogeneous relation. Then we define:

$$R \text{ reflexive} : \Leftrightarrow I \subset R \Leftrightarrow R \cup I = R \Leftrightarrow \forall x: (x, x) \in R;$$

$$R \text{ irreflexive} : \Leftrightarrow R \subset \bar{I} \Leftrightarrow R \cap \bar{I} = R \Leftrightarrow \forall x: (x, x) \in \bar{R}.$$

Transposition of a Relation.

A relation can be transposed this is a characteristic trail of a relation algebra as opposed to mere subset algebra. We define the transpose or converse of a relation R as follows

$$\backslash \backslash \quad R^T = \{(x, y) | (y, x) \in R\}.$$

Next we recall the notions of symmetry for homogeneous relations which we shall then reexpress using the transpose:

Definition 4.

be homogeneous relation. Then we have

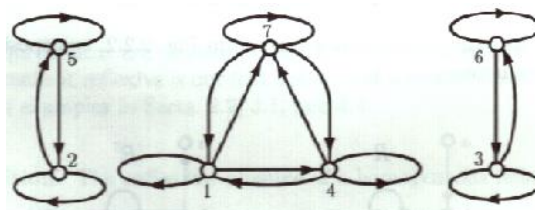
$$R \text{ symmetric} : \Leftrightarrow R^T \subset R \Leftrightarrow R^T = R;$$

$$R \text{ asymmetric} : \Leftrightarrow R \cap R^T \subset O \\ \Leftrightarrow R^T \subset \bar{R}.$$



These notions can easily be expressed in terms of relations; the proposition below could also serve as a definition.

Example 2. Reflexive and symmetric relation as a graph



For drawing symmetric relations one therefore prefers undirected graphs, as in Fig. 2, where the (undirected) edges represent pairs of arrows pointing in opposite directions.

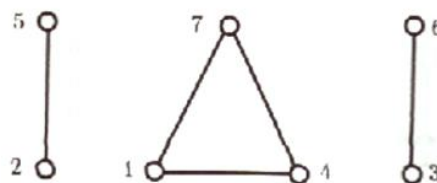


Fig. 2: Irreflexive and symmetric relation as an undirected graph

A simple graph $G = (V, \Gamma)$ consists of a set V of points and an irreflexive, symmetric relation Γ on V called the adjacency relation of the graph. So $\Gamma = \Gamma^T \subset \bar{I}$ holds.

HETEROGENEOUS RELATIONS

We now pass from homogeneous to heterogeneous relations. In terms of matrices this amounts to passing from square matrices to general rectangular ones.

Definitions 5.

1) Given possibly different sets X and Y , a subset R of $X \times Y$ is called a heterogeneous relation between X and Y . For representing a heterogeneous relation one can use a rectangular Boolean matrix and also a bipartitioned graph.

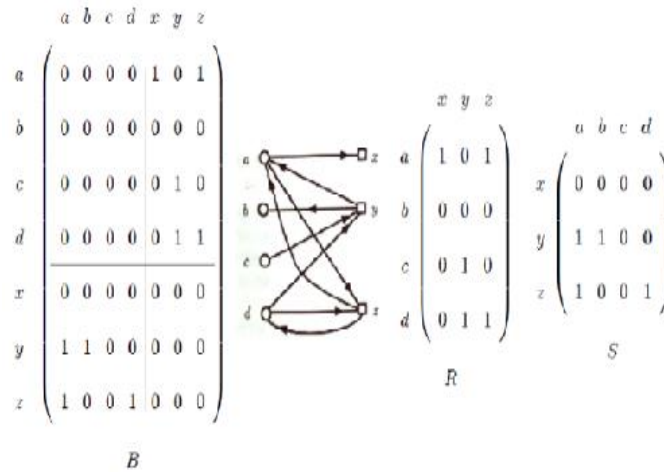
2) A quadruple (X, Y, R, S) is called a bipartitioned graph, if

a) X and Y are nonempty disjoint point sets.

b) $R \subset X \times Y$ and $S \subset Y \times X$ are heterogeneous relation.

Example 3. If $V = \{a, b, c, d, e, x, y, z\} =$

$X \cup Y$. Then associated relation of a bipartitioned graph



Thus, a bipartitioned graph (X, Y, R, S) leads to a graph (V, B) with point]= [set

V = X ∪ Y and an associated matrix of special form

$$B = \begin{pmatrix} O & R \\ S & O \end{pmatrix}.$$

HEYPERGRAHS

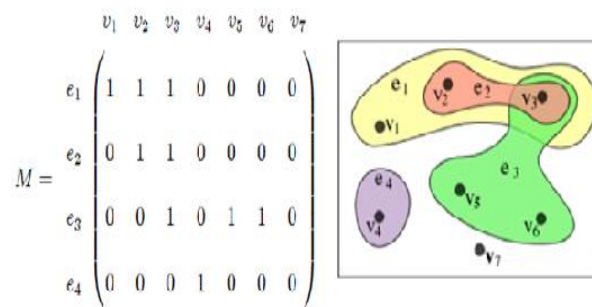
Basically, a hypergraph is just a heterogeneous relation disguised as a graph which will be called incidence in this context.

Definition 6.

The triple $G = (P, V, M)$ is called a hypergraph, if

- a) P is a set of hyperedges.
- b) V is a set of points .
- c) $M \subset P \times V$ is a relation called the incidence.

Example 4. A hypergraph, with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$; $P = \{e_1, e_2, e_3, e_4\}$



and

SOME APPLICATIONS OF HYPERGRAPHS

We will show some examples of complex systems for which hypergraph representation is necessary. Specially, we will more investigate the food web case. Most of applications are taken from [1,4, 7].

a) Numerical Linear Algebra.

The runtime of linear algebra computations can vary dramatically depending on the sparsity of input matrices and their patterns of non-zero values, which can affect the sparsity of intermediate matrices appearing during computations. In particular, many linear algebra operations (such as matrix-vector multiply, matrix- matrix

multiply, eigenvalue problems, etc.) are faster for block-diagonal matrices. Computing these operations on matrices with permuted rows or columns will give an answer that differs by the same permutation. Therefore the result can be interpreted in the same way, and it is desirable to swap rows in order to bring non-zero elements in input matrices as close to the diagonal as possible before applying such operations. One way to achieve this is through recursive calls to hypergraph partitioning on a hypergraph representation of the matrix. Fig. 3, shows a small example of a sparse block-diagonal matrix with its corresponding hypergraph .

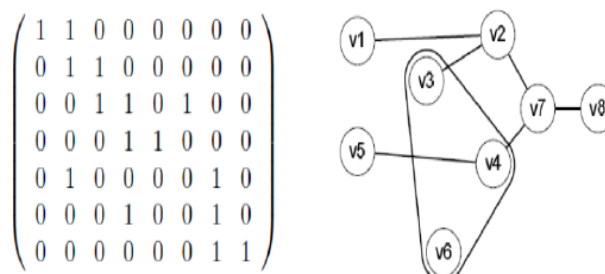


Fig. 3: An example of a nearly block- diagonal matrix and corresponding Hypergraph.

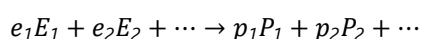
b) Social Networks

In social networks nodes represent people or groups of people, normally called

actors, that are connected by pairs according to some pattern of contact or interactions between them. Such patterns can be of friendship, collaboration, business relationships, etc. There are some cases in which hypergraph representations of the social network are indispensable. These are, for instance, the commercial transactions in social networks in which it is necessary to consider the coordinated actions of more than two actors, such as a buyer, a seller and a broker. In hypergraphs the nodes represent actors related by a common process, which is represented by a hyper edge, such as a commercial transaction.

c) Reaction and Metabolic Networks

A chemical reaction is a process in which a set of chemical compounds known as educts, E_i react in certain stoichiometric proportions, e_i to be transformed into a set of other chemical compounds named products, P_i which are produced in certain stoichiometric quantities p_i :



A chemical reaction can be described as a weighted directed hyperedge in a directed hypergraph where nodes are the chemicals and hyper-edges are the reactions. The absence of a well-developed theory for the structural analysis of (directed) hypergraphs means that two alternative representations of a chemical reaction are commonly used. The first is the bipartite graph, in which a set of nodes represents educts and products and the other set represents the reaction itself. The other representation consists of the substrate graph, which considers educts and products as nodes two nodes are connected if the corresponding chemical compounds take part in the same reaction.

d) Protein Complex Networks

The systematic characterization of multi-protein complexes in the whole proteome of an organism requires the data to be organized in the form of protein membership lists of the protein complexes. The most common forms of this organization are the protein-protein interaction graphs and the complex intersection graphs. In the first representation the nodes of the graph represent proteins and an edge links two proteins that interact with each other. This representation however, does not take into account the multi-protein complexes. In the complex intersection graph the nodes represent complexes, and a link exists between two complexes if they have one or more proteins in common. Clearly, this second representation does not provide information about proteins. A natural way of accounting for the information about both proteins and common protein membership in the complexes, such as common regulation, localization, turnover, or architecture, is to use a hypergraph

representation. In the protein hypergraphs each protein is represented by a node and each complex by a hyperedge. These kinds of hypergraphs can be visualized as bipartite graphs.

d) Food webs

Trophic relation in ecological systems are normally represented through the use of food webs, which are oriented graphs D (digraphs) whose nodes represent species and links represent trophic relations between species. Another way of representing food webs is by means of competition graphs $C(D)$, which have the same set of nodes as the food web but in which two nodes are connected if and only if the corresponding species compete for the same prey in the food web. In the competition graph we can only know if two linked species have some common prey, but we cannot know the composition of the whole group of species that compete for common prey. In order to solve this problem a competition hypergraph $CH(D)$ has been proposed in which nodes represent species in the food web and hyperedges represent groups of species that compete for common prey. It has been shown that in many cases competition hypergraphs yield a more detailed description of the predation relations among the species in the food web than competition graphs. A food web and its competition graph and hypergraph are illustrated in Fig.4.

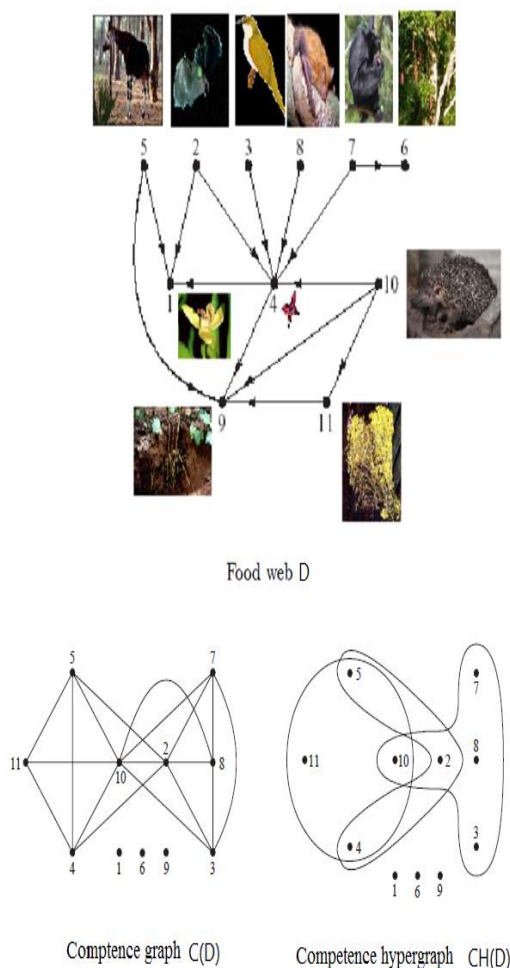


Fig.4 Food webs are commonly considered to be acyclic digraphs; however, in literature competition graphs of digraphs with cycles or loops also are investigated. Roberts [9] observed, that every graph G together with $|E(G)|$ isolated vertices is the competition graph of an acyclic digraph. He introduced the competition number $k(G)$ as the smallest k , such that G together with k isolated vertices is the competition graph of an acyclic digraph. Surveys of the large literature around competition graphs can be found in [5,6]. Obviously, the edges of $CH(D)$

correspond to certain cliques in $C(D)$, and this proves to be very useful in the following. If $M = (m_{ij})$ is the adjacency matrix of digraph D , then the competition graph $C(D)$ is the row graph $RG(M)$ (see [6,8]). To find a similar characterization for competition hypergraphs, we define the row hypergraph $RH(M)$. The vertices of this hypergraph correspond to the rows of M , i.e. to the vertices v_1, v_2, \dots, v_n of D , and the edges correspond to certain column s ; in detail

$$E(RH(M)) = \{ \{v_{i_1}, \dots, v_{i_k}\} \mid k \geq 2 \wedge \exists j \in \{1, \dots, n\}: m_{ij} = 1 \Leftrightarrow i = \{i_1, \dots, i_k\} \}$$

This notion yields immediately the following result.

Lemma 1. Let D be a digraph with adjacency matrix M . Then the competition hypergraph $CH(D)$ is the row hypergraph $RH(M)$.

Note that any permutation of rows or columns in M does not change the row hypergraph $RH(M)$.

Conversely, for a competition hypergraph H with n vertices and t edges we call each $(n \times n)$ – matrix M with entries 0 or 1 a competition matrix of H if H the same $RH(M)$. Such a competition matrix is said to be standardized if $e_j \in E(H)$ corresponds to column j of M for $j = 1, \dots, t$ and all entries are 0 in columns $t + 1, \dots, n$.

For a graph G , let us call a collection $\mathcal{E} = \{C_1, \dots, C_t\}$ an edge cover of G , if each $C_i \subseteq V(G)$ generates a clique in G or $C_i = \emptyset$, and every edge of G is contained in at least one of the cliques in \mathcal{C} .

Competition hypergraphs of acyclic digraphs.

We start with a well-known property of acyclic digraphs (see for instance [6]).

Lemma 2. A digraph D is acyclic iff its vertices can be labelled such that the adjacency matrix M of D is strictly lower triangular.

Using Lemmas 1 and 2, it follows that the edges of a competition hypergraph of an acyclic digraph D correspond to the columns of a strictly lower triangular adjacency matrix M of D . As a consequence we obtain for a competition hypergraph $CH(D)$ of an acyclic digraph D with n vertices:

$$\begin{aligned} |E(CH(D))| &\leq n - 2, \\ \sum_{e \in E(CH(D))} |e| &\leq \frac{n(n-1)}{2} - 1 \\ \forall k \in \{2, \dots, n\}: |\{e \in E(CH(D)) \mid |e| \geq k\}| &\leq n - k \end{aligned}$$

Dutton and Brigham [7] as well as Lundgren and Maybee [8] proved the following characterization for competition graphs.

Theorem 3 (Dutton and Brigham [7], Lundgren and Maybee [8]).

A graph G with n vertices is a competition graph of an acyclic digraph iff there is an edge cover $\mathcal{E} = \{C_1, \dots, C_n\}$ and a vertex labelling v_1, v_2, \dots, v_n of G , such that $v_i \in C_j$ implies $i > j$.

In the next theorem we give the corresponding result for competition hypergraphs.

Theorem 4. A hypergraph H with n vertices and $E(H) = \{e_1, \dots, e_t\}$ is a competition hypergraph of an acyclic digraph iff its vertices can be labelled v_1, v_2, \dots, v_n , such that $v_i \in e_j$ implies $i > j$.

Proof. Suppose $H = CH(D)$ for some acyclic digraph D . By Lemma 2 there is a vertex labelling v_1, v_2, \dots, v_n of $V(D)$, which generates a strictly lower triangular adjacency matrix M of D . With Lemma 1 follows $CH(D) = RH(M)$. Using

the notation $e_j = \{v_i \mid i \in \{1, \dots, n\} \wedge m_{ij} = 1\}$ for $t = 1, \dots, t$, we obtain the required condition: $v_i \in e_j$ implies $i > j$.

Conversely, if the condition of the theorem is true, there is a vertex labelling of $V(H)$, such that H has a strictly lower triangular standardized competition matrix M . By Lemmas 1 and 2 this is the adjacency matrix of an acyclic digraph D with $H = CH(D)$.

7. Conclusion.

We presented homogeneous and heterogeneous relations and their relationship to graphs. I took a special type of graphs which are the hypergraphs; I proved some of their properties and their applications. For future work, we would like to apply Review (software for relations) to illustrate our results.

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