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Some Properties of Meromrphic Univalent Functions with Negative Coefficients Defined by Dziok-Srivastava Operator

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Abstract: The main aim of the present investigation is to introduce a new class of meromorphic univalent functions with negative coefficients defined by Dziok-Srivastava operator. Some geometric properties are introduced, like coefficient estimate, integral operator, Feket-Szegö bounds for this class of meromorphic functions.

Keywords: Dziok-Srivastava operator, integral operator, Feket-Szegö inequality.

Introduction

Let $\mathcal{U}^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$ be the punctured unit disk in the complex plane. we denote by $\sum$ the class of meromorphic functions in $\mathcal{U}^*$ with $\mathcal{G}(0) = \mathcal{G}'(0) - 1 = 0$ of the form

$$\mathcal{G}(z) = \frac{1}{z} - a_l z^l - a_{l+1} z^{l+1} - \dots\dots\dots, 0 < |z| < 1, l \geq 0 \ . \qquad 1.1$$

A function $\mathcal{G} \in \sum$ is meromorphic starlike of order η $(0 \leq \eta < 1)$ if

$$-R_e\left\{\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right\} > \eta \,, \mathcal{G}(z) \neq 0 \; for \; z \in \mathcal{U}^* \ ,$$

the class of all such family of functions is denoted by $\sum^*(\eta)$ .A function $\mathcal{G} \in \sum$ is meromorphic convex of order η $(0 \leq \eta < 1)$ if

$$-R_e\left\{1 + \frac{z\mathcal{G}''(z)}{\mathcal{G}'(z)}\right\} > \eta \,, \mathcal{G}'(z) \neq 0 \; for \; z \in \mathcal{U}^*$$

The class of all such functions is denoted by $\sum^l(\eta)$. Let $\mathcal{G}(z)$ be a function given by 1.1 and $\chi(z) = z^{-1} - \sum_{l=0}^{\infty} \mathscr{b}_l z^l$ we define the hadamard product of $\mathcal{G}$ and χ by

$$(\mathcal{G} * \chi)(z) = z^{-1} - \sum_{l=0}^{\infty} a_l \mathscr{b}_l z^l$$

For complex parameters μ_t and ρ_s,where$(t = 1,2, \dots, t; j = 1,2, \dots \dots, s \; and \; \rho_s \neq 0, -1, -2, \dots)$, the generalized hypergeometric function ${}_t\psi_s\ (z)$ is defined as

$${}_t\psi_s\ (\mu_1, \mu_2, \dots, \mu_t, \rho_1, \rho_2, \dots, \rho_s)(z) = \sum_{l=0}^{\infty} \frac{(\mu_1)_l (\mu_2)_l \dots\dots (\mu_t)_l}{(\rho_1)_l (\rho_2)_l \dots\dots (\rho_s)_l} \frac{z^l}{l!} \ (t \leq s + 1, t, s \in \mathbb{N}_0 = \{0,1,2, \dots\dots\}; z \in \mathcal{U}^*) \text{, were}$$

$$(v)_l = \frac{\Gamma(v+l)}{l} = \begin{cases} 1 & if \ l=0, v \in \mathbb{C}^* = \mathbb{C}\backslash\{0\} \\ v(v+1)\dots\dots\dots(v+l-1) & l \in \mathbb{N}; v \in \mathbb{C} \end{cases}, \qquad 1.2$$

Is the Pochhammer symbol defined in terms of Gamma function.

Let the function ${}_t\psi_s\ (\mu_1, \mu_2, \dots, \mu_t, \rho_1, \rho_2, \dots, \rho_s; z)$ be defined as

$$\mathcal{R}(\mu_1, \mu_2, \dots, \mu_t, \rho_1, \rho_2, \dots, \rho_s; z) = z^{-1} \ {}_t\psi_s\ (\mu_1, \mu_2, \dots, \mu_t, \rho_1, \rho_2, \dots, \rho_s; z) \ .$$

The Liu-Srivastava linear operator $\mathcal{K}(\mu_1, \mu_2, \dots, \mu_t, \rho_1, \rho_2, \dots, \rho_s): \sum \rightarrow \sum$ is defined by

$$\mathcal{K}(\mu_1, \mu_2, \dots, \mu_t, \rho_1, \rho_2, \dots, \rho_s)\mathcal{G}(z) = \mathcal{R}(\mu_1, \mu_2, \dots, \mu_t, \rho_1, \rho_2, \dots, \rho_s; z) * \mathcal{G}(z)$$

$$= z^{-1} - \sum_{l=0}^{\infty} a_l \Gamma_l z^l$$

Where

$$\Gamma_l = \left| \frac{(\mu_1)_l (\mu_2)_l \dots\dots (\mu_t)_l}{(\rho_1)_l (\rho_2)_l \dots\dots (\rho_s)_l} \frac{1}{(l)!} \right| \ , \qquad 1.3$$

for simplicity, we use a shorter symbol $\mathcal{K}_s^t(\mu_1)$ instead of $\mathcal{K}(\mu_1, \mu_2, \dots, \mu_t, \rho_1, \rho_2, \dots, \rho_s)$.

Some interesting subfamilies of analytic functions associated with the generalized hypergeometric function, were considered recently by Srivastava et al. [7]. The family $\sum^*(\eta)$ and various other subfamilies of $\sum$ have been studied rather extensively in [4] , [5].By using of the generalized Dziok-Srivastava operator $\mathcal{K}_s^t$, we define a new subfamily of functions in $\sum$ as follows:

For $0 \le \eta < 1$ and $\xi \in \mathbb{C} - (0,1]$,we let $\mathcal{S}(\xi,\eta)$,denote a subfamily of $\sum$ consisting functions of the form 1.1 satisfying the condition

$$R_e\left\{\frac{\xi z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'' + z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_s^t\mathcal{G}(z)+\xi\left(z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'\right)}\right\} > \eta \quad . \qquad 1.4$$

Coefficient inequalities, properties of certain integral operator, as well as the Fekete- Szegö like inequality are discussed for a new class of meromorphic functions $\mathcal{S}(\xi,\eta)$.

Coefficient inequalities for a function in the class $\mathcal{S}(\xi,\eta)$

In the following theorem we introduce a necessary and sufficient condition for function $\mathcal{G}$ to be in $\mathcal{S}(\xi,\eta)$.

Theorem 1 Let $\mathcal{G} \in \sum$ given by 1.1 .Then $\mathcal{G} \in \mathcal{S}(\xi,\eta)$ if and only if

$$\sum_{l=0}^{\infty} \Upsilon(l,\xi,\eta)\Gamma_l a_l < (2\xi-1)(1-\eta) \quad . \qquad 2.1$$

Where $\Upsilon(l,\xi,\eta) = l\left[1+\xi\left((l-1)-\eta l\right)\right] + \eta(1-\xi), 0 \le \eta < 1$ and $\xi \in \mathbb{C} - (0,1]$. 2.2

The result is sharp for the function

$$\mathcal{K}_s^t\mathcal{G}(z) = z^{-1} - \frac{(2\xi-1)(1-\eta)}{\Upsilon(l,\xi,\eta)\Gamma_l} z^l, l = 0,1,2,\ldots . \qquad 2.3$$

Proof. If $\mathcal{G} \in \mathcal{S}(\xi,\eta)$ then

$$R_e\left\{\frac{\xi z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'' + z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_s^t\mathcal{G}(z)+\xi\left(z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'\right)}\right\} =$$

$R_e\left\{\frac{2\xi-1-\sum_{l=1}^{\infty}(l+\xi l(l-1))a_l\Gamma_l z^{l+1}}{2\xi-1+\sum_{l=1}^{\infty}(1-\xi(l^2+1))a_l\Gamma_l z^{l+1}}\right\} > \eta$,since $z \to 1^-$ we have

$$\frac{2\xi-1-\sum_{l=1}^{\infty}(l+\xi l(l-1))a_l\Gamma_l}{2\xi-1+\sum_{l=1}^{\infty}(1-\xi(l^2+1))a_l\Gamma_l} > \eta \ .$$

This shows that 2.1 holds.

Conversely assume that 2.1 holds ,since $R_e(w) > \eta$ if and only if $|w-(1+\eta)| < |w+(1-\eta)|$.

It is sufficient to show that

$$\left|\frac{\frac{\xi z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'' + z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_s^t\mathcal{G}(z)+\xi\left(z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'\right)} - (1+\eta)}{\frac{\xi z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'' + z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_s^t\mathcal{G}(z)+\xi\left(z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'\right)} + (1-\eta)}\right| =$$

$$\left|\frac{-\eta(2\xi-1)-\sum_{i=0}^{\infty}(l-\xi l-\xi+1-\xi\eta l^2-\eta\xi+\eta)a_l\Gamma_l z^{l+1}}{2(2\xi-1)-\eta(2\xi-1)-\sum_{i=0}^{\infty}(l+2\xi l^2-\xi l+\xi-1-\xi\eta l^2-\eta\xi+\eta)a_l\Gamma_l z^{l+1}}\right| \le 1 \ .$$

Thus we have $\mathcal{G} \in \mathcal{S}(\xi,\eta)$. ■

Integral operators for a function in the class $\mathcal{S}(\xi,\eta)$

The integral transformation of functions of the class $\mathcal{S}(\xi,\eta)$ will be shown in this section.

Theorem 2 If the function $\mathcal{G}(z)$ is given by 1.1 be in $\mathcal{S}(\xi,\eta)$.Then the integral operator

$$\mathcal{F}(z) = d\int_0^1 \mathfrak{n}^d \mathcal{G}(\mathfrak{n}z)\, d\mathfrak{n} \quad (0 < \mathfrak{n} \le 1, 0 < d < \infty) ,$$

is in $\mathcal{S}(\xi,\eta)$ such that

$$\sigma \leq \psi(l)\ ,$$

where $\psi(l) = \frac{\{l\{1+\xi[(l-1)-\eta l]\}+\eta(1-\xi)\}-dl[1+\xi(l-1)](1-\eta)}{d(1-\eta)[(1-\xi)-\xi l^2]+(l+d+1)\Upsilon(l,\xi,\eta)}$.

The result is sharp for the function $\mathcal{G}(z) = z^{-1} - \frac{(2\xi-1)(1-\eta)}{\Upsilon(0,\xi,\eta)}$.

Proof. Let $\mathcal{G} \in \mathcal{S}(\xi,\eta)$. Then

$$\mathcal{F}(z) = d\int_0^1 \text{n}^d \mathcal{G}(\text{n}z)\,d\text{n} = z^{-1} - \sum_{l=0}^{\infty} \frac{d}{l+d+1}\mathcal{G}_l z^l \quad .$$

We will prove that

$$\sum_{l=0}^{\infty} \frac{d\Upsilon(l,\xi,\sigma)\Gamma_l}{(l+d+1)(2\xi-1)(1-\sigma)} a_l \leq 1 \quad . \qquad 3.1$$

From the fact that $\mathcal{G} \in \mathcal{S}(\xi,\eta)$,we have

$$\sum_{l=0}^{\infty} \frac{\Upsilon(l,\xi,\eta)\Gamma_l}{(2\xi-1)(1-\eta)} a_l \leq 1 \ .$$

Note that 3.1, holds if

$$\frac{d\Upsilon(l,\xi,\sigma)\Gamma_l}{(l+d+1)(2\xi-1)(1-\sigma)} \leq \frac{\Upsilon(l,\xi,\eta)\Gamma_l}{(2\xi-1)(1-\eta)}\ ,$$

solving for σ ,we have

$$\sigma \leq \frac{\Upsilon(l,\xi,\eta)-dl[1+\xi(l-1)](1-\eta)}{d(1-\eta)[(1-\xi)-\xi l^2]+(l+d+1)\Upsilon(l,\xi,\eta)} = \psi(l)\ .$$

A simple arithmetic will show that $\psi(l)$ is increasing and $\psi(l) \geq \psi(0)$.■

Theorem 3 Let $\mathcal{G}(z)$, given by 1.1,be in $\mathcal{S}(\xi,\eta)$,and

$\mathcal{W}(z) = \frac{1}{d}\left[(d+1)\mathcal{G}(z) + z\mathcal{G}'(z)\right] = z^{-1} - \sum_{l=0}^{\infty} \frac{(l+d+1)}{d}\mathcal{G}_l z^l\ , d > 0\ .$

Then $\mathcal{W}(z)$ is in $\mathcal{S}(\xi,\eta)$ for $|z| \leq \upsilon(\xi,\eta,\delta)$ where

$$\upsilon(\xi,\eta,\delta) = \ {}_l inf\left(\frac{d(1-\delta)\Upsilon(l,\xi,\eta)}{(1-\eta)(l+d+1)\Upsilon(l,\xi,\delta)}\right)^{\frac{1}{l+1}} \quad (l = 0,1,2,\dots.)\ .$$

The result is sharp for the function $\mathcal{G}_l(z) = z^{-1} - \frac{(2\xi-1)(1-\eta)}{\Upsilon(l,\xi,\eta)\Gamma_l} z^l \quad (l = 0,1,2,\dots.)$.

Proof . Let $\mathcal{W} = \frac{\xi z^2\left(\mathcal{K}_\varsigma^t \mathcal{G}(z)\right)'' + z\left(\mathcal{K}_\varsigma^t \mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_\varsigma^t \mathcal{G}(z) + \xi\left(z^2\left(\mathcal{K}_\varsigma^t \mathcal{G}(z)\right)'' + z\left(\mathcal{K}_\varsigma^t \mathcal{G}(z)\right)'\right)}$.

To show that

$$\left|\frac{\mathcal{W}-(1+\delta)}{\mathcal{W}+(1-\delta)}\right| < 1. \qquad 3.2$$

After simplifying the inequality, we note that this inequality convinced if

$$\sum_{l=0}^{\infty} \frac{\Upsilon(l,\xi,\delta)(l+d+1)\Gamma_l}{d(2\xi-1)(1-\delta)} a_l\,|z|^{l+1} \leq 1\ , \qquad 3.3$$

since $\mathcal{G} \in \mathcal{S}(\xi,\eta)$,by theorem 1 ,we have

$$\sum_{l=0}^{\infty} \Upsilon(l,\xi,\eta)\Gamma_l a_l \leq (2\xi-1)(1-\eta)\ .$$

The inequality in 3.3 is convinced if

$$\frac{\Upsilon(l,\xi,\delta)(l+d+1)\Gamma_l}{d(2\xi-1)(1-\delta)} a_l |z|^{l+1} \leq \frac{\Upsilon(l,\xi,\eta)\Gamma_l}{(2\xi-1)(1-\eta)} a_l\ .$$

By solving this inequality for $|z|$,we have

$$|z| \leq \left(\frac{d(1-\delta)\Upsilon(l,\xi,\eta)\Gamma_l}{\Upsilon(l,\xi,\delta)(l+d+1)(1-\eta)}\right)^{\frac{1}{l+1}}.$$

Thus we get the required result. ■

The Coefficient bounds for the class of meromorphic functions $\mathcal{S}(\xi,\eta)$

Several researchers have presented studies on the inequality of Feket-Szegö for analytic functions, these inequalities were studied be researchers for classes of meromorphic functions, including [1] ,[2] ,[6] .

Definition 3 Let $\mathcal{S}(\xi,\eta)$ be the class of functions $\mathcal{G} \in \sum$ for which

$$\frac{\xi z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'' + z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_s^t\mathcal{G}(z)+\xi\left(z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'\right)} \prec \mathcal{H}(z)(z \in \mathcal{U}^*, \xi \in \mathbb{C}-(0,1], R(\xi) \geq 0) \quad , \qquad 4.1$$

where $\prec$ denotes subordination between analytic functions. In the following theorem we proved the bounds for the class $\mathcal{S}(\xi,\eta)$. To prove our result, we need the following Lemma .

Lemma 1 [3] If $\mathcal{P}(z) = 1 + \mathcal{C}_1 z + \mathcal{C}_2 z^2 + \dots\dots$ is a function with positive real part in $\mathcal{U}^*$,then for any complex number λ,

$$\left|\mathcal{C}_2 - \lambda {\mathcal{C}_1}^2\right| \leq 2max\{1, |1-2\lambda|\} \quad . \qquad 4.2$$

Theorem 4 Let $\psi(z) = 1 + \mathcal{B}_1 z + \mathcal{B}_2 z^2 + \dots\dots$.If $\mathcal{G}(z)$ is given by 1.1 belongs to $\mathcal{S}(\psi)$,then for any complex number λ,

(1) $\left|a_1 - \lambda {a_0}^2\right| \leq \frac{1}{2}\left|\frac{(2\xi-1)}{(1-\xi)}\right|\frac{|\mathcal{B}_1|}{\Gamma_1} max\left\{1, \left|\frac{\mathcal{B}_2}{\mathcal{B}_1} - \left(1 - \frac{2(2\xi-1)\Gamma_1}{(1-\xi)}\lambda\right)\mathcal{B}_1\right|\right\}, \mathcal{B}_1 \neq 0 \ .$ 4.3

(2) $\left|a_1 - \lambda {a_0}^2\right| \leq \left|\frac{(2\xi-1)}{(1-\xi)}\right|\frac{1}{\Gamma_1}, \mathcal{B}_1 = 0\ .$ 4.4

The bounds obtained are sharp .

Proof. If $\mathcal{G}(z) \in \mathcal{S}(\psi)$,then there is a Schwarz function such that $\mathcal{W}(0) = 0$, $|\mathcal{W}(z)| < 1$ and analytic in $\mathcal{U}^*$such that

$$\frac{\xi z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'' + z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_s^t\mathcal{G}(z)+\xi\left(z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'\right)} = \psi(\mathcal{W}(z)) \ . \qquad 4.5$$

Define the function $\mathcal{P}_1(z) = \frac{1+\mathcal{W}(z)}{1-\mathcal{W}(z)} = 1 + \mathcal{C}_1 z + \mathcal{C}_2 z^2 + \dots\dots$.

Since $\mathcal{W}(z)$ is Schwarz function, it is clear that $R(\mathcal{P}_1(z)) > 0$ and $\mathcal{P}_1(0) = 1$,define

$$\mathcal{P}(z) = \frac{\xi z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'' + z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_s^t\mathcal{G}(z)+\xi\left(z^2\left(\mathcal{K}_s^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_s^t\mathcal{G}(z)\right)'\right)} = 1 + \mathfrak{b}_1 z + \mathfrak{b}_2 z^2 + \dots\dots \ ,$$

since $\mathcal{W} = \frac{\mathcal{P}_1 - 1}{\mathcal{P}_1 + 1}$,

therefore $\psi(\mathcal{W}(z)) = \psi\left(\frac{\mathcal{P}_1-1}{\mathcal{P}_1+1}\right)$,

that is $\mathcal{P}(z) = \psi\left(\frac{\mathcal{P}_1-1}{\mathcal{P}_1+1}\right)$, 4.6

since $\psi(z) = 1 + \mathcal{B}_1 z + \mathcal{B}_2 z^2 + \dots\dots$,

therefore

$$\psi\left(\frac{\mathcal{P}_1-1}{\mathcal{P}_1+1}\right) = 1 + \frac{1}{2}\mathcal{B}_1\mathcal{C}_1 z + \left[\frac{1}{2}\mathcal{B}_1\left(\mathcal{C}_2 - \frac{1}{2}{\mathcal{C}_1}^2\right) + \frac{1}{4}\mathcal{B}_2{\mathcal{C}_1}^2\right] z^2 + \dots\dots \ . \qquad 4.7$$

Form 4.6 and 4.7, we obtain

$$1+\mathfrak{b}_1 z+\mathfrak{b}_2 z^2+\cdots=1+\frac{1}{2}\mathcal{B}_1\mathcal{C}_1 z+\left[\frac{1}{2}\mathcal{B}_1\left(\mathcal{C}_2-\frac{1}{2}\mathcal{C}_1{}^2\right)+\frac{1}{4}\mathcal{B}_2\mathcal{C}_1{}^2\right]z^2+\ldots\ldots,$$

thus, we conclude that

$$\mathfrak{b}_1=\frac{1}{2}\mathcal{B}_1\mathcal{C}_1 \quad \text{and} \quad \mathfrak{b}_2=\frac{1}{2}\mathcal{B}_1\left(\mathcal{C}_2-\frac{1}{2}\mathcal{C}_1{}^2\right)+\frac{1}{4}\mathcal{B}_2\mathcal{C}_1{}^2 .$$

From the other hand, since

$$\frac{\xi z^2\left(\mathcal{K}_\varsigma^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_\varsigma^t\mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_\varsigma^t\mathcal{G}(z)+\xi\left(z^2\left(\mathcal{K}_\varsigma^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_\varsigma^t\mathcal{G}(z)\right)'\right)}=1+\frac{1-\xi}{2\xi-1}a_0 z+\left(\left(\frac{1-\xi}{2\xi-1}\right)^2 a_0{}^2-\left(\frac{2(1-\xi)}{2\xi-1}\right)\Gamma_1 a_1\right)z^2+\cdots,$$

and $\mathcal{P}(z)=\dfrac{\xi z^2\left(\mathcal{K}_\varsigma^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_\varsigma^t\mathcal{G}(z)\right)'}{(\xi-1)\mathcal{K}_\varsigma^t\mathcal{G}(z)+\xi\left(z^2\left(\mathcal{K}_\varsigma^t\mathcal{G}(z)\right)''+z\left(\mathcal{K}_\varsigma^t\mathcal{G}(z)\right)'\right)}=\psi(\mathcal{W})$.

So, we get

$$1+\mathfrak{b}_1 z+\mathfrak{b}_2 z^2+\cdots=1+\frac{1-\xi}{2\xi-1}a_0 z+\left(\left(\frac{1-\xi}{2\xi-1}\right)^2 a_0{}^2-\left(\frac{2(1-\xi)}{2\xi-1}\right)\Gamma_1 a_1\right)z^2+\cdots ,$$

That is

$$\mathfrak{b}_1=\frac{1-\xi}{2\xi-1}a_0 \quad \text{and} \quad \mathfrak{b}_2=\left(\left(\frac{1-\xi}{2\xi-1}\right)^2 a_0{}^2-\frac{2(1-\xi)}{2\xi-1}\Gamma_1 a_1\right) . \qquad 4.8$$

First, substitute about a_0 in $\mathfrak{b}_2$ and then about $\mathfrak{b}_1$, $\mathfrak{b}_2$ in 4.8,so we have

$$a_0=\frac{2\xi-1}{2(1-\xi)}\mathcal{B}_1\mathcal{C}_1 \quad \text{and} \quad a_1=-\frac{1}{4\Gamma_1}\left(\frac{2\xi-1}{1-\xi}\right)\mathcal{B}_1\mathcal{C}_2+\frac{1}{8\Gamma_1}\left(\frac{2\xi-1}{1-\xi}\right)\mathcal{C}_1{}^2\left(\mathcal{B}_1+\mathcal{B}_1{}^2-\mathcal{B}_2\right),$$

therefore

$$a_1-\lambda a_0{}^2=-\frac{1}{4\Gamma_1}\left(\frac{2\xi-1}{1-\xi}\right)\mathcal{B}_1\left(\mathcal{C}_2-\mathcal{C}_1{}^2 M\right),$$

where $M=\frac{1}{2}\left[1+\left(1-2\left(\frac{2\xi-1}{1-\xi}\right)\Gamma_1\lambda\right)\mathcal{B}_1-\frac{\mathcal{B}_2}{\mathcal{B}_1}\right]$,

Thus the result 4.3 follows by application of lemma 1,from the other hand if $\mathcal{B}_1=0$,then $a_0=0$ and

$$a_1=\frac{-1}{8\Gamma_1}\left(\frac{2\xi-1}{1-\xi}\right)\mathcal{C}_1{}^2\mathcal{B}_2 .$$

Since $\mathcal{P}(z)$ has a real part ,$|\mathcal{C}_2|\le 2$,from this ,we get

$$\left|a_1-\lambda a_0{}^2\right|\le\left|\frac{(2\xi-1)}{(1-\xi)}\right|\frac{|\mathcal{B}_2|}{2\Gamma_1}.$$

Now $\psi(z)$ have positive real part , $|\mathcal{B}_2|\le 2$,therefore ,we have

$$\left|a_1-\lambda a_0{}^2\right|\le\left|\frac{(2\xi-1)}{(1-\xi)}\right|\frac{1}{\Gamma_1} .$$

The bounds are sharp for the functions $\mathcal{G}_1(z)$ and $\mathcal{G}_2(z)$ defined by

$$\frac{\xi z^2\left(\mathcal{K}_\varsigma^t\mathcal{G}_1(z)\right)''+z\left(\mathcal{K}_\varsigma^t\mathcal{G}_1(z)\right)'}{(\xi-1)\mathcal{K}_\varsigma^t\mathcal{G}_1(z)+\xi\left(z^2\left(\mathcal{K}_\varsigma^t\mathcal{G}_1(z)\right)''+z\left(\mathcal{K}_\varsigma^t\mathcal{G}_1(z)\right)'\right)}=\psi(z^2)\text{ ,where }\mathcal{G}_1(z)=\frac{1-2z-z^2}{z(1-z^2)} ,$$

$$\frac{\xi z^2\left(\mathcal{K}_\varsigma^t\mathcal{G}_2(z)\right)''+z\left(\mathcal{K}_\varsigma^t\mathcal{G}_2(z)\right)'}{(\xi-1)\mathcal{K}_\varsigma^t\mathcal{G}_2(z)+\xi\left(z^2\left(\mathcal{K}_\varsigma^t\mathcal{G}_2(z)\right)''+z\left(\mathcal{K}_\varsigma^t\mathcal{G}_2(z)\right)'\right)}=\psi(z)\text{ ,where }\mathcal{G}_2(z)=\frac{1-3z}{z(1-z)} .$$

Clearly that $\mathcal{G}_1(z),\mathcal{G}_2(z)$ are in $\sum$.■

References

1. Aouf MK, EL- Ashwah RM,and Zayed HM. Feket- Szegö inequalities for certain class of meromorphic functions, Journal of Egyptian Mathematical Society ,2013;21:197-200 .
2. Dziok J , Darus M ,and Sokol J. Coefficients inequalities for class of meromorphic functions ,Turkish Journal of Mathematics,2018;40(42): 2506-2512 .
3. Keogh FR, Merkes EP. A coefficient inequality for certain classes of analytic functions, proceedings of American Mathematical Society, 1969;20(1):8-12 .
4. Mahmoud MS, Juma ARS,and AL-Saphory RAM. On a subclass of meromorphic univalent functions involving hypergeometric function, Journal of AL-Qadisiyah for computer Science and Mathematics, 2019;11(3):12-20
5. Mayilvaganan S, Magesh N ,and Gatti NB. On certain subclasses of meromorphic functions with positive coefficients associated with Liu-Srivastava linear operator , Int.J. of pure and Appli. Mathematics, 2017;113(13):132-141.
6. Silverman H, Suchithra K , Stephen BA, and Gangadharan A. Coefficient bounds for certain classes of meromorphic functions, Journal of inequality and Applications, 2008: 1-9.
7. Srivastava HM , Li SH, and Tang H. Certain classes of K-uniformly close-to-convex functions and other related functions defined by using the Dziok-Srivastava operator, Bull. Math. Anal. Appl.,2009;1(3) :49-63.