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Survey about Generalizing Distances

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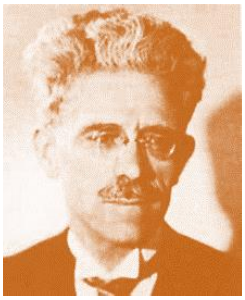
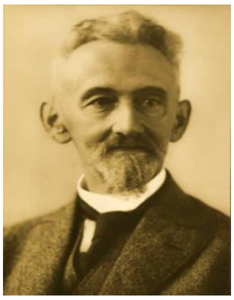
As known, in general topology the talking be about "nearness". This is exactly needed to discuss subjects such convergence and continuity. The simple way to study about nearness is to correspond the set with a distance function to inform us how far apart two elements of are. The metric concept introduced by a French mathematician Maurice René Fréchet (1878 – 1973) in 1906 in his work on some points of the functional calculus. However, the name is due to a German mathematician Felix Hausdorff (1868 –1942) who is considered to be one of the founders of modern topology. In addition to these contribution, he contributed significantly to set theory, descriptive set theory, measure theory, and functional analysis.



Maurice René Fréchet



Felix Hausdorff

A metric space is a set X equipped with a concept of distance. The objects of X can be thought of as points in space, with the distance between points given by a non-negative distance formula, such that, for all $\tilde{a},\tilde{b},\tilde{c}$ in Ω: the distance from $\tilde{a}$ to $\tilde{b}$ is zero if and only if $\tilde{a}$ and $\tilde{b}$ are identical (coincidence), the distance from $\tilde{a}$ to $\tilde{b}$ is the same as distance from $\tilde{b}$ to $\tilde{a}$, (symmetry) and the distance from $\tilde{a}$ to $\tilde{b}$ plus that from $\tilde{b}$ to $\tilde{c}$ is greater than or equal to the distance from $\tilde{a}$ to $\tilde{c}$ (the triangular inequality). The triangle inequality is importance in metric arguments since it uses to proving of: the continuity of a metric d in both variables, the relative topology is Hausdorff, a sequence may converge to at most one point, each open ball is an open set and each convergent sequence is a Cauchy sequence. Though the article, Ω will be a non-empty set [1].

Definition (1): [2] "Let $d:\Omega\, x\, \Omega \to [0,\infty)$ be a function such that $\forall\, \tilde{a},\tilde{b},\tilde{c} \in \Omega$: (1) $d(\tilde{a},\tilde{b}) = 0 \Leftrightarrow \tilde{a} = \tilde{b}$, (2) $d(\tilde{a},\tilde{b}) = d(\tilde{b},\tilde{a})$, (3) $d(\tilde{a},\tilde{c}) \le d(\tilde{a},\tilde{b}) + d(\tilde{b},\tilde{c})$, then d (and the ordered pair (Ω,d))is called metric or distance (respectively, metric space)."

The distance is employ to construct a sub base for a topology, the metric by the concept of ball." Every metric space is Hausdorff, T_2. A pseudo metric space is a generalized metric space in which the distance between two distinct points can be zero, i.e., for any $\tilde{a} \in \Omega$ $d(\tilde{a},\tilde{a}) = 0$. Pseudo metric spaces are not necessarily Hausdorff. The difference between pseudo metrics and metrics is entirely topological. That is, a pseudo metric is a metric if and only if the topology it generates is T_0 "

The distance function has provided great support for all sciences via four main model of distances: for $\tilde{a} = (\alpha_i)_{i=1}^n, \tilde{b} = (\beta_i)_{i=1}^n \in \mathcal{R}^n$ Euclidean distance is the shortest distance between two points

$$d(\tilde{a},\tilde{b}) = \sqrt{\sum_{i=1}^{n}(\alpha_i - \beta_i)^2}$$

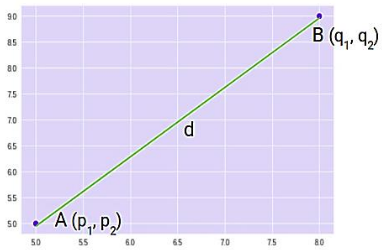
"Although it is a common distance measure, Euclidean distance is not scale in-variant which means that distances computed might be skewed depending on the units of the features. Typically, one needs to normalize the data before using this distance measure.

Moreover, as the dimensionality increases of your data, the less useful Euclidean distance becomes. This has to do with the curse of dimensionality which relates to the notion that higher-dimensional space does not act as we would, intuitively, expect from 2- or 3-dimensional space. "[3]

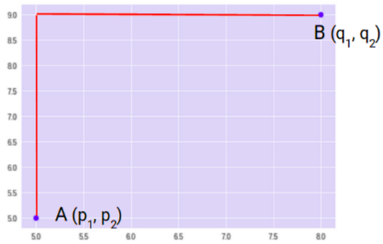
Manhattan distance is the sum of absolute differences between points across all the dimensions

$$d(\tilde{a},\tilde{b}) = \sum_{i=1}^{n}|\alpha_i - \beta_i|.$$

"When your dataset has discrete and/or binary attributes, Manhattan seems to work quite well since it takes into account the paths that realistically could be taken within values of those attributes. Take Euclidean distance, for example, would create a straight line between two vectors when in reality this might not actually be possible."[4]



Euclidean distance between $A(p_1,p_2)$and$B(q_1,q_2)$

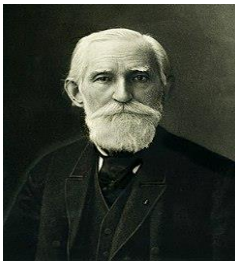


Manhattan distance between $A(p_1,p_2)$and$B(q_1,q_2)$

Minkowski distance is the generalization of Euclidean and Manhattan distances

$$d(\tilde{a},\tilde{b}) = \left(\sum_{i=1}^{n}|\alpha_i - \beta_i|^p\right)^{\frac{1}{p}}, p \geq 1$$

which introduced by the German mathematician Hermann Minkowski.



Pafnuty Lvovich Chebyshev

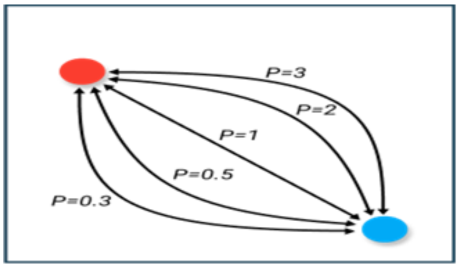
Hermann Minkowski

"The upside to p is the possibility to iterate over it and find the distance measure that works best for your use case. It allows you a huge amount of flexibility over your distance metric, which can be a huge benefit if you are closely familiar with p and many distance measures" [5-7]. The limiting case $p \to \pm\infty$, yield the Chebyshev distance

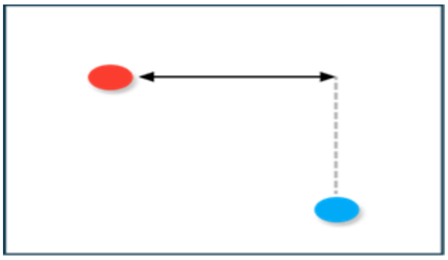
$$d(\tilde{a},\tilde{b}) = \lim_{p\to\infty}\left(\sum_{i=1}^{n}|\alpha_i - \beta_i|^p\right)^{\frac{1}{p}} = max_{i=1}^{n}|\alpha_i - \beta_i|$$

or $$d(\tilde{a},\tilde{b}) = \lim_{p\to-\infty}\left(\sum_{i=1}^{n}|\alpha_i - \beta_i|^p\right)^{\frac{1}{p}} = min_{i=1}^{n}|\alpha_i - \beta_i|$$

"As mentioned before, Chebyshev distance can be used to extract the minimum number of moves needed by aking to go from one square to another. Moreover, it can be a useful measure in games that allow unrestricted 8-way movement", see [5-7]

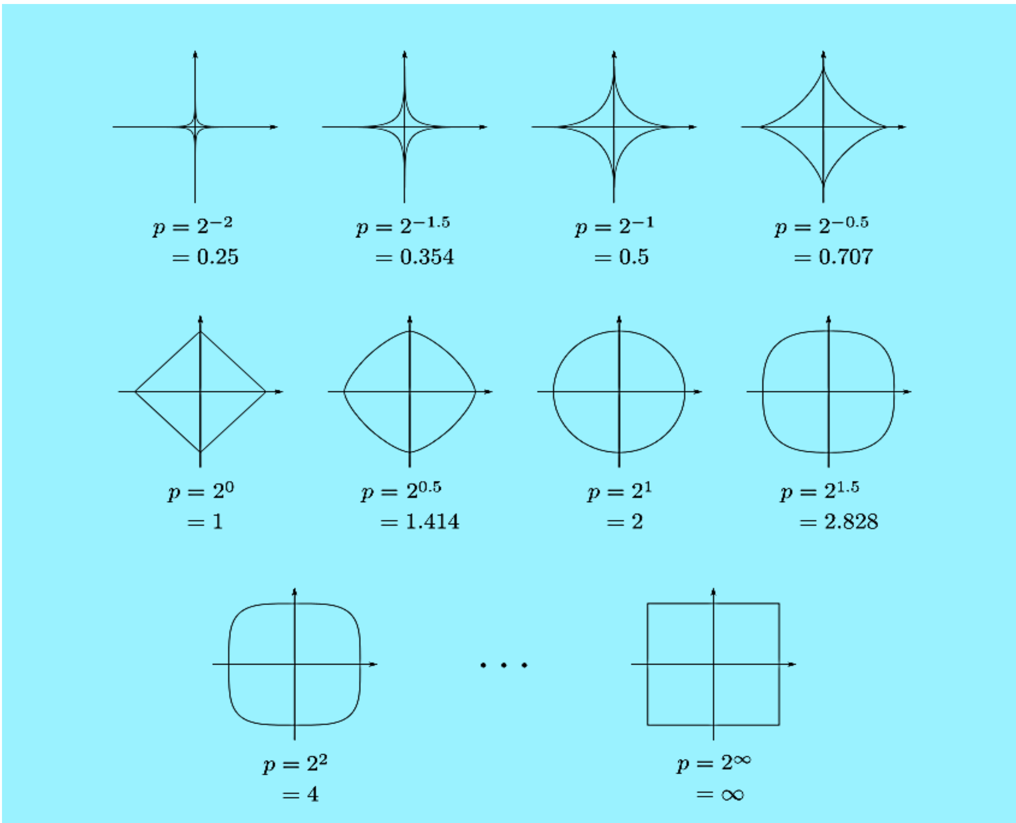


Minkowski's distance up on *p*



Chebyshev's distance

To imagine Minkowski's distance, follow drawing types of unit balls up on p.



Unit ball up on p

Finally, Hamming distance between two equal-length strings of symbols is the number of positions at which the corresponding symbols are different. As example,

- **0000** and **1111** is 4.
- **2173896** and **2233796** is 3.

"Typical use cases include error correction/detection when data is transmitted over computer networks. It can be used to determine the number of distorted bits in a binary word as a way to estimate error. Moreover, you can also use Hamming distance to measure the distance between categorical variables" [4].

As is known, the above distances have been developed to the state of the infinite dimension, where their effect in spaces like normed, Hilbert … etc. was clear. This great importance of metric distances encouraged researchers to find new generalizations to define the distance function itself some will be mentioned here. Firstly,

Definition (2): [8] **"**A mapping $d : \Omega \, x \, \Omega \to [0, \infty)$ is called a quasi-distance if for any $\tilde{a}, \tilde{b}, \tilde{c} \in \Omega$;we have $d(\tilde{a}, \tilde{b}) = 0 \Leftrightarrow \tilde{a} = \tilde{b}$ and $d(\tilde{a}, \tilde{b}) \leq d(\tilde{a}, \tilde{c}) + d(\tilde{c}, \tilde{b})$. Also, the pair (Ω, d) is called a quasi-metric space."

An example of a quasi-metric which is not metric in [5]**,** consider $\Omega = \mathcal{R}$ and

$$d(\tilde{a}, \tilde{b}) = \begin{cases} \tilde{a} - \tilde{b}, & \tilde{a} \geq \tilde{b} \\ 1, & \tilde{a} < \tilde{b} \end{cases}$$

Remark (1):

(1) Every quasi-metric has a conjugate quasi-metric. The function ρ^{-} defined by $\rho^{-}(\tilde{a}, \tilde{b}) = \rho(\tilde{b}, \tilde{a})$ for all $\tilde{a}, \tilde{b} \in \Omega$ is also a quasi-metric on Ω and is called the conjugate quasi-metric of ρ. Also, the mapping $\rho^{s}(\tilde{a}, \tilde{b}) = \max\{\rho(\tilde{a}, \tilde{b}), \rho^{-}(\tilde{a}, \tilde{b})\}$ is a metric on Ω. The inequalities $\rho(\tilde{a}, \tilde{b}) \leq \rho^{s}(\tilde{a}, \tilde{b})$ and $\rho^{-}(\tilde{a}, \tilde{b}) \leq \rho^{s}(\tilde{a}, \tilde{b})$ hold for all $\tilde{a}, \tilde{b} \in \Omega$.

(2) In a quasi-metric space, the limit of a sequence is not necessarily uniqueness. [8

Definition (3): [9] A function $d : \Omega \, x \, \Omega \to [0, \infty)$ is called a semi-distance ⟺for any $\tilde{a}, \tilde{b} \in \Omega$, we have **(**1) $d(\tilde{a}, \tilde{b}) = 0 \Leftrightarrow \tilde{a} = \tilde{b}$ **, (**2) $d(\tilde{a}, \tilde{b}) = d(\tilde{b}, \tilde{a})$ where limiting point are defined in usual way. The space (Ω, d) by Frechet is called E-space and by Menger is called semi-metric.

Example (1) [10] Let $\Omega = \{\frac{1}{k} : k \in N\} \cup \{0\}$. Define $d : \Omega \times \Omega \to [0, \infty)$ by $d(0,1) = d(1,0) = 1$; $d(\frac{1}{k}, 1) = d(1, \frac{1}{k}) = , \frac{2}{3}$, for $k \geq 2$; d(1, 1) = 0; and $d(\tilde{a}, \tilde{b}) = |\tilde{a} - \tilde{b}|$, for $\tilde{a}, \tilde{b} \in \Omega - \{1\}$, d is semi-distance

Various generalizations of standard distance have been studied. One of these distances is distance b or (b −metric). It is the generalization of trigonometric inequalities which has been studied by Czerwik [11]

Definition (4): [12] "Let $s \geq 1$, a function $d : \Omega \, x \, \Omega \to [0, \infty)$ be a s −metric on $\Omega \Leftrightarrow \forall \tilde{a}, \tilde{b}, \tilde{c} \in \Omega$: (1) $d(\tilde{a}, \tilde{b}) = 0$ if and only if $\tilde{a} = \tilde{b}$, (2) $d(\tilde{a}, \tilde{b}) = d(\tilde{b}, \tilde{a})$and (3) $d(\tilde{a}, \tilde{c}) \leq s[d(\tilde{a}, \tilde{c}) \leq d(\tilde{a}, \tilde{b}) + d(\tilde{b}, \tilde{c})$]. And $(\Omega, d)$is called a s −metric space (or metric type space)

The type of s-metric spaces is effectively larger than that of the usual metric spaces. if $s = 1$, then (Ω, d) is metric space. The following examples illustrates this.

Example (2): [13] "Let $\Omega = \{-1, 0, 1\}$. Define $d : \Omega \, x \, \Omega \to R^{+}$ by$d(\tilde{a}, \tilde{b}) = d(\tilde{b}, \tilde{a})$ $\forall \tilde{a}, \tilde{b} \in \Omega$.

$d(\tilde{a}, \tilde{a}) = 0, \tilde{a} \in \Omega$, $d(-1, 0) = 3$and $d(-1, 1) = d(0,1) = 1$.

Then (Ω, d) is a s − metric space, but not a metric since the triangle inequality is not hold, i.e.,

$d(-1, 1) + d(1,0) = 1 + 1 = 2 < 3 = d(-1, 0)$.And $d(-1, 0) \leq 3/2\,(d(-1, 1) + d(1,0))$, so, $(\Omega, d)$is a s − metric space with $r = 3/2$.

On the other hand, we have the following example."

Example (3): [14] **"**Let (Ω, d)be a metric space and $\rho(\tilde{a}, \tilde{b}) = (d(\tilde{a}, \tilde{b}))^{s}$ where$s > 1$. Then $(\Omega, \rho)$is a s −metric with $s = 2^{b-1}$.

Remark (2): [15] In general, a s −distance function is not continuous. It should be noted, s −distance need not be jointly continuous in both variables. For this claim, the reader is referred to see the following example.

Example (4): [16] or [17]" Let $\Omega = N \cup \{\infty\}$, and $d : \Omega \, x \, \Omega \to R$ be defined by

$$d(m,n) = \begin{cases} 0, \text{ if } m = n \\ |\frac{1}{m}+\frac{1}{n}|, \text{ if } m,n \text{ is even or } mn = \infty \\ 5, \text{ if } m,n \text{ is odd and} (m \neq n) \\ 2 \quad \text{otherwies.} \end{cases}$$

Then, for all $m, n, \tilde{a} \in \Omega$, $d(m,\tilde{a}) \leq 5/2d(m,n) + d(n,\tilde{a})$ holds.

Thus, $(\Omega, d)$is a s −metric space (with s = 5/2). Let $\tilde{a}_n = 2n$, $\forall n \in N$. Then $d(2n, \infty) = 1/2n \to 0$ as $n \to \infty$, that is, $\tilde{a}_n \to \infty$, but $d(\tilde{a}_n, 1) = 2$ not implies $5 = d(\infty, 1)$ as $n \to \infty$."

Since then, several works have dealt with other theoretical field in such spaces, see[17],[18],[19],[11],[20],[21],[22],[23],[24],[25],[26],[27]and[28]. In 2017, Kamran and et al. [29] introduced an extension of s-distance, where they expanded the triangle inequality by using the real-valued function d_θ: :Ω ×:Ω → [0,∞), $d(\tilde{a},\tilde{c}) \leq \theta(\tilde{a},\tilde{c})[d(\tilde{a},\tilde{b}) + d(\tilde{b},\tilde{c})]$. Also see [30-31].

Below we mention a generalization of the condition (3) in Definition (1)

Definition (5): [32] "A function $d : \Omega \, x \, \Omega \to [0,\infty)$ is called a rectangular distance if for different elements $\tilde{a}, \tilde{b}, m$ and $n \in \Omega$:(1) $d(\tilde{a},\tilde{b}) = 0 \Leftrightarrow \tilde{a} = \tilde{b}$, (2) $d(\tilde{a},\tilde{b}) = d(\tilde{b},\tilde{a})$ and (3) $d(\tilde{a},\tilde{b}) \leq d(\tilde{a},n) + d(n,m) + d(m,\tilde{b})$.

Then$(\Omega, d)$is called a rectangular or Branciari's space.

Remark (3): [33] topologically, the Branciari metric and the usual metric spaces are different. In particular:

(i) The Branciari distance does not have to be continuous (in both variables).
(ii) It does not have to be the metric of Branciari Hausdorff.
(iii) An open ball does not have to be an open set.
(iv) The convergent sequence may not be a Cauchy sequence.

For many results and definitions Branciari's spaces, see [24, 29, 38, 39, 44, 69]. Similar to the above generalizations Similar to the above generalizations, a generalization is given for the rectangular distance

Definition (6): [34] " Let $s \geq 1$, and $d : \Omega \times \Omega \to [0,\infty)$ be a function, d is is called a b −rectangular $\tilde{a}, \tilde{b}m$ and $n \in \Omega$ iff: (1) $d(\tilde{a},\tilde{b}) = 0$ if and only if $\tilde{a} = \tilde{b}$, (2) $d(\tilde{a},\tilde{b}) = d(\tilde{b},\tilde{a})$ and (3) $d(\tilde{a},\tilde{b}) \leq s[d(\tilde{a},n) + d(n,m) + d(m,\tilde{b})]$.And $(\Omega, d)$is called a s −rectangular metric space. "

The following example shows that s −rectangular distance not necessary rectangular.

Example (5): Let $\Omega = \mathbb{N}$, define $d: \Omega \times \Omega \to \Omega$ by

1) $$d(\tilde{a},\tilde{b}) = \begin{cases} 0 & if \ \tilde{a} = \tilde{b} \\ 4\beta & if \ \tilde{a},\tilde{b} \in \{1,2\} \ and \ \tilde{a} \neq \tilde{b} \\ \beta & if \ \tilde{a} \ or \ \tilde{b} \notin \{1,2\} and \ \tilde{a} \neq \tilde{b} \end{cases}$$

where $\beta > 0$ is a scalar. d is s- rectangular distance with $s = \frac{4}{3} > 1$, but d is not a rectangular, since $d(1,2) = 4\beta > 3\beta = d(1,3) + d(3,4) + d(4,2)$

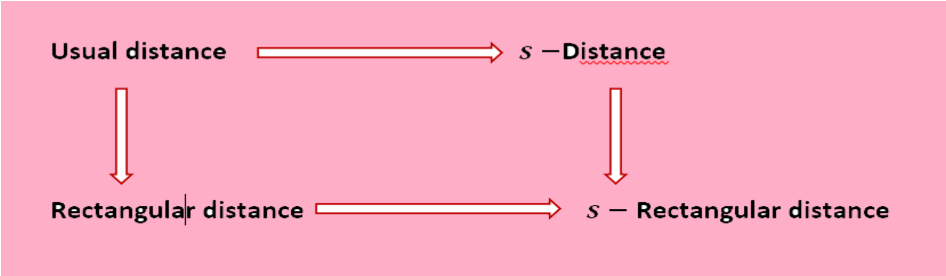
2) If $d(\tilde{a},\tilde{b}) = d(\tilde{b},\tilde{a}) \; \forall \; \tilde{a},\tilde{b} \in \Omega$, $\beta > 0$ and

$$d(\tilde{a},\tilde{b}) = \begin{cases} 0 & if \ \tilde{a} = \tilde{b} \\ 10\beta & if \ \tilde{a} = 1, \tilde{b} = 2 \\ \beta & if \ \tilde{a} \in \{1,2\} and \ \tilde{b} \in \{3\} \\ 2\beta & if \ \tilde{a} \in \{1,2,3\} and \ \tilde{b} \in \{4\} \\ 3\beta & if \ \tilde{a} \ or \ \tilde{b} \notin \{1,2,3,4\} \ and \ \tilde{a} \neq \tilde{b} \end{cases}$$

Clear d is s- rectangular distance with $r = 2 > 1$, but $(\Omega, d)$is not a rectangular, as

$$d(1,2) = 10\beta > 5\beta = d(1,3) + d(3,4) + d(4,2).$$

Below, a diagram and example for inclusions where their inverse inclusions do not hold

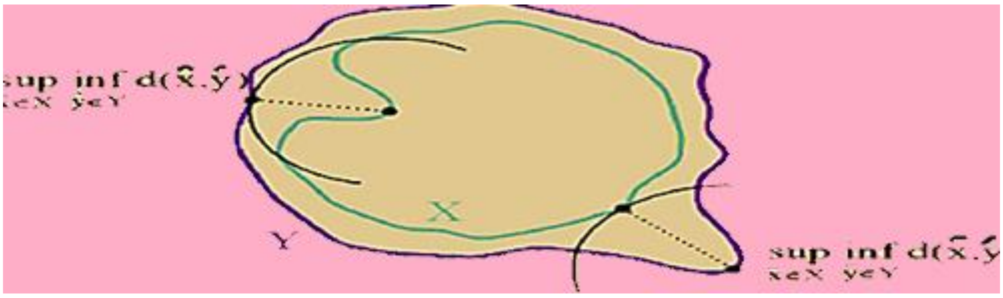


We are now talking about a type of distance between two sets presented by Felix Hausdorff. "It is the greatest of all the distances from a point in one set to the closest point in the other set". [35]

Definition (7) [35] For $\emptyset \neq A \subset \Omega, \emptyset \neq B \subset \Omega$, the Hausdorff distance between A and B is

$$H(A,B) = \max\{ \sup_{\tilde{x} \in A} d(\tilde{x}, B), \sup_{\tilde{y} \in B} d(\tilde{y}, A)\}$$

where $d(\tilde{x},B) = \inf \{d(\tilde{x}, \tilde{y}): \tilde{y} \in B\}$ and by convention $\inf \emptyset = \infty$ and $H(\emptyset,\emptyset) = 0$.



Another form of Hausdorff distance can be defined by neighborhoods of sets.

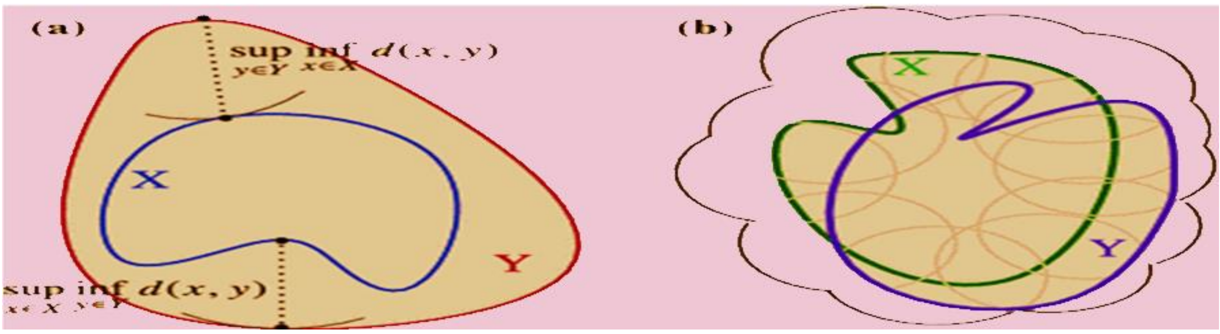
Definition (8) [35] the ε-neighborhood of a non-empty subset A of the metric space (Ω,d) is

$$N_\varepsilon(A) = \{\tilde{x} \in \Omega : d(\tilde{x}, \tilde{a}) < \varepsilon \text{ for some } \tilde{a} \in A\}.$$

Therefore,

$\cap_{\varepsilon>0} N_\varepsilon(A) = \bar{A}$ and $N_\varepsilon(\cup_{\varepsilon>0} A_n) = \cup_{\varepsilon>0} N_\varepsilon(A_n)$.

$$H(A,B) = \inf\{\varepsilon > 0, A \subset N\varepsilon(B) \text{ and } B \subset N\varepsilon(A)\}$$



Hausdorff's distance between two subsets X and Y (in the plane).

Example (6): Let $A = [0,1]\cup[6,8] B = [-1,5]$. Then $d(A,B) = 3 = d(8,5)$.

$d(B,A) = 2.5 = d(3.5,1) = d(3.5,6)\ h(A,B) = h(B,A) = 3$.

"Many mathematicians have tried to extend the concept of distance to defined distance of three or more points. Gahler ([36], [37]) first defined 2−metric space as follows:

Definition (9): [39] A function d on $\Omega\times\Omega\times\Omega$ is said to be a 2 −distance on Ω if

i) for distinct elements $\tilde{a},\tilde{b}\in\Omega$, $\exists\tilde{c}\in\Omega$ such that $d(\tilde{a},\tilde{b},\tilde{c})\neq 0$
ii) $d(\tilde{a},\tilde{b},\tilde{c}) = 0$ when at least two of $\tilde{a},\tilde{b},\tilde{c}$ are equal,
iii) $d(\tilde{a},\tilde{b},\tilde{c}) = d(\tilde{a},\tilde{c},\tilde{b}) = d(\tilde{b},\tilde{c},\tilde{a})$ for all $\tilde{a},\tilde{b},\tilde{c}\in\Omega$,
iv) $d(\tilde{a},\tilde{b},\tilde{c})\leq d(\tilde{a},\tilde{b},\tilde{d}) + d(\tilde{a},\tilde{d},\tilde{b}) + d(\tilde{d},\tilde{b},\tilde{c})$ for all $\tilde{a},\tilde{b},\tilde{c},\tilde{d}\in\Omega$.

And then the pair$(\Omega,d)$is called a 2 −metric space.

Remark (4): [38] "A 2 −metric is not a continuous function of its variables, every2 −metric is non-negative and it might assumed that every 2 −metric space contains at least three distinct points."

Remark (5): [39] Geometrically, Geometrically, $d(\tilde{a},\tilde{b},\tilde{c})$ is interpret as the area of the triangle spanned by $\tilde{a},\tilde{b}$ and $\tilde{c}$. In Definition (1.1.17), if condition (iv) is canceled then d is called a semi2−distance.

After that Dhage [40] gave another definition of distance together which is

Definition (10): [40] "Let Ω be a nonempty set, and let R denote the real numbers. Afunction $D{:}\,\Omega^3\to R$ satisfying the following axioms:

1) $D(\tilde{a},\tilde{b},\tilde{c})\leq 0$ for all $\tilde{a},\tilde{b},\tilde{c}\in\Omega$

2) $D(\tilde{a},\tilde{b},\tilde{c}) = 0$ if and only if $\tilde{a}=\tilde{b}=\tilde{c}$

3) $D(\tilde{a},\tilde{b},\tilde{c}) = D(\tilde{a},\tilde{c},\tilde{b}) = \cdots$ (Symmetry in all three variables),

4) $D(\tilde{a},\tilde{b},\tilde{c})\leq D(\tilde{a},\tilde{b},a)+D(\tilde{a},a,\tilde{c})+D(a,\tilde{b},\tilde{c})$ for all $\tilde{a},\tilde{b},\tilde{c}\in\Omega$ (Rectangle inequality),

is called a generalized distance, or a D −distance on Ω.

Remark (6): $D(\tilde{a},\tilde{b},\tilde{c})$ may be interpreted as a measure of the perimeter of the triangle with vertices at $\tilde{a},\tilde{b}$ and $\tilde{c}$.

"Unfortunately, most of the claims concerning the fundamental topological properties of D −metric spaces are incorrect (see [41]). This claim provided inspiration for the formation of more general concept called a G −distance by Mustafa and Sims [42]."

Definition (11): [42] "Let $\omega{:}\,\Omega^3\to[0,\infty)$ be function satisfying the following condition:

1-$\omega(\tilde{a},\tilde{b},\tilde{c}) = 0$ if and only if $\tilde{a}=\tilde{b}=\tilde{c}$.

2-$\omega(\tilde{a},\tilde{a},\tilde{b}) > 0, \forall\tilde{a},\tilde{b}\in\Omega$ with $\tilde{a}\neq\tilde{b}$

3- $\omega(\tilde{a},\tilde{a},\tilde{b})\leq\omega(\tilde{a},\tilde{b},\tilde{c})$ for all $\tilde{a},\tilde{b},\tilde{c}\in\Omega$ with $\tilde{b}\neq\tilde{c}$

4- $\omega(\tilde{a},\tilde{b},\tilde{c}) = \omega(\tilde{a},\tilde{c},\tilde{b}) = \cdots$, (symmetry in all three variables),

5- $\omega(\tilde{a},\tilde{b},\tilde{c})\leq\omega(\tilde{a},a,a)+\omega(a,\tilde{b},\tilde{c})$ for all, $\tilde{b},\tilde{c},a\in\Omega$.

then ω is called generalized distance on Ω and the pair (Ω,ω) is called a G − metric space ."

Example (7): [43]

"Consider $\Omega = R^{+}$, with usual distance $d(\tilde{a},\tilde{b}) = |\tilde{a}-\tilde{b}|$, for all $\tilde{a},\tilde{b}$ in Ω. Define $\omega{:}\,\Omega^3\to R^{+}$

$\omega(\tilde{a},\tilde{b},\tilde{c}) = |\tilde{a}-\tilde{b}|+|\tilde{b}-\tilde{c}|+|\tilde{c}-\tilde{a}|$ for all $\tilde{a},\tilde{b},\tilde{c}\in\Omega$

1. $\omega(\tilde{a},\tilde{b},\tilde{c}) = 0 \Leftrightarrow |\tilde{a}-\tilde{b}| + |\tilde{b}-\tilde{c}| + |\tilde{c}-\tilde{a}| = 0$, since $|\tilde{a}-\tilde{b}| + |\tilde{b}-\tilde{c}| + |\tilde{c}-\tilde{a}| = 0$we have $\tilde{a} = \tilde{b} = \tilde{c}$, then $\omega(\tilde{a},\tilde{b},\tilde{c}) = 0$ if and only if $\tilde{a} = \tilde{b} = \tilde{c}$

2. $\omega(\tilde{a},\tilde{a},\tilde{b}) > 0$. (By Definition of absolute value)

3. $\omega(\tilde{a},\tilde{a},\tilde{b}) = |\tilde{a}-\tilde{a}| + |\tilde{a}-\tilde{b}| + |\tilde{b}-\tilde{a}|$

$$\le |\tilde{a}-\tilde{b}| + |\tilde{b}-\tilde{c}| + |\tilde{c}-\tilde{a}|$$

$$= \omega(\tilde{a},\tilde{b},\tilde{c}).$$

4. $\omega(\tilde{a},\tilde{b},\tilde{c}) = \omega(\tilde{a},\tilde{c},\tilde{b}) = \cdots$,(symmetry in all three variables),

5. $\omega(\tilde{a},\tilde{b},\tilde{c}) \le \omega(\tilde{a},a,a) + \omega(a,\tilde{b},\tilde{c})$ for all, $\tilde{b},\tilde{c},a \in \Omega$.

Then $\omega$is a g −metric on Ω. "

Definition (12): [44] The G-distance ω is called symmetric if $\omega(\tilde{a},\tilde{b},\tilde{b}) = \omega(\tilde{a},\tilde{a},\tilde{b})$ for all $\tilde{a},\tilde{b} \in \Omega$.

Example(8):[45]"Let $\Omega = \{\tilde{a},\tilde{b}\}$ and $\omega(\tilde{a},\tilde{a},\tilde{a}) = \omega(\tilde{b},\tilde{b},\tilde{b}) = 0$, $\omega(\tilde{a},\tilde{a},\tilde{b}) = 3$, $\omega(\tilde{a},\tilde{b},\tilde{b}) = 3$ and extend ω to all of Ω^3 by symmetry in the variables. Then the G is symmetric, since $\omega(\tilde{a},\tilde{b},\tilde{b}) = \omega(\tilde{a},\tilde{a},\tilde{b})$."

Example (9) [45]"Let $\Omega = \{\tilde{a},\tilde{b}\}$ and $\omega(\tilde{a},\tilde{a},\tilde{a}) = \omega(\tilde{b},\tilde{b},\tilde{b}) = 0$, $\omega(\tilde{a},\tilde{a},\tilde{b}) = 1$, $\omega(\tilde{a},\tilde{b},\tilde{b}) = 2$ and by symmetry expand ω to all of $\Omega \, x \, \Omega \, x \, \Omega$. Then ω is a g −distance, but $\omega(\tilde{a},\tilde{b},\tilde{b}) \ne \omega(\tilde{a},\tilde{a},\tilde{b})$."

Proposition (1): [42**]** For a G −distance ω, then the following are equivalent:

1) (Ω,ω) is symmetric.
2) $\omega(\tilde{a},\tilde{b},\tilde{b}) \le \omega(\tilde{a},\tilde{b},a), \forall \tilde{a},\tilde{b},a \in \Omega$.
3) $\omega(\tilde{a},\tilde{b},\tilde{c}) \le \omega(\tilde{a},\tilde{b},a) + \omega(\tilde{c},\tilde{a},b), \forall \tilde{a},\tilde{b},\tilde{c},a,b \in \Omega$.

Sometimes, a usual distance drives form G −distance such as

Proposition (2): [46]

i) every G –distance (Ω,ω) on Ω defines a distance d_ω on Ω given by
ii) $d_\omega(\tilde{a},\tilde{b}) = \omega(\tilde{a},\tilde{b},\tilde{b}) + \omega(\tilde{b},\tilde{a},\tilde{a})$ for all $\tilde{a},\tilde{b} \in \Omega$.
iii) if ω is symmetric then $d_\omega(\tilde{a},\tilde{b}) = 2\omega(\tilde{a},\tilde{b},\tilde{b})$.
iv) if ω isnot symmetric then $\frac{3}{2}\omega(\tilde{a},\tilde{b},\tilde{b}) \le d_\omega(\tilde{a},\tilde{b}) \le 3\omega(\tilde{a},\tilde{b},\tilde{b})$.

Subsequently, many theoretical and applied studies appeared based on that distance. We will defer it to a later time, but you can see some of them as follows: [47-52].

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