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Jordan Generalized Centralizerhomo on Prime Rings

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Abstract

In this article we introduce the concepts of generalized centralizerhomo and Jordan generalized centralizerhomo on prime ring. The aim object of this paper we prove: Every Jordan generalized centralizerhomo of 2-torsion free prime ring R is Jordan generalized triple centralizerhomo on R.

Keywords: Prime ring, centralizer, homomorphism, generalized centralizerhomo , Jordan generalized

1. Introduction

The objective of the definition of generalized left centralizerhomo is to generalize the definition of left centralizerhomo on rings and to specify the relation between two concepts: generalized left centralizerhomo and Jordan generalized left centralizerhomo with in certain conditions.

The definition of prime ring and semiprime ring was presented in [1]. The definition of $2-$torsion free was presented in [2] while the definition of centralizerhomo was presented in[3].On the other hand, [3] was first to introduce the concepts of left centralizerhomo, Jordan left centralizerhomo and generalization on rings, many results were found by the researchers,

one of these results was that "every Jordan left centralizer of a $2-$torsion free prime ring R is a centralizer". For more information of left centralizer and homomorphism see [4,5,6,7]

One important question can be answered in this paper whether there is a relation between a concept of Jordan generalized left centralizerhomo and a concept of Jordan triple generalized left centralizerhomo within certain conditions.

In this paper we refer to the left centralizerhomo by centralizerhomo for short.

2. Generalized Centralizerhomo on Prime Rings.

Definition 2.1: Let Ψ be an additive mapping on a ring R then Ψ is called generalized centralizerhomo on R into itself associated with centralizerhomo g on R is

$\Psi(\varkappa u)= g(\varkappa)u + \Psi(\varkappa)g(u)$, $\forall\, \varkappa, u \in R$.

Ψ is called Jordan generalized centralizerhomo on R associated with Jordan centralizerhomo g on R if for all a$\in$ R ,then $\Psi(\varkappa 2)= g(\varkappa)\varkappa + \Psi(\varkappa)g(\varkappa)$

Ψ is called Jordan generalized triple centralizerhomo on R associated with Jordan triple centralizerhomo g on R if for all $\varkappa, u \in R$,then $\Psi(\varkappa u\varkappa) = g(\varkappa)u\varkappa + \Psi(\varkappa)g(u)g(\varkappa)$.

Example 2.2: Let R ={(0 a 0 0): a $\in$ Q} then R is a ring we define an additive mapping Ψon R by Ψ(0 a 0 0) = (0 3a^2 0 0) and we defend centralizerhomo h on R by h(0 a 0 0) = (0 2a 0 0) , for all a$\in$Q. Then $\Psi$is generalized centralizerhomo on R.

Lemma1 : Let R be a ring and Ψ is Jordan generalized centralizerhomo on R then,

$\Psi(\varkappa u+u\varkappa) = g(\varkappa) u + \Psi(\varkappa)g(u) + g(u) \varkappa + \Psi(u)g(\varkappa) \qquad \forall\, \varkappa, u \in R$

Proof:

$\Psi((\varkappa+ u)(\varkappa+ u)) = g(\varkappa + u)(\varkappa + u)+ \Psi(\varkappa + u)g(\varkappa + u)$

$= (g(\varkappa)+g(u))(\varkappa+ u) + (\Psi(\varkappa)+ \Psi(u)) (h(\varkappa)+g(u))$

$= g(\varkappa) \varkappa +g(\varkappa) u +g(u) \varkappa +g(u) u + \Psi(\varkappa)g(\varkappa)+ \Psi(\varkappa)g(u)+ \Psi(u)g(\varkappa)+ \Psi(u)g(\varkappa) \quad ...(1)$

$\Psi((\varkappa+u)(\varkappa+u)) = \Psi(\varkappa^2+\varkappa u+u\varkappa+u^2)$

$= \Psi(\varkappa^2)+\Psi(u^2)+\Psi(\varkappa u+u\varkappa)$

$= g(\varkappa)\varkappa+\Psi(\varkappa)g(\varkappa)+g(u)u+\Psi(u)g(u)+\Psi(\varkappa u+u\varkappa)$ ...(2)

Comparing (1) and (2) we get

$\Psi(\varkappa+u\varkappa) = g(\varkappa)u+\Psi(\varkappa)g(u)+g(u)\varkappa+\Psi(u)g(\varkappa)$

Definition 2.3: Let be Jordan generalized centralizerhomo on a ring R then we define the mapping φ on R × R into R by $\varphi(\varkappa, u) = \Psi(\varkappa u)-g(\varkappa)u-\Psi(\varkappa)g(u)$ ∀ ϰ, u ∈ R.

In the following lemma we present the properties of φ:

Lemma 2: Let Ψ be Jordan generalized centralizerhomo on a ring R then

i) $\varphi(\varkappa, u) = -\varphi(u, \varkappa)$

ii) $\varphi(\varkappa+c, u) = \varphi(\varkappa, u)+\varphi(c, u)$

iii) $\varphi(\varkappa, u+c)=\varphi(\varkappa, u)+\varphi(\varkappa,c)$

Holds for all ϰ, u ∈ R

Proof: i) By lemma 1 we get

$\Psi(\varkappa u+u\varkappa) = g(\varkappa)u+\Psi(\varkappa)g(u)+g(u)\varkappa+\Psi(u)g(\varkappa)$

$\Psi(\varkappa u)-g(\varkappa)u-\Psi(\varkappa)g(u) = -\Psi(u\varkappa)+g(u)\varkappa+\Psi(u)g(\varkappa)$

$\varphi(\varkappa, u) = -\varphi(u, \varkappa)$

ii) $\varphi(\varkappa+c, u)= \Psi((\varkappa+c)u)-g(\varkappa+c)u-\Psi(\varkappa+c)g(u)$

$= \Psi(\varkappa u+cu)-g(\varkappa)u-g(c)u-(\Psi(\varkappa)+\Psi(c))g(u)$

$= \Psi(\varkappa u)-g(\varkappa)u-\Psi(\varkappa)g(u)+\Psi(cu)-g(c)u-\Psi(c)g(u)$

$= \varphi(\varkappa, u)+\varphi(c, u)$

iii) by using the same way of (ii) we get (iii).

Remark 2.4: Let R be a ring. Then Ψ is generalized centralizerhomo on R if and only if $\varphi(\varkappa, u) = 0$, ∀ ϰ, u ∈ R.

3. The Main Results

Lemma 3: Let Ψ be Jordan generalized centralizerhomo on a ring R where homomorphism

acts as centralizer on R then $\varphi(\varkappa, u)r[\varkappa,b]=0$ ∀ ϰ, u ∈ R

Proof: Let $w= \varkappa ur u\varkappa + u\varkappa r\varkappa u$

$\Psi(w) = \Psi(\varkappa(uru)\varkappa + u(\varkappa r\varkappa)u)$

$= g(\varkappa)uru\varkappa+\Psi(\varkappa)g(uru)g(\varkappa)+g(u)\varkappa r\varkappa u+\Psi(u)g(\varkappa r\varkappa)g(u)$

$= g(\varkappa)uru\varkappa+\Psi(\varkappa)g(u)g(r)g(u)g(\varkappa)+g(u)\varkappa r\varkappa u+\Psi(u)g(\varkappa)g(r)g(\varkappa)g(u)$ ...(1)

On the other hand

$\Psi(w) = \Psi((\varkappa u)r(u\varkappa)+(u\varkappa)r(\varkappa u))$

$= g(\varkappa u)\varkappa u+\Psi(\varkappa u)g(r)g(u\varkappa)+g(u\varkappa)r\varkappa u+\Psi(u\varkappa)g(r)g(\varkappa u)$ ...(2)

Comparing (1) and (2) we get

$0= (\Psi(\varkappa u)-g(\varkappa)u-\Psi(\varkappa)g(u))r\varkappa u+(\Psi(u\varkappa)-g(u)\varkappa-\Psi(u)g(\varkappa))ru\varkappa$

$= \varphi(\varkappa, u)r\varkappa u-\varphi(u, \varkappa)ru\varkappa$

= φ(ϰ, u)r[ϰ, u]

Lemma 4: Let Ψ be Jordan generalized centralizergomo on a prime ring R then for every ϰ, u,c,d∈ R , φ(ϰ, u)r[c,d]=0

Proof: replace ϰ +c for ϰ in Lemma 3 ,then

φ(ϰ+c, u)r[ϰ +c, u] =0

φ(ϰ, u)r[ϰ, u] + φ(ϰ, u)r[c, u] + φ(c, u)r[ϰ, u] + φ(c, u)r[c, u] =0

By Lemma 3 we get

φ(ϰ, u)r[c, u] + φ(c, u)r[ϰ, u] = 0

Therefore

0= φ(ϰ, u)r[c, u]r φ(ϰ, u)r[c, u]

= - φ(ϰ, u)r[c, u]r φ(c, u)r[ϰ, u]

By primness of R

φ(ϰ, u)r[c, u] =0 ...(1)

Replace u +d for b in Lemma 3 we have

φ(ϰ, u +d)r[ϰ, u +d] =0

φ(ϰ, u)r[ϰ, u] + φ(ϰ, u)r[ϰ,d] + φ(ϰ,d)r[ϰ, u] + φ(ϰ,d)r[ϰ,d] =0

By Lemma 3

φ(ϰ, u)r[ϰ,d] + φ(ϰ,d)r[ϰ, u] =0

Therefore

0= φ(ϰ, u)r[ϰ,d] r φ(ϰ, u)r[ϰ,d]

= - φ(ϰ, u)r[ϰ,d] r φ(ϰ,d)r[ϰ, u]

By primness of R φ(ϰ, u)r[ϰ,d] =0 ...(2)

Now, φ(ϰ, u)r[ϰ+c, u +d]=0

φ(ϰ,b)r[ϰ,b] + φ(ϰ,b)r[ϰ,d] + φ(ϰ,b)r[c,b] + φ(ϰ,b)r[c,d]=0

By (1), (2) and Lemma 3 we obtain

φ(ϰ,b)r[c,d] =0

Theorem 5 : Every Jordan generalized centralizerhomo on 2 – torsion free prime ringR is generalized centralizerhomo on R.

Proof: Let Ψbe Jordan generalized centralizerhomo on 2 – torsion free prime ringR.

By lemma 4 and since R is prime then we get either φ(ϰ, u) =0 , for all ϰ, u ∈ R or [c,d] =0 for all c,d∈ R.

If [c,d] ≠0, for all c,d∈ R then φ(ϰ, u) =0, for all ϰ, u ∈ R , then by remark 2.4 we get Ψ is centralizerhomo on R.

If [c,d] =0 for all c,d∈ R then R is commutative and hence by Lemma 1 we get

Ψ(2ϰ u) = 2(g(ϰ) u + Ψ(ϰ)g(u))

Since R is 2 – torsion free we get Ψis generalized centralizerhomo.

Proposition 6: Every Jordan generalized centralizerhomo of $2-$torsion free ring R is Jordan generalized triple centralizerhomo on R.

Proof: Let Ψbe Jordan generalized centralizerhomo on R.

Replace ϰ u +uϰ by u in Lemma 1 we get

Ψ(ϰ (ϰu+uϰ)+ (ϰu+uϰ) ϰ)= g(ϰ)(ϰu+uϰ) + Ψ(ϰ)g(ϰu+uϰ)+ g(ϰu+uϰ) ϰ + Ψ(ϰu+uϰ)g(ϰ)

=g(ϰ) ϰ u +g(ϰ)u ϰ + Ψ(ϰ)(g(ϰ)g(u)+g(u)g(ϰ)) + g(ϰ)u ϰ +g(u)ϰϰ +

(Ψ(ϰ)g(u)+ Ψ(u)g(ϰ))g(ϰ)

= g(ϰ)ϰu +g(ϰ)uϰ + Ψ(ϰ)g(ϰ)g(u)+Ψ(ϰ)g(u)g(ϰ)+ Ψ(ϰ)g(u)g(ϰ)+ Ψ(u)g(ϰ)g(ϰ) ...(1)

On the other hand

Ψ(ϰ(ϰu+uϰ)+ (ϰu+uϰ)ϰ) = Ψ(ϰϰu + ϰuϰ + ϰuϰ + uϰϰ)

= Ψ(ϰϰu) + Ψ(2 ϰuϰ) + Ψ(uϰϰ)

= g(ϰ) ϰu + Ψ(ϰ)g(ϰ)g(u)+ g(u) ϰϰ + Ψ(u)g(ϰ)g(ϰ)+ 2Ψ(ϰuϰ) ...(2)

Comparing (1) and (2) we get

2 Ψ(ϰuϰ) = g(ϰ)uϰ + Ψ(ϰ)g(u)g(ϰ) + g(ϰ)uϰ + Ψ(ϰ)g(u)g(ϰ)

Since R is 2-torsion free we get

Ψ(ϰuϰ) = g(ϰ)uϰ + Ψ(ϰ)g(u)g(ϰ)

Hence Ψ is Jordan generalized triple centralizerhomo.

Corollary 7: Every generalized centralizerhomo on $2-$torsion free ring R is Jordan generalized triple centralizerhomo on R.

Proof: Let Ψ be generalized centralizerhomo on a ring R, then Ψ is Jordan generalized centralizerhomo on R, hence by Proposition 6 we obtain Ψ is Jordan generalized triple centralizedhomo on R.

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