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SECOND ORDER PARTIAL DERIVATIVES

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Abstract

The rules for calculating partial derivatives and differentials are the same as for calculating the derivative of a function of one variable, except that when finding partial derivatives per one variable, the other variables are considered as constants.

Keywords: Partial derivatives, Function of one variable, Function of two variables, Lagrange's theorem.

1. Introduction

If all unknown functions that enter the differential equation depend on only one independent variable, and therefore the equation does not contain partial derivatives, this equation is called an ordinary differential equation [1]. If unknown functions that depend on several independently variables enter the equation, and partial derivatives of unknown functions appear in the equation, that equation is called a partial differential equation [2]. We will consider here only ordinary differential equations, i.e. equations in which unknown functions appear that depend on only one independently variable [3]. If only one unknown function appears in an ordinary differential equation, with its derivatives, then such an equation has a general form.

For function $z = f(x, y)$ partial derivatives by variables and x and y are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

2. Partial Excerpts

Partial excerpts partial excerpts $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ by variables *x*, *y* are partial derivatives of the **second order** after *x* , *y* respectively [4].

$\frac{\partial^2 z}{\partial x^2}$ - partial derivative of the second order function *z* as per *x*

$\frac{\partial^2 z}{\partial y^2}$ - partial derivative of the second order function *z* as per *y*

$\frac{\partial^2 z}{\partial x \partial y}$ - mixed partial derivative of the second order function *z* as per *x* , y (partial excerpt partial excerpt

$\frac{\partial z}{\partial y}$ per variable *x*)

2.1. Third-order partial derivatives

by variable *x*

$$f'''_{xxx},\ f'''_{xyx},\ f'''_{yxx},\ f'''_{yyx}$$

by variable *y*

$$f'''_{xxy},\ f'''_{xyy},\ f'''_{yxy},\ f'''_{yyy}$$

In the case of functions of one independent variable, we have proved that the continuity of a given function at that point follows from the assumption of the existence of the first derivative at some point [5],[6]. However, for functions of two or more variables from the existence of both (or all) partial variable derivatives at some point, the continuity of the function at that point cannot follow.

Example: Function $z = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

at all points $(x,y) \in \Re^2$ has both partial derivatives. In dots $(x,y) \neq (0,0)$ the denominator of a given function as well as the denominator of the first practical derivatives is different from, while in the coordinate origin we have that

$$z_x'(0,0) = \lim_{\Delta x \to 0} \frac{z(\Delta x, 0) - z(0,0)}{\Delta x} = 0.$$

$$z_y'(0,0) = \lim_{\Delta y \to 0} \frac{z(0, \Delta y) - z(0,0)}{\Delta y} = 0.$$

However, the given function has an interruption at point (0,0) because it is along the lines $y = kx,\ k \neq 0$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{k^2}{1+k^2} \neq 0, \qquad k \neq 0$$

The existence of partial derivatives affects the behavior of the function only in the directions of the coordinate axes but not in all other directions [7],[8].

Theorem: If a function $z = f(x,y)$ in closed area D it has limited partial derivatives, which means that there is a large enough real number $M > 0$ which does not depend on *x* and *y*, such

that

$$\left|f_x'(x,y)\right| < M, \ \left|f_y'(x,y)\right| < M, \ \ \forall (x,y) \in D \qquad (*)$$

then it is a function $z = f(x,y)$ continuous in areas D.

Proof: Suppose the points $(x,y),(x+\Delta x, y+\Delta y)$ are the points of rectilinear orthogonal segments, which connect these points with the point $(x+\Delta x, y)$ too $\in D$, which is certainly fulfilled if it is (x,y) inner point of the area D, a Δx i Δy small enough.

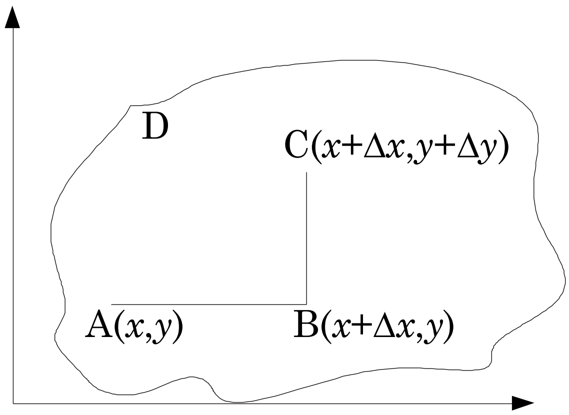

Figure 1. Graph of a given function

In that case, the total increment of a given function can be written in the form

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = \begin{bmatrix} f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) \\ - f(x, y+\Delta y) + f(x,y) \end{bmatrix}.$$

If we apply Lagrange's mean value theorem to each of the expressions in parentheses, treating the expression in the first bracket as the increment of a given function by a variable, and the expression in the second bracket as the increment of a function by a variable

$$f(x+\Delta x, y+\Delta y) - f(x,y) = f_y'(x+\Delta x, y+\eta\Delta y)\Delta y + f_x'(x+\theta\Delta x, y+\Delta y)\Delta x,$$

$$0 < \eta < 1, 0 < \theta < 1 \qquad (**)$$

where f_y' takes at some point a vertical segment connecting the points $f(x+\Delta x, y)$ i $f(x+\Delta x, y+\eta\Delta y)$ a f_x' at some point of the horizontal segment between the points (x,y) i $(x+\Delta x, y)$. If we use the restrictions now (*) in the equation $(**)$ we get that

$$\left| f(x+\Delta x, y+\Delta y) - f(x,y) \right| < M\left(|\Delta x| + |\Delta y| \right)$$

which means that the total increment functions $f(x,y)$ we can make it arbitrarily small if the increments are independently variable Δx i Δy small enough, follows to function $f(x,y)$ whose partial derivatives are f_x' i f_y' bounded in closed area D, continuous at each point of that area[9].

Theorem: If the partial derivatives are mixed f''_{xy}, f''_{yx} parametric functions $f(x, y)$ continuous functions by *x* and *y* at each point of some domain D, then at every interior point of that domain holds

$$f''_{xy} = f''_{yx}$$

Proof: If the point $P(x, y)$ the inner point of area D will then be for sufficiently small increments Δx i Δy $(\Delta x, \Delta y \neq 0)$ and points $P_1(x+\Delta x, y)$, $P_2(x, y+\Delta y)$, $P_3(x+\Delta x, y+\Delta y)$, as well as a rectangle $P\,P_1P_2P_3$ lie within area D.

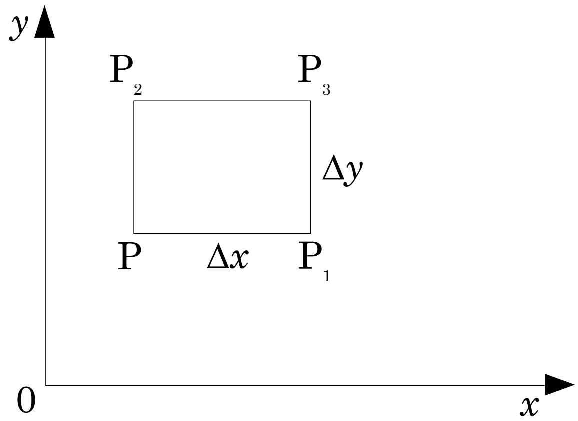

Figure 2. View of a rectangle lying inside area D.

Let's form an expression

$$A = f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x, y). \quad (*)$$

Let's introduce an auxiliary function $\varphi(x) = f(x, y+\Delta y) - f(x, y)$ variables *x* (*y* will be considered a parameter). In that case we can write that it is

$$A = \varphi(x+\Delta x) - \varphi(x)$$

that is, after applying Lagrange's mean value theorem

$$A = \varphi'(x+\theta\,\Delta x)\Delta x, \quad 0 < \theta < 1.$$

However, $\varphi'(x) = f'_x(x, y+\Delta y) - f'_x(x, y)$ and since in the formulation of the theorem it is assumed that there is a mixed partial derivative f''_{xy} by re-applying Lagrange's theorem they obtained

$$\varphi'(x) = f'_x(x, y+\Delta y) - f'_x(x, y) = f''_{xy}(x, y+\eta\Delta y)\Delta y$$

ie.

- $$A = f''_{xy}(x+\theta\,\Delta x, y+\eta\,\Delta y)\Delta x\Delta y, \quad 0 < \eta < 1, 0 < \theta < 1$$

Similarly, starting from $(*)$ we can introduce an auxiliary function by meaningful $\psi(x) = f(x+\Delta x, y) - f(x, y)$ where the parameter is, then it is

$$A = \psi(y+\Delta y) - \psi(y)$$

and by applying Lagrange's theorem we have twice that it is

$$\bullet\bullet \quad A = f''_{yx}(x + \theta_1 \Delta x, y + \eta_1 \Delta y)\Delta x \Delta y, \;\; 0 < \eta_1 < 1, 0 < \theta_1 < 1$$

By comparing equations ● and ● ● it turned out to be

$$f''_{xy}(x + \theta \Delta x, y + \eta \Delta y) = f''_{xy}(x + \theta_1 \Delta x, y + \eta_1 \Delta y)$$

whence assuming yes $\Delta x \to 0, \; \Delta y \to 0$, and given the assumption of continuity of mixed partial derivatives f''_{xy} and f''_{yx} it immediately follows that it is

$$f''_{xy}(x, y) = f''_{xy}(x, y), \;\; \forall (x, y) \in \operatorname{int} D.$$

Lagrange's theorem, one of the most important results of infinite group theory, states that the order of a subgroup must divide the order of the group[10],[11]. This theorem provides a powerful tool for finite group analysis; gives us an idea of exactly which subgroups we can expect an infinite group to possess.

3. CONCLUSION

This paper first analyzes the goals, tasks, methods, and then focuses on teaching strategies and processes, finally comes to reflection, therefore formation of a complete, detailed and complete text design on the proof of Lagrange's mean value theorem and its application.

It is the basis of the whole differential science to form a set of mean value theorems with Roll's theorem, Lagrange's mean value theorem and Cauchy's mean value theorem, especially Lagrange's mean value theorem. It established a quantitative relationship between function value and the derivative value, thus can be used by the derivative of the mean value theorem to study the function and mean value theorem is mainly used for the theoretical analysis and proof, for example as the derivative judge function monotonicity, take extreme, concave and convex, inflection point, and other important functional state to provide important theoretical basis, to grasp the function of all kinds of geometric features of images. In short, the mean value theorem in differential calculus is a bridge between the value of derivative and the value of function, and a tool for inferring the integrity of functions by using the local properties of derivatives. Lagrange's mean value theorem as a connecting theorem in differential mean value theorem, we need to be able to skillfully apply it, which is of great significance to the study of higher mathematics.

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