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**Results on a faster iterative scheme for a generalized monotone asymptotically α-non-expansive mapping**

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**Abstract**

This article devoted to present results on convergence of Fibonacci-Halpern scheme (shortly, FH) for monotone asymptotically $\alpha_n$ -nonexpansive mapping (shortly, *ma* $\alpha_n$ *-n* mapping) in partial ordered Banach space (shortly, POB space). Which are auxiliary theorem for demi-close's proof of this type of mappings, weakly convergence of increasing FH-scheme to a fixed point with aid monotony of a norm and $\sum_{n=1}^{+\infty}\lambda_n = +\infty, \lambda_n = \min\{h_n,(1-h_n)\}$ where $\{h_n\} \subset (0,1)$ is associated with FH-scheme for an integer $n > 0$ more than that, convergence amounts to be strong by using Kadec-Klee property and finally, prove that this scheme is weak- $w^2$ stable up on suitable status.

**Keywords**: Banach space, fixed point, monotone mapping, α-nonexpansive mapping, iterative scheme.

**Introduction**

Let A be a normed space and $G: D \subseteq A \to D$, a mapping G is called nonexpansive if

$$\|Gr - Ge\| \le \|r - e\| \quad \forall r, e \in D \quad (1)$$

Aoyama et al. [8] presented a class of $\lambda$ -hybrid mappings in a Hilbert space, meaning, a mapping G is called $\lambda$ -hybrid if

$$\|Gr - Ge\|^2 \le \|r - e\|^2 + 2(1-\lambda)\langle r - Gr, e - Ge\rangle \quad (2)$$

and showed a fixed point theorem. Obviously, a nonexpansive mapping is $\lambda$ -hybrid mapping (if $\lambda = 1$). Aoyama and Kohsaka[7] also presented the class of $\alpha$ -nonexpansive mappings, meaning, a mapping G is $\alpha$ -nonexpansive if for all $r, e \in D(G)$

$$\|G^n r - G^n e\|^2 \le \alpha_n \|G^n r - e\|^2 \le \alpha_n \|G^n e - r\|^2 + (1-2\alpha_n)\|r - e\|^2 \quad (3)$$

where $\alpha$<1 and gave fixed point results. A nonexpansive mapping and is $\alpha$-nonexpansive ($\alpha = 0$) and a $\lambda$ -hybrid mapping is $\frac{1-\lambda}{2-\lambda}$ -nonexpansine if $\lambda$ <2 in Hilbert space.

The concept of a monotone nonexpansive mapping is introduced by Bachar and Khamisi [10] in a POB space with the order "$\precsim$" and then common approximate fixed points are realized of monotone nonexpansive semigroups. Recalling, a mapping $G: D \subseteq A \to D$ is said to be monotone nonexpansive if G is monotone ($Gr \precsim Ge$ if $r \precsim e$) and

$$\|Gr - Ge\| \le \|r - e\| \text{ with } r \precsim e \quad (4)$$

Note that, the continuity of monotone nonexpansive mapping may be not achieved, see [33] or [4]. At the beginning of studying the existence of fixed point for the nonexpansive mapping G, Mann formed the following iterative scheme which was later known by his name, Mann' iteration:

for any $a_1 \in D, a_{n+1} = \beta_n a_n + (1-\beta_n)Ga_n \quad \forall n \geq 1$ (5)

where $\beta_n \in (0,1)$ is a sequence with certain conditions.

Later, many researchers introduced results on convergence of the Mann scheme and its modified versions for differe classes of mappings such as nonexpansine, pseudo-contractions, total asymptotically nonexpansive mappings ... etc. For example, see [1-3], [5-6],[12],[14],[18] and see [22-23],[24],[26], [31], [34], [35]. Recently, there are some convergence theorems of such a scheme in an POB (A,≾). Dehaish and Khamsi [13] obtained the weak convergence of the Mann scheme for a monotone nonexpansive mapping provided $\alpha n$ $\in [a,b] \subset (0,1)$, but their result do not entail $\beta_n = \dfrac{1}{n+1}$. Motivated by the above findings, Song et al. [28] considered the weak convergence of the Mann iteration scheme for a monotone nonexpansive mapping G, $\{r_n\}$ defined by

$r_{n+1} = \beta_n r_n + (1-\beta_n)Gr_n$ for integer $n \geq 1$,where $\{\beta_n\} \subset (0,1)$ (6)

with condition $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$, which include $\beta_n = \dfrac{1}{n+1}$ as a special case. Here, we present *ma* $\alpha_n$ *-n* mapping there is the existence theorem of fixed points for a *ma* $\alpha_n$ *-n* mapping G and showed the weak\strong convergence of the FH-scheme to a fixed point with $\sum_{n=1}^{+\infty} \lambda_n = +\infty, \lambda_n = \min\{h_n, (1-h_n)\}$ where $\{h_n\} \subset (0,1)$ for $n \geq 1$.

**Theorem (1.1):** Let D be a nonempty and closed convex subset of a uniformly convex Banach space and $G: D \to D$ be a monotone nonexpansive mapping. Assume that A satisfies Opial condition and the sequence $\{r_n\}$ is define by (6) with $r_1 \precsim Gr_1$ (or $Gr_1 \precsim r_1$). If $F(G) \neq \emptyset$ and $s \precsim r_1$ (or $r_1 \precsim s$ )for some $s \in F(G)$. Then $\{r_n\}$ weakly converges to a fixed point $r^*$ of G.

During 2010-2020, Abed and Malih[19-21] established weak and strong convergence results of random Fibonacci-Mann and random Fibonacci-Ishikawa scheme to random fixed points of monotone random asymptotically nonexpansive mappings.

In this paper, indicate to a Banach space by A with the norm $\|.\|$, its dual $A^*$ and the partial order "≾". Let $F(G) = \{r \in A, Gr = r\}$ is the set of all fixed point of mapping G. Let D be closed convex subset of A and $[r,e] = \{t \in D: r \precsim t \precsim e\}$is an order interval for all $r,e \in D$ which is closed and convex. The convexity of $[r,e]$ implies that $r \precsim tr + (1-t)e \precsim e$ for all $r,e \in D$ with $r \precsim e$. The fixed point set with depending on partial orders denoted by

$F_{\leq}^{r}(G) = \{s \in F(G): s \leq r\}$ for some $r \in D$ and $F_{\geq}^{r}(G) = \{s \in F(G): s \geq r\}$, for some $r \in D$.

Sometime, we assume a norm $\|.\|$ is monotone which define by [27], i.e. $\|r\| \leq \|e\|$ for all $r,e \in A$ and $0 \leq r \leq e$

In the following the definition of a monotone asymptotically $\alpha_n$ -nonexpansive mapping:

**Definition (1.2):** Let $G: A \to A$ be a mapping G is called *ma* $\alpha_n$ *-n* mapping if for $r,e \in A$ with $r \precsim e$,

$\|G^n r - G^n e\|^2 \leq \alpha_n \|G^n r - e\|^2 \leq \alpha_n \|G^n e - r\|^2 + (1-2\alpha_n)\|r-e\|^2$.

And then prove some convergence and stability results about FH-scheme

$r_0 \in D$ and $h_n \subset (0,1)$, $r_{n+1} = h_n r_n + (1-h_n)G^{f(n)} r_n$ (7)

where $\{f_i\}$ is sequence of Fibonacci numbers and $f(i) = f(i-1) + f(i-2), i \geq 1$.

**Definition (1.3):** [30] A Banach space $(A, \|.\|)$ is said to be uniformly convex (shortly, UCBS) if $\forall$ $\varepsilon$ >0, ∃δ>0 and for r, e ∈ A if $\|r\| \leq 1, \|e\| \leq 1$ and $\|r-e\| \geq \varepsilon$ then $\|r+e\| \leq 2(1-\delta)$

**Definition (1.4):** [21] Let A be a Banach space. Then a function $\delta_A : [0,2] \to [0,1]$ is said to be the modulus of convexity of A if

$$\delta_A(\varepsilon) = \inf\left\{1 - \left\|\frac{r+e}{2}\right\| : \|r\| \leq 1, \|e\| \leq 1, \|r-e\| \geq \varepsilon\right\}.$$

**Definition (1.5):** [17] Let A be a Banach space satisfying Kadec-Klee property if for every sequence $\{r_n\}$ in A converging weakly to (r) together with $\|r_n\|$ converging strongly to $\|r\|$ imply that $\{r_n\}$ converges strongly to a point $r \in A^{**}$.

Any uniformly convex Banach space is reflexive and has the Kadec-Klee property [9].

**Definition (1.6):** [11] A mapping $G : B \to A$ is said to be demi closed with respect to $s \in A$ if for any sequence $\{r_n\} \in B, \{r_n\}$ converges weakly to $r$ and $G(r_n)$ converges strongly to $s$ .Then $r \in B$ and $G(r) = s$.

**Lemma (1.7)**: [30] Let A be a reflexive Banach space, ∅ ≠D⊂A and A be a closed , assume that $f : D \to (-\infty, \infty)$ is coercive and proper convex lower semi-continuous function. Then there exists $r \in D$ such that $f(r) = \inf_{e \in D} f(e)$

**Proposition (1.8):** [25] Let A be a uniformly convex Banach space with the modulus of convexity $\delta_A(.)$ .Then ∀ t > 0 and $r, e \in A$ with $\|r\| \leq t, \|e\| \leq t$,

$$\|\beta r + (1-\beta)e\| \leq t\left[1 - 2\min\{\beta, 1-\beta\}\,\delta_A\left(\frac{\|r-e\|}{t}\right)\right], \forall \beta \in (0,1)$$

If, $\beta = \frac{1}{2}$ then $\left\|\frac{r+e}{2}\right\| \leq t\left[1 - \delta_A\left(\frac{\|r-e\|}{t}\right)\right]$

**Proposition (1.9):** [29] Let A be POB space and $\{r_n\}$ , $\{e_n\}$ are two sequence in A such that $r_n \precsim e_n$, for an integer $n > 0$.

If $\{r_n\}$ and $\{e_n\}$ weakly converges to r and e respectively, then $r \precsim e$.

**Fixed point result**

Starting with following proposition

**Proposition (2.1):** Let D be a nonempty closed convex subset of POB space $(A, \precsim)$ and $G : D \to D$ be *ma* $\alpha_n$ -*n* mapping, then

(1) $\|G^n r - G^n s\| \leq \|r - s\|$ $s \in F(G)$

(2) For every $r, e \in D$ with $r \precsim e$ (or, $e \precsim r$)

$$\|G^n r - G^n e\|^2 \le \|r-e\|^2 + \frac{2\alpha_n}{1-\alpha_n}\|G^n r - r\|^2 + \frac{2|\alpha_n|}{1-\alpha_n}\|G^n r - r\|(\|r-e\| + \|G^n r - G^n e\|)$$

Proof (1): Let $s \in F(G)$, by the definition of *ma* $\alpha_n$ *-n* mapping

$$\|G^n r - G^n s\|^2 \le \alpha_n \|G^n r - s\|^2 + \alpha_n \|G^n s - r\|^2 + (1-2\alpha_n)\|r-e\|^2$$

$$\le \alpha_n \|G^n r - s\|^2 + (1-\alpha_n)\|r-e\|^2$$

$$(1-\alpha_n)\|G^n r - s\|^2 \le (1-\alpha_n)\|r-e\|^2$$

$\|G^n r - G^n s\|^2 \le \|r-s\|^2$ .Then $\|G^n r - G^n s\| \le \|r-s\|$

Proof (2): If $\alpha_n$ >0

$$\|G^n r - G^n e\|^2 \le \alpha_n \|G^n r - e\|^2 + \alpha_n \|G^n e - r\|^2 + (1-2\alpha_n)\|r-e\|^2$$

$$\le \alpha_n(\|G^n r - r\| + \|r-e\|)^2 + \alpha_n(\|G^n e - G^n r\| + \|G^n r - r\|)^2 + (1-2\alpha_n)\|r-e\|^2$$

$$\le \alpha_n \|G^n r - r\|^2 + 2\alpha_n \|G^n r - r\|\|r-e\| + \alpha_n \|r-e\|^2 + \alpha_n \|G^n e - G^n r\|^2$$

$$+2\alpha_n \|G^n e - G^n r\|\|G^n r - r\| + \alpha_n \|G^n r - r\|^2 + (1-2\alpha_n)\|r-e\|^2$$

$$\|G^n r - G^n e\|^2 \le \|r-e\|^2 + \frac{2\alpha_n}{1-\alpha_n}\|G^n r - r\|^2 + \frac{2\alpha_n}{1-\alpha_n}\|G^n r - r\|(\|r-e\| + \|G^n r - G^n e\|)$$

If $\alpha_n$ < 0

$$\|G^n r - G^n e\|^2 \le \alpha_n(\|G^n r - r\| - \|r-e\|)^2 + \alpha_n(\|G^n e - G^n r\| - \|G^n r - r\|)^2 + (1-2\alpha_n)\|r-e\|^2$$

$$\le \alpha_n \|G^n r - r\|^2 - 2\alpha_n \|G^n r - r\|\|r-e\| + \alpha_n \|r-e\|^2 + \alpha_n \|G^n e - G^n r\|^2$$

$$-2\alpha_n \|G^n e - G^n r\|\|G^n r - r\| + \alpha_n \|G^n r - r\|^2 + (1-2\alpha_n)\|r-e\|^2$$

$$(1-\alpha_n)\|G^n r - G^n e\|^2 \le (1-\alpha_n)\|r-e\|^2 + 2\alpha_n \|G^n r - r\|^2 - 2\alpha_n \|G^n r - r\|\left(\|r-e\| + \|G^n r - G^n e\|\right)$$

$$\|G^n r - G^n e\|^2 \le \|r-e\|^2 + \frac{2\alpha_n}{1-\alpha_n}\|G^n r - r\|^2 + \frac{-2\alpha_n}{1-\alpha_n}\|G^n r - r\|(\|r-e\| + \|G^n r - G^n e\|).$$

Then, for all $r, e \in D$ with $r \precsim e$

$$\|G^n r - G^n e\|^2 \le \|r-e\|^2 + \frac{2\alpha_n}{1-\alpha_n}\|G^n r - r\|^2 + \frac{2|\alpha_n|}{1-\alpha_n}\|G^n r - r\|(\|r-e\| + \|G^n r - G^n e\|)$$

If $\alpha_n = 0$

$$\|G^n r - G^n e\|^2 \le \alpha_n \|G^n r - e\|^2 + \alpha_n \|G^n e - r\|^2 + (1-2\alpha_n)\|r-e\|^2$$

$\|G^n r - G^n e\|^2 \le \|r-e\|^2$ .Then $\|G^n r - G^n e\| \le \|r-e\|$

**Theorem (2.2):** Let A be UCBS and $\varnothing \ne D \subset A$, D is closed convex. Let $G: D \to D$ be a *ma* $\alpha_n$ *-n* mapping, and the norm $\|\cdot\|$ is monotone, If $\{r_n\}$ in D is weakly converges to r with $r_n \precsim G^n r_n \precsim r$ (or $r \precsim G^n r_n \precsim r_n$ ) and $\lim_{n\to\infty}\|r_n - G^n r_n\| = 0$, then $Gr = r$ .

**Proof**: Suppose that $r_n \precsim G^n r_n \precsim r$ , for an integer $n$ > 0. Let $K = \{e \in D; r_n \le e\}$ .Then $K = \bigcap_{n=1}^{\infty} K_n$ where $K_n = \{e \in D; r_n \le e\}$ since $r \in K_n$ , then $K_n$ is nonempty. Let $e_1, e_2 \in K_n$ , that mean $r_n \le e_1, r_n \le e_2$ , and $\lambda r_n \le \lambda e_1, (1-\lambda) r_n \le (1-\lambda) e_2$ , by combining two inequalities, getting $r_n \le \lambda e_1 + (1-\lambda) e_2$ ,then $\lambda e_1 + (1-\lambda) e_2 \in K_n$ ,so $K_n$ is convex .

Now, let *e* be a limit point of $K_n$, then $\exists e_m \subset K_n \ni e_m \to e$ , since $\forall m, r_n \le e_m$ and $e_m$ is increasing sequence $e_m \le e \forall m$ , then $r_n \le e$ , so $e \in K_n$ that implies $K_n$ is closed. Since $\{r_n\}$ is weakly converges then $\{r_n\}$ is bounded. From $\lim_{n\to\infty}\|r_n - G^n r_n\| = 0$ the sequences $\{r_n\}$ and $\{G^n r_n\}$ are equivalent, then $\{G^n r_n\}$ is bounded.

Now, put a function as follows $\varphi(e) = \limsup_{n\to\infty}\|r_n - e\| \forall e \in K$ . Clearly, $\varphi$ is proper, coercive, convex and continuous function. By Lemma (3.1.1) $\exists z \in K$ such that $\varphi(z) = \limsup_{n\to\infty}\|r_n - z\| = \inf_{e\in K}\varphi(e) = t$ . By definition of $K$ and Proposition (3.1.3), we get $r_n \precsim z$, $r_n \precsim r \precsim z$ , and hence $0 \le r - r_n \le z - r_n \ \forall n$, that mean $\|r - r_n\| \le \|z - r_n\|$ and so, $\varphi(r) \le \varphi(z)$ . Then, $\varphi(r) = \varphi(z) = \lim_{n\to\infty}\|r_n - r\| = \inf_{e\in D} = t.$

Since G is monotone and $r_n \precsim G^n r_n \precsim G^n r$ , hence $G^n r \in K$ . Convexity of *K* gives that $\frac{r + G^n r}{2} \in K$ , and

$$t = \varphi(r) \le \varphi(\frac{r+G^n}{2}) \text{ and } t = \varphi(r) \le \varphi(G^n r) \tag{7}$$

By Proposition (3.1.5) getting

$$\|G^n r_n - G^n r\|^2 \le \|r_n - r\|^2 + \frac{2\alpha_n}{1-\alpha_n}\|G^n r_n - r_n\|^2 + \frac{2|\alpha_n|}{1-\alpha_n}\|G^n r_n - r_n\|(\|r_n - r\| + \|G^n r_n - G^n r\|) \Rightarrow$$

$$(\limsup_{n\to\infty}\|G^n r_n - G^n r\|)^2 \le (\limsup_{n\to\infty}\|r_n - r\|)^2 \Rightarrow \qquad \limsup_{n\to\infty}\|G^n r_n - G^n r\| \le \limsup_{n\to\infty}\|r_n - r\| = \varphi(r).$$

The inequality $\|r_n - G^n r\| \le \|r_n - G^n r_n\| + \|G^n r_n - G^n r\|$ implies

$$\varphi(G^n r) = \limsup_{n\to\infty}\|r_n - G^n r\|$$

$$\le \limsup_{n\to\infty}\|G^n r_n - G^n r\| \le \limsup_{n\to\infty}\|r_n - r\| = \varphi(r) = t \tag{8}$$

So, the inequality $\left\|r_n - \frac{r+G^n r}{2}\right\| \le \frac{1}{2}\|r_n - r\| + \frac{1}{2}\|r_n - G^n r\|$ implies

$$\varphi(\left\|\frac{r+G^n r}{2}\right\|) \le \frac{1}{2}\limsup_{n\to\infty}\|r_n - r\| +$$

$$\frac{1}{2}\limsup_{n\to\infty}\|r_n - G^n r\| \le \limsup_{n\to\infty}\|r_n - r\| = \varphi(r) = t \qquad (9)$$

Then by (7),(8) and (9),gives that $\varphi(r) = \varphi(G^n r) = \varphi(\frac{r+G^n r}{2}) = t \ge 0$

To prove $r = G^n r$, assume that $t = 0$. Then $\lim_{n\to\infty}\|r_n - G^n r\| = \lim_{n\to\infty}\|r_n - r\| = 0$, that mean $r = G^n r$.

If $t = 0$, then ∀є>0,∃j>0 such that $\|r_n - G^n r\| < t+\varepsilon$ and $\|r_n - r\| < t+\varepsilon \quad \forall n > j$

Proposition (3.1.2) yields

$$\left\|r_n - \frac{r+G^n r}{2}\right\| = \left\|\frac{1}{2}(r_n - r) + \frac{1}{2}(r_n - G^n r)\right\| \le (t+\varepsilon)\left(1-\delta_A\left(\frac{\|r-G^n r\|}{t+\varepsilon}\right)\right) \qquad (10)$$

without loss of generality restrict t є >1 without loss of generality. So (10) can be rewritten as follow

$$\left\|r_n - \frac{r+G^n r}{2}\right\| \le (t+\varepsilon)\left(1-\delta_A\left(\frac{\|r-G^n r\|}{t+1}\right)\right),$$

subsequently,

$$t = \varphi(\frac{r+G^n r}{2}) = \limsup_{n\to\infty}\left\|r_n - \frac{r+G^n r}{2}\right\| \le (t+\varepsilon)\left(1-\delta_A\left(\frac{\|r-G^n r\|}{t+1}\right)\right)$$

$$\Rightarrow t\delta_A\left(\frac{\|r-G^n r\|}{t+1}\right) \le (t+\varepsilon)\delta_A\left(\frac{\|r-G^n r\|}{t+1}\right) \le t+\varepsilon-r = \varepsilon.$$

Since $\varepsilon$ is arbitrary, $\delta_A\left(\frac{\|r-G^n r\|}{t+1}\right) = 0$, which imply $r = G^n r$. Then $Gr = r, \forall n$

If $r \precsim G^n r_n \precsim r_n$ ∀n > 0, we need the set $K = \{e \in D; r_n \ge e\}$ .The rest of the proof is the same.

**Convergence results**

**Theorem (3.1):** Let A be UCBS. Let $\varnothing \ne D \subset A$, D is closed convex. Let $G: D \to D$ be *ma* $\alpha_n$*-n* mapping. Suppose that the norm $\|.\|$ is monotone and the sequence $\{r_n\}$ define by (7) with $r_1 \precsim Gr_1$ and $F^r_{\ge}(G) \ne \varnothing$.

If the iteration condition $\{h_n\} \subset (0,1)$ satisfy $\sum_{n=1}^{+\infty} \lambda_n = +\infty, \lambda_n = \min\{h_n, (1-h_n)\}$ for an integer n >0, then $\{r_n\}$ weakly converges to a some fixed point $r \in F_{\succeq}^{r_1}(G)$ and $r_n \precsim r \ \forall n.$

**Proof** : Firstly ,we prove that $r_n \precsim r_{n+1} \precsim s$ ,where $s \in F_{\succeq}^{r_1}(G)$

We used the mathematical induction in proved .Since $s \in F_{\succeq}^{r_1}(G)$ that mean $r_1 \precsim s$. Hence G is monotone then $r_n \precsim G^{f(n)} r_1 \precsim G^{f(n)} s \precsim s$ ,and by definition (7)

$r_2 = h_1 r_1 + (1-h_1) G^{f(1)} r_1,\ G^{f(1)} r_1 = G r_1$

$r_1 \precsim r_2 \precsim G r_1 \precsim s$. Assume that $r_n \precsim s$ then $G^{f(n)} r_n \precsim G^{f(n)} s = s$,and from definition (7) getting

$r_n \precsim r_{n+1} \precsim G^{f(n)} r_n \precsim s.$

Then the sequence $\{r_n\}$ is increasing and bounded, since s is upper bound.

Secondly, to prove that $\lim_{n\to\infty} \|r_n - s\|$ exists, from Proposition (2.1),and by define of G, getting

$$\begin{aligned}\|r_{n+1} - s\| &\le \|h_n(r_n - s) + (1-h_n)(G^{f(n)} r_n - s)\| \\ &\le h_n \|r_n - s\| + (1-h_n)\|G^{f(n)} r_n - s\| \\ &\le h_n \|r_n - s\| + (1-h_n)\|r_n - s\| \\ &\le \|r_n - s\| \ldots \le \|r_1 - s\|\end{aligned}$$

So $\forall s \in F_{\succeq}^{r_1}(G)$ , $\{\|r_n - s\|\}$ is bounded and non-increasing, which mean $\lim_{n\to\infty} \|r_n - s\|$ exists by [27, Theorem 2]. Hence, the sequences $\{r_n\}$ and $\{G^{f(n)} r_n\}$ are bounded w.r.t norm,

Since A is UCBS then it is reflexive, and $\{r_n\}$ is bounded, so by [30,Theorem 9], then $\{r_n\}$ is weakly sequentially compact. Implying ∃ $\{r_{n_l}\} \subset \{r_n\}$ such that $\{r_{n_l}\}$ is weakly converge to r. For any fixed n, there exists large enough $n_l$ such that $r_n \precsim r_{n_i}$ By Proposition (1.9) $r_n \precsim r$.

To show that $\{r_n\}$ converges to *r* weakly. If not, then there is a subsequence $\{r_{n_j}\}$ of $\{r_n\}$ where $\{r_{n_j}\}$ weakly converge to *w*, such that $w \ne r$. For any fixed n, ∃ $n_j$ such that $r_{n_i} \precsim r_{n_j}$. And $r_{n_i} \precsim w$ (by Proposition (1.9)). Since $\{r_{n_l}\}$ weakly converges to *r*, thus $r \precsim w$.Using the same method of proof to have $w \precsim r$ .Then $w = r$ , which is a contradiction. Then $r_n \xrightarrow{w} r$ .

Thirdly, to prove $\liminf_{n\to\infty} \|r_n - G^{f(n)} r_n\| = 0$, assume that $\lim_{n\to\infty} \|r_n - s\| = c$ ,if $c = 0$ the conclusion is trivial. If c > 0, then there exists $u, v \in$ ,and some $M$ >0 such that

$0 \le u \le \|r_n - s\| \le v, \forall n > M$ . Otherwise, by Proposition (1.8), let $t = \|r_n - s\|$ and $\beta = h_n, \forall n > M$

$$\|r_{n+1} - s\| \le \|h_n(r_n - s) + (1-h_n)(G^{f(n)} r_n - s)\|$$

$$\leq \|r_n - s\| \left( 1 - 2\min\{h_n, 1-h_n\} \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{\|r_n - s\|} \right) \right)$$

$$\leq \|r_n - s\| \left( 1 - 2\lambda_n \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) \right),$$

that mean

$$u\lambda_n \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) \leq 2\|r_n - s\| \lambda_n \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) \leq \|r_n - s\| - \|r_{n+1} - s\|$$

Then, getting

$$\sum_{n=M+1}^{i} u\lambda_n \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) \leq \sum_{n=M+1}^{i} (\|r_n - s\| - \|r_{n+1} - s\|) = \|r_{M+1} - s\| - \|r_i - s\|,$$

and therefore

$$\sum_{n=M+1}^{+\infty} u\lambda_n \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) \leq \lim_{i\to+\infty} \|r_{M+1} - s\| - \|r_i - s\| < +\infty .$$

Hence, $\liminf_{n\to\infty} \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) = 0$ hold, if not then $\liminf_{n\to\infty} \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) > 0$

$\exists q > 0$ and $p$>0, then $\delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) \geq q > 0 \forall n > p$

$\Rightarrow u\lambda_n \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) \geq uq\lambda_n$ .by condition $\sum_{n=1}^{+\infty} u\lambda_n = +\infty \Rightarrow \sum_{n=1}^{+\infty} \lambda_n \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) = +\infty$,

which contradiction. So, $\liminf_{n\to\infty} \delta_A \left( \frac{\|r_n - G^{f(n)} r_n\|}{v} \right) = 0$

The properties of modulus of convexity implies

$$\liminf_{n\to\infty} \|r_n - G^{f(n)} r_n\| = 0$$

It is easy to see $\{r_n\}$ is weakly converges, Since $\exists\{r_{nj}\} \subset \{r_n\}$, where $\lim_{n\to\infty} \|r_{nj} - G^{f(n)} r_{nj}\| = 0$ and $\{r_{nj}\}$ weakly converges to $r$ .so $\{r_n\}$ weakly converges to *r*. Then, by Theorem (2.2), $Gr = r$ i.e., $r \in F_{\geq}^{r_1}(G)$.

A same proved method using in the following

**Theorem(3.2) :** Let A be UCBS. Let $\varnothing \neq D \subset A$, D is closed convex and $G : D \to D$ be *ma* $\alpha_n$ -*n* mapping. Suppose that the norm $\|.\|$ is monotone and the sequence $\{r_n\}$ define by (7) with $Gr_1 \precsim r_1$ and $F_{\leq}^{r}(G) \neq \varnothing$.

If the iteration condition $\{h_n\} \subset (0,1)$ satisfy $\sum_{n=1}^{+\infty} \lambda_n = +\infty, \lambda_n = \min\{h_n, (1-h_n)\}$ for every positive integer n, then $\{r_n\}$ weakly converges to a some fixed point $r \in F_{\leq}^{r_1}(G)$ and $r \precsim r_n$.

Recall normal cone to state the next corollary, a cone P is called normal [27], if ∃K > 0, such that $0 \leq r \leq e \Leftrightarrow \|r\| \leq K\|e\|$ for all $r, e \in A$

**Corollary (3.3):** Let A be UCBS $(A, \precsim)$ w.r.t. the normal cone P, and D be a nonempty closed convex subset of A. Let $G: D \to D$ be *a ma* $\alpha_n$ *-n* mapping. Suppose that the sequence $\{r_n\}$ define by (4) with $Gr_1 \precsim r_1$ and $F_{\leq}^{r}(G) \neq \varnothing$ or $r_1 \precsim Gr_1$ and $F_{\geq}^{r}(G) \neq \varnothing$. If the iteration condition $\{h_n\} \subset (0,1)$ satisfy $\sum_{n=1}^{+\infty} \lambda_n = +\infty, \lambda_n = \min\{h_n, (1-h_n)\}$ for an integer n>0, then $\{r_n\}$ weakly converges to a some fixed point $r \in F(G)$ .

**Theorem (3.4):** Let A be UCBS, $\varnothing \neq D \subset A$, D is closed convex and $G: D \to D$ be *ma* $\alpha_n$ *-n* mapping .Suppose that the norm $\|.\|$ is monotone and the sequence $\{r_n\}$ define by (7) with $0 \leq Gr_1 \precsim r_1$ and $F_{\geq}^{r}(G) \neq \varnothing$. If the iteration condition $\{h_n\} \subset (0,1)$ satisfy $\sum_{n=1}^{+\infty} \lambda_n = +\infty, \lambda_n = \min\{h_n, (1-h_n)\}$ for an integer n >0, then $\{r_n\}$ strongly converges to a some fixed point $r \in F_{\geq}^{r_1}(G)$ and $r \precsim r_n$.

**Proof:** Depending on the Theorem (3.1) that $\{r_n\}$ weakly converges to $r$, since $r \in F_{\geq}^{r_1}(G)$ then $r_1 \precsim r$ and $r_1 \precsim G^{f(n)}r_1 \precsim G^{f(n)}r = r$, from definition (4)

$r_2 = h_1 r_1 + (1-h_1)G^{f(n)}r_1 = Gr_1$ so $r_1 \precsim r_2 \precsim Gr_1$.

Let $r_n \precsim r$ , then $G^{f(n)}r_1 \precsim G^{f(n)}r = r$

and by definition (4) we have $r_n \precsim r_{n+1} \precsim G^{f(n)}r_n \precsim r$.Then

$0 \precsim r_1 \precsim r_n \precsim r_{n+1} \precsim r$, for an integer n > 0.

Since the $\|.\|$ is monotone, then $0 \leq \|r_1\| \leq \|r_n\| \leq \|r_{n+1}\| \leq \|r\|, \forall n$

Note that, the sequence $\{\|r_n\|\}$ of real number is bounded and monotone increasing. Then $\lim_{n\to\infty}\|r_n\|$ exists and $\lim_{n\to\infty}\|r_n\| \leq \|r\|$

Hence, $\|r\| \leq \liminf_{n\to\infty}\|r_n\| = \lim_{n\to\infty}\|r_n\| \leq \|r\|$, which imply $\lim_{n\to\infty}\|r_n\| = \|r\|$ , by the weakness of lower semi-continuity of the norm. Since A is UCBS, then it has Kadec-Klee property ,i.e.,

$r_n \xrightarrow{w} r$ and $\|r_n\| \to \|r\|$ implies $\lim_{n\to\infty} r_n = r$ .

**Theorem (3.5):** Let A be UCBS, $\varnothing \neq D \subset A$, D is closed convex and $G: D \to D$ be *ma* $\alpha_n$ *-n* mapping. Suppose that the norm $\|.\|$ is monotone and the sequence $\{r_n\}$ define by (7) with $Gr_1 \precsim r_1$ and $F_{\leq}^{r}(G) \neq \varnothing$. If the iteration condition $\{h_n\} \subset (0,1)$ satisfy $\sum_{n=1}^{+\infty} \lambda_n = +\infty, \lambda_n = \min\{h_n, (1-h_n)\}$ for every positive integer n, then $\{r_n\}$ strongly converges to a some fixed point $r \in F_{\leq}^{r_1}(G)$ and $r \precsim r_n \forall n$.

**Proof:** Depending on the Theorem( 3.1) that $\{r_n\}$ weakly converges to $r$ ,since $r \in F_{\geq}^{r_1}(G)$

then $r \precsim r_1$ and $r = G^{f(n)}r \precsim G^{f(n)}r_1 \precsim r_1$, from definition (4)

$r_2 = h_1 r_1 + (1-h_1)G^{f(n)}r_1 = Gr_1$. So, $r \precsim Gr_1 \precsim r_2 \precsim r_1$. Let, $r \precsim r_n$ then $G^{f(n)}r = r \precsim G^{f(n)}r_n$, and by definition (7) we have $r \precsim G^{f(n)}r_n \precsim r_{n+1} \precsim r_n$. Then

0$\precsim r \precsim r_{n+1} \precsim r_n \precsim r_1$for an integer n > 0.

Then, $0 \precsim -r_1 \precsim -r_n \precsim -r_{n+1} \precsim -r, \forall n$.

Since the norm $\|.\|$ is monotone, then $0 \leq \|r_1\| \leq \|r_n\| \leq \|r_{n+1}\| \leq \|r\|, \forall n$

The rest of the proof is the same as Theorem (3.4).

**Corollary (3.6):** Let A be UCBS w.r.t. the normal cone P and $\varnothing \neq D \subset A$, D is closed convex. Let $G: P \to P$ be a *ma* $\alpha_n$ *-n* mapping. Suppose that $\{r_n\}$ as in (7) with $r_1 = 0$ and $F(G) \neq \varnothing$. If the iteration condition $\{h_n\} \subset (0,1)$ satisfy $\sum_{n=1}^{+\infty} \lambda_n = +\infty, \lambda_n = \min\{h_n, (1-h_n)\}$ for every positive integer n, then $\{r_n\}$ strongly converges to $r \in F(G)$.

**Proof:** It's clear that $F(G) = F_{\geq}^{0}(G) = F_{\geq}^{1}(G)$. Since $r_1 = 0$ and $G(P) \subset P$, then$r_1 = 0 \precsim G0 = Gr_1$.

Consequently, the conclusion comes directly from Theorem (3.4).

**Stability of FH-iterative Scheme**

Recall the following definitions:

**Definition (4.1):** [25] A sequence $\{e_n\}$ is an approximate of the sequence $\{r_n\}$ ⇔there exists a decreasing sequence of positive number $\{\eta_n\}$ converging to $\eta \geq 0$ such that $\|r_n - e_n\| \leq \eta, \forall n \geq k$ for any $k \in$

**Definition (4.2):** [25] Let $(A, \| \|)$ be a normed space, $G: A \to A$ be a mapping and $\{r_n\}$ defined by $r_0 \in A$ and $r_{n+1} = f(G, r_n), n \geq 0$. Suppose that $\{r_n\}$ converges to fixed point s of G. If for any approximate sequence $\{e_n\} \subset A$ of $\{r_n\}$, $\lim_{n\to\infty} \|e_{n+1} - f(G, e_n)\| = 0$ implies $\lim_{n\to\infty} e_n = s$, then $\{r_n\}$ is said to be weakly stable w.r.t. G.

**Definition (4.3):** [16] The sequences $\{r_n\}$ and $\{e_n\}$ are called equivalent if $\lim_{n\to\infty} \|r_n - e_n\| = 0$

**Definition (4.4):** [32] Let $\{r_n\}$ be iterative scheme converges strongly to $s \in F(G)$. If for any equivalent sequence $\{e_n\} \subset A$ of $\{r_n\}$, $\lim_{n\to\infty} \|e_{n+1} - f(G, e_n)\| = 0$ implies $\lim_{n\to\infty} e_n = s$, then the iteration sequence $\{r_n\}$ is said to be weak-w[2] stable w.r.t G.

**Example (4.5):** Let $G:[0,1]\to[0,1]$, define by $Gr = \begin{cases} 0, r \in [0,\frac{1}{2}] \\ \frac{1}{2} \quad r \in (\frac{1}{2}, 1] \end{cases}$

where [0,1] is endowed with the usual metric. G is continuous at every point of [0,1] except at $\frac{1}{2}$ and 0 is the only fixed point of G. We will show that the Mann iteration is weak –stable. Let $r_0 \in [0,1]$ and

$r_{n+1} = h_n Gr_n + (1-h_n)r_n$ , $h_n \in (0,1)$ ,with $h_n = \dfrac{1}{n+2}$ $\forall n = 0,1,2...$

$r_0 = 0, Gr_0 = 0, h_0 = \dfrac{1}{2} \Rightarrow r_1 = (1-\dfrac{1}{2})\cdot 0 + \dfrac{1}{2}\cdot 0 = 0$

then $r_n = 0$.

Suppose that $\{e_n\}$ approximate sequence of $\{r_n\}$ .Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ for $n \to \infty$

such that $|r_n - e_n| \leq \eta_n$, $n \geq k$ .Then $-\eta_n \leq r_n + e_n \leq \eta_n$ ,which mean $-\eta_n + \eta_n \leq r_n + e_n + \eta_n$

$0 \leq r_n + e_n + \eta_n$, $0 \leq e_n \leq r_n + \eta_n \forall n \geq k$ .Since $r_n = 0 \Rightarrow 0 \leq e_n \leq \eta_n \forall n \geq k_1 = \max\{2,k\}$ .Choose $\{\eta_n\}$ such that $\eta_n \leq \dfrac{1}{2}, n \geq k_1 \Rightarrow 0 \leq e_n \leq \dfrac{1}{2}$ .So $Ge_n = 0$ . Then $\varepsilon_n = |e_{n+1} - f(G,e_n)|$

$\left|\dfrac{n+3}{2n+n} - (1-h_n)e_n + Ge_n h_n\right| = \left|\dfrac{2n-2}{4n^2+4n}\right|$ and $\lim_{n\to\infty}\left|\dfrac{2n-2}{4n^2+4n}\right| = 0$

Now, $\lim_{n\to\infty} \varepsilon_n = 0$ which implies $\lim_{n\to\infty} e_n = 0$ , so the Mann iteration is weakly stable w.r.t G.

**Theorem (4.6):** Let A be UCBS and $\varnothing \neq D \subset A$, D is closed convex and $G: D \to D$ be *ma* $\alpha_n$ *-n* with fixed point s. Suppose that $\{r_n\}$ define by (4) with $r_0 \precsim Gr_0, h_n \in (0,1)$ and $s \precsim r_0$. If $\{e_n\}$ be any equivalent sequence of $\{r_n\}$ with $r_n \precsim e_n$ (or $e_n \precsim r_n$ ),then $\{r_n\}$ is weak- $W^2$ stable w.r.t G.

**Proof:** Consider $\{e_n\}$ to be an equivalent sequence of $\{r_n\}$ . Let $r_n \precsim e_n$ by monotonicity of G $G^{f(n)}r_n \precsim G^{f(n)}e_n$.

Set $\varepsilon_n = \|e_{n+1} - f(G,e_n)\|$. Let $\varepsilon_n \to 0$ as $n \to \infty$ .Then

$$\|e_{n+1} - s\| \leq \|e_{n+1} - f(G,e_n)\| + \|f(G,e_n) - r_{n+1}\| + \|r_{n+1} - s\|$$

$$\leq \varepsilon_n + \left\|(h_n e_n + (1-h_n)G^{f(n)}e_n) - (h_n r_n + (1-h_n)G^{f(n)}r_n)\right\| + \|r_{n+1} - s\|$$

$$\leq \varepsilon_n + h_n\|e_n - r_n\| + \left\|G^{f(n)}e_n - G^{f(n)}r_n\right\| + \|r_{n+1} - s\|$$

$$\leq \varepsilon_n + h_n\|e_n - r_n\| + (1-h_n)\left[\left\|G^{f(n)}e_n - G^{f(n)}s\right\| + \left\|G^{f(n)}s - G^{f(n)}r_n\right\|\right] + \|r_{n+1} - s\|$$

$$\leq \varepsilon_n + h_n\|e_n - r_n\| + (1-h_n)\left[\alpha_{f(n)}\left\|G^{f(n)}e_n - s\right\| + \alpha_{f(n)}\|e_n - s\| + (1-2\alpha_{f(n)})\|e_n - s\|\right]$$

$$+\left[\alpha_{f(n)}\left\|G^{f(n)}r_n - s\right\| + \alpha_{f(n)}\|r_n - s\| + (1-2\alpha_{f(n)})\|r_n - s\|\right] + \|r_{n+1} - s\|$$

Let $\lim n \to \infty$ on both side. Then $\lim_{n\to\infty}\|e_{n+1} - s\| = 0$ . So $\{r_n\}$ is weak - $w^2$ stable w .r. t G.

In the following example, we present a compression between the behaviors of FH-scheme and two different iterative schemes.

**Example (2.10):** Let $G: R \to R, G(s) = \frac{s+3}{2}$

be a function with fixed point s=3.Consider the following three

$x_1 \in [0, \infty), x_{n+1} = h_n x_n + (1 - h_n)G^{f(n)}G(x_n)$

$y_1 \in [0, \infty), y_{n+1} = h_n y_n + (1 - h_n)G^n(y_n)$ (see, [22])

$z_1 \in [0, \infty), z_{n+1} = h_n z_n + (1 - h_n)G(z_n)$ (see, [22])

Fix $x_1 = y_1 = z_1 = 20$ and $h_n = \frac{1}{\sqrt{n+1}}$. By using Math lap, we show in tables (1-2)

and figures (1-2) that $\{x_n\}$ is faster than $\{y_n\}$ and $\{z_n\}$ where

In case 1 $x_1 = y_1 = z_1 = 0.1$.

case 2 $x_1 = y_1 = z_1 = -1.5$.

case 3 $x_1 = y_1 = z_1 = 20$.

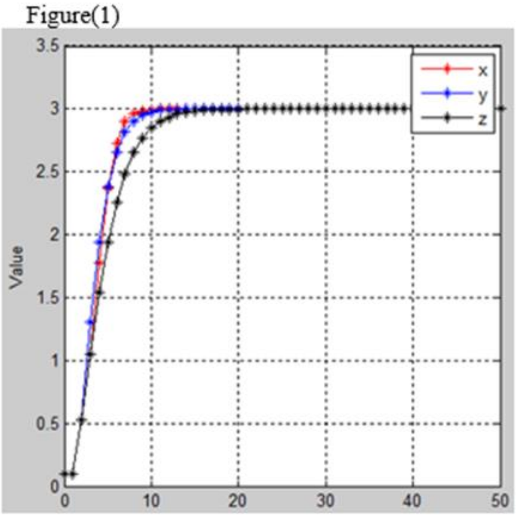
Table (1)

| n | $x_n$ | $y_n$ | $z_n$ |
|---|---|---|---|
| 0 | 0.10000000 | 0.10000000 | 0.10000000 |
| 1 | 0.10000000 | 0.10000000 | 0.10000000 |
| 2 | 0.52469517 | 0.52469517 | 0.52469517 |
| 3 | 1.04778863 | 1.30933536 | 1.04778863 |
| 4 | 1.77986789 | 1.94333460 | 1.53584147 |
| 5 | 2.37003128 | 2.38141730 | 1.94052494 |
| 6 | 2.73116681 | 2.65595283 | 2.25399803 |
| ... | ... | ... | ... |
| 19 | 2.99999998 | 2.99998406 | 2.99759740 |
| 20 | 3.00000000 | 2.99999334 | 2.99853008 |
| ... | ... | ... | ... |
| 27 | 3.00000000 | 2.99999999 | 2.99995773 |
| 28 | 3.00000000 | 3.00000000 | 2.99997487 |
| ... | ... | ... | ... |
| 42 | 3.00000000 | 3.00000000 | 2.99999999 |
| 43 | 3.00000000 | 3.00000000 | 2.99999999 |
| 44 | 3.00000000 | 3.00000000 | 3.00000000 |
| ... | ... | ... | ... |
| 48 | 3.00000000 | 3.00000000 | 3.00000000 |
| 49 | 3.00000000 | 3.00000000 | 3.00000000 |

Table (2)

| n | $x_n$ | $y_n$ | $z_n$ |
|---|---|---|---|
| 0 | -1.50000000 | -1.50000000 | -1.50000000 |
| 1 | -1.50000000 | -1.50000000 | -1.50000000 |
| 2 | -0.84099026 | -0.84099026 | -0.84099026 |
| 3 | -0.02929351 | 0.37655487 | -0.02929351 |
| 4 | 1.10669156 | 1.36034679 | 0.72802987 |
| 5 | 2.02246233 | 2.04013029 | 1.35598697 |
| 6 | 2.58284505 | 2.46613370 | 1.84241073 |
| ... | ... | ... | ... |
| 20 | 2.99999999 | 2.99998967 | 2.99771909 |
| 21 | 3.00000000 | 2.99999573 | 2.99861068 |
| ... | ... | ... | ... |
| 28 | 3.00000000 | 2.99999999 | 2.99996100 |
| 29 | 3.00000000 | 3.00000000 | 2.99997688 |
| ... | ... | ... | ... |
| 43 | 3.00000000 | 3.00000000 | 2.99999999 |
| 44 | 3.00000000 | 3.00000000 | 2.99999999 |
| 45 | 3.00000000 | 3.00000000 | 3.00000000 |
| ... | ... | ... | ... |
| 48 | 3.00000000 | 3.00000000 | 3.00000000 |
| 49 | 3.00000000 | 3.00000000 | 3.00000000 |
| 50 | 3.00000000 | 3.00000000 | 3.00000000 |

Figure(1)




Figure(2)

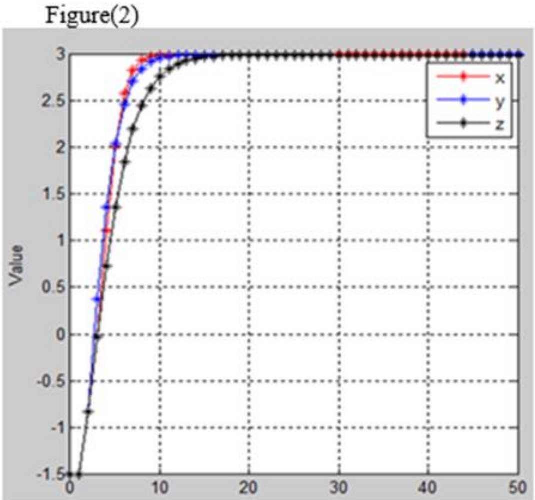
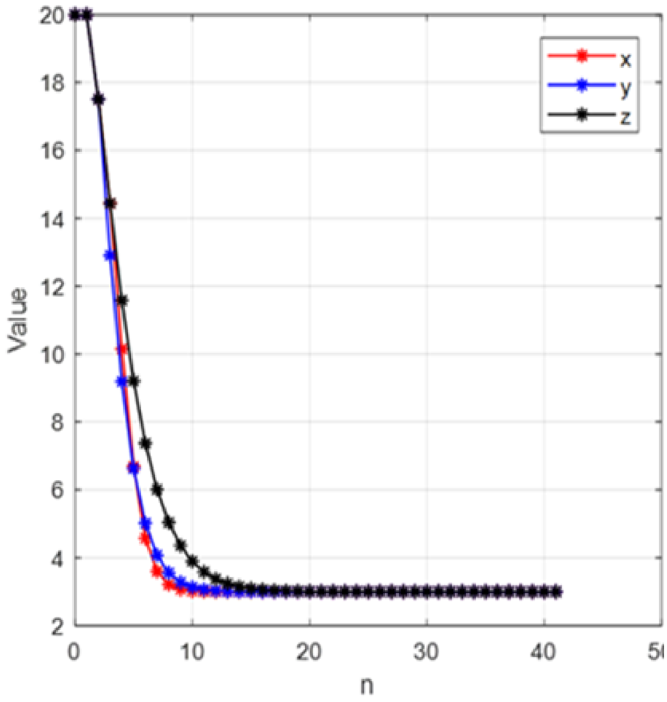


Table (3)

| n | $x_n$ | $y_n$ | $z_n$ |
|---|---|---|---|
| 0 | 20.00000000 | 20.00000000 | 20.00000000 |
| 1 | 20.00000000 | 20.00000000 | 20.00000000 |
| 2 | 17.51040764 | 17.51040764 | 17.51040764 |
| 3 | 14.44399770 | 12.91079273 | 14.44399770 |
| .. | ... | .. | .. |
| 20 | 3.00000003 | 3.00003902 | 3.00861677 |
| 21 | 3.00000001 | 3.00001614 | 3.00524855 |
| 22 | 3.00000000 | 3.00000043 | 3.00069495 |
| .. | ... | ... | ... |
| 30 | 3.00000000 | 3.00000000 | 3.00005164 |
| ... | ..... | ... | ... |
| 42 | 3.00000000 | 3.00000000 | 3.00000008 |
| 43 | 3.00000000 | 3.00000000 | 3.00000005 |
| 44 | 3.00000000 | 3.00000000 | 3.00000003 |
| 45 | 3.00000000 | 3.00000000 | 3.00000001 |
| 46 | 3.00000000 | 3.00000000 | 3.00000000 |
| 47 | 3.00000000 | 3.00000000 | 3.00000000 |

Figure(3)




**References**

1. Abdul Jabber, M. F., Abed, S. S. (2020). The convergence of iteration scheme to fixed points in modular spaces, Iraqi Journal of Science, vol.60, no,10, pp2197-2202.

2. Abed, S. S., Abdul Jabber, M. F. (2020). Approximating fixed points in moduler spaces, Karbala International Journal of Modern Science, vol.6,no.2, pp121-128.

3. Abed, S. S., Abdul Jabbar, M. F.( 2021). Some Results on Normalized Duality Mappings and Approximating Fixed points in Convex Real Moduler Spaces, Baghdad journal of science 18(4):1218-1225.DOI:http://dx.doi.org/10.21123/bsj.2021.18.4.1218.

4. Abed, S. S., Abed, A. N. Convergence and stability of iterative scheme for a monotone a total asymptotically nonexpansive mapping, Accepted Iraqi Journal of science.

5. Abed, S. S., Mohamed Hasan, Z. M. (2019). Common fixed point of a finite-step iteration algorithm under total asymptotically quasi-nonexpansive maps, Baghdad Science Journal ,16(3),654-660.

6. Alber, YA. I., Chidume, C.E., Zegeye, H. (2006). Approximating fixed points of total asymptotically nonexpansive mappings", Fixed point Theory and Applications , article Id 10673,1-12.

7. Aoyama, K., Kohsaka, F. (2011). Fixed point theorem for $\alpha$ -non-expansive mapping in Banach spaces .Nonlinear Anal.74,437-4391.

8. Aoyama, K., Iemoto, S., Kohsaka, F., Takahashi, W. (2010). Fixed point and ergodic theorems for $\lambda$ -hybrid mapping in Hilbert spaces . J. Nonlinear convex Anal.11, 335-343.

9. Aunpam, S., Mohammed, I. (2014). Approximating Fixed Point of Generalized Nonexpansinve Mappings Via Faster Iteration Schemes"Fixed point theory, No.4.605-623.

10. Bachar, M., Khamsi, M. A. (2015). On common approximate fixed points of monotone nonexpansive semigroups in Banach spaces. Fixed point Theory Appl, 2015,160.

11. Browder, F. E. (1966). Semicontractive and Semiaccretive Nonlinear Mappings in Banach Spaces, Bull. Amer . Math. Soc 74,660-665.

12. Chidume, C. E., Ali, B. (2007). Weak and strong convergence theorems for finite families of asymptotically nonexpansive mapping in Banach spaces, J. Math, Anal. Appl . 330,377-387.

13. Dehaish, B. A., Khamsi, M. A. (2015). Mann iteration process for momotone nonexpansive mappings. Fixed Point Theory. Appl.2015,177.

14. Feng, G. U. (2006). Convergence of the implicit iterative process with errors for a finite family of asymptotically nonexpansive mappings, Acta Math. Sci. 26,1131-1143.

15. Goebel, K., Krik, W. A. (1972). Fixed point theorem for Asymptotically Nonexpansive Mapping , Proc. Amer. Math. Soc.35-171-174.

16. Harder, A. M., Hicks, T. L. (1988). Stability result for fixed point iteration procedures, Math . Japonica,33,693-706.

17. Khan, S. H., Kim, H. K. (2010). Common Fixed Point of two Nonexpansive Mappings Modified Faster Iteration Scheme, Bull. Korean. Math. Soc. 47,No.5,PP .973-985,DOI:10.4134/BKMS.

18. Lim, T .C., Xu, H. X. (1994). Fixed point theorems for asymptotically nonexpansive mappings. Nonl. Anal 22,1345-1355.

19. Malih, S. H., Abed, S. S. (2019). Approximating random fixed points under a new iterative sequence. J. of Inter. Math. Vol. 22, No. 8, pp. 1407–1414,DOI 10.1080/09720502.2019.1700927.

20. Malih, S. H., Abed, S. S. (2021). Convergence and stability of some random iterative scheme, accepted in ICPAS conference and accepted publication in IOP Journal of Physics.

21. Malih, S. H., Abed, S. S.(2021). Convergence of random iterative scheme to a common random fixed points, accepted in ICPAS conference and accepted publication in IOP Journal of Physics.

22. Mohamed Hasan, Z. M., Abed, S. S. (2019). Weak convergence of two iteration scheme in Banach spaces, Engineeeing and Technology Journal 37,B02,1-12.http://dx.doi.oeg/10.30684/eti.37.2B.1

23. Na, J., Tang, Y. (2014). Weak and strong convergence theorems of fixed points for total asymptotically nonexpansive multi-vaued mappings in Banach spaces.J. Appl . Math ,sci. 8,1903-1913.

24. Reich, S. (1980). Strong convergence theorems for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl.75,287-292.

25. Sahu, D. R., ORegan, D., Agarwal, R. P. (2009). Fixed Point Theory for Lipschitzain-type Mappings with Applications, Topological Fixed Point Theory and Its Applications ,Springer Science + Business. Media, LLC.

26. Saluja, G. S. (2014). Convergence to common fixed points for generalized asymptotically quasi-nonexpansive mappings. Bull. of the Inter. Math. Virtual Institute, 4,69-79doi:1.7251/BIMVI140169S.

27. Samanta, K. T., Sanjay, R., Bivas, D. (2010). Cone Normed Linear Spaces, West Bengal, India.

28. Song, Y. S., Promluang, K., Kumam, P., Cho, Y. J. (2016). Some convergence theorem of the Mann iteration for monotone α-non-expansive mapping ,Appl. Math. Comput., 287/288,74-82.

29. Sun, J. (2008). Nonlinear Functional Analysis and Its Application, Science Publishing House, Beijing.

30. Takahashi, W. (2009). Nonlinear Functional Analysis-Fixed Point Theory and its Applications , Yokohama Publishers Inc., Yokohama.

31. Tan, K. K., Xu, H. K., (1994). Fixed point iteration processes for asymptotically nonexpansive mappings, Proc. Amer. Math .Soc. 122,733-739.

32. Timis, I. (2010). On the weak stability of Picard iteration for some contractive type mappings, Annal. Uni. Craiva, Math. Comput. Sci .Series,37,106-114.

33. Uddin, I., Garodia, C., Nieto, J. J. (2018). Mann iteration for monotone nonexpansive mappings in ordered CAT(0) space with an application to integral equations ,Uddin et al. Journal of Ineqalities and Applications ,2018:339.

34. Weiping, G., Cho, Y. J. (2008). On the strong convergence of the implicit iterative processes with errors for a finite family of asymptotically nonexpansive mappings, Appl. Math. Lett. 21,1046-1052.

35. Zhou, Y. Y., Chang, S. S. (2002). convergence of the implicit iterative process with errors for a finite family of asymptotically nonexpansive mappings in Banach spaces, Numer. Anal. Appl. 23,911-921.