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Metallic Ratios, Pythagorean Triples & p≡1(mod 4) Primes : Metallic Means, Right Triangles and the Pythagoras Theorem

Dr. Chetansing Rajput

M.B.B.S. Nair Hospital (Mumbai University) India, Asst. Commissioner (Govt. of Maharashtra)

Email: chetansingkrajput@gmail.com Website: https://goldenratiorajput.com/

Lecture Link 1 : https://youtu.be/vBfVDaFnA2k

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Abstract

This paper synergizes the newly discovered geometry of all Metallic Means and the recently published mathematical formulae those provide the precise correlations between different Metallic Ratios. The paper illustrates the concept of the "Triads of Metallic Means", and aslo the close correspondence between Metallic Ratios and the Pythagorean Triples as well as Pythagorean Primes.

Keywords: Metallic Mean, Pythagoras Theorem, Fibonacci, Pell, Lucas, Pi, Phi, Silver Ratio, Right Triangle, Metallic Numbers, Metallic Ratio Triads, 3 6 9, Pythagorean Triples, Bronze Ratio, Golden Ratio, Pascal's Triangle, Metallic Ratio

Introduction

This paper brings together following, recently discovered, new aspects of Mertallic Ratios.

1) The Generalised Geometric Construction of all Metallic Ratios: cited by the Wikipedia in its page on "Metallic Mean" [6]. This generalised geometric substantiation of all Metallic Means was published in January 2021 [7]

2) The Mathematical Formula that provides the precise correlation between different Metallic Means. This explicit formula has been recently published in the month of May 2021 [1].

These couple of important aspects of Metallic Means: viz. the generalised geometric constructions of all Metallic Means and the concerned mathematical formulae, were brought together in the work mentioned in Reference [2].

The prime objective of current paper is to further synergize the these two newly discovered aspects of Metallic Means.

The *synergism* between above two features of Metallic Means unveils an intriguing pattern of Metallic Ratios, which asserts that the mathematical implications of these Means have not been fully appreciated so far. The abovementioned Geometry and Mathematics synergically enable us to recognize the full worth of these Metallic Means, as described in this paper.

As a brief introduction, each Metallic Mean $\boldsymbol{\delta_n}$ is the root of the simple Quadratic Equation $\mathbf{X^2 - nX - 1 = 0}$, where **n** is any positive natural number.

Thus, the fractional expression of the nth Metallic Ratio is $\boldsymbol{\delta_n} = \frac{\mathbf{n + \sqrt{n^2 + 4}}}{\mathbf{2}}$

Moreover, each Metallic Ratio can be expressed as the continued fraction:

$\boldsymbol{\delta_n} = \mathbf{n} + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \dots}}}$; And hence, $\boldsymbol{\delta_n} = \mathbf{n} + \frac{1}{\delta n}$ .....References: [3], [4], [5]

GEOMETRIC CONSTRUCTION OF ALL METALLIC MEANS :

Each Metallic Ratio can be constructed geometrically with a special Right Angled Triangle. Any nth Metallic Mean can be represented by the Right Triangle having its catheti **1** and $\frac{\mathbf{n}}{\mathbf{2}}$. Hence, the right triangle with one of its catheti = **1** may substantiate any Metallic Mean, having its second cathetus = $\frac{\mathbf{n}}{\mathbf{2}}$, where n = 1 for Golden Ratio, n = 2 for Silver Ratio, n = 3 for Bronze Ratio, and so on. Such Right Triangle provides the precise value of nth Metallic Mean by the generalised formula:

$$\textbf{The n}^{\textbf{th}}\textbf{ Metallic Mean } (\boldsymbol{\delta_n}) = \textbf{Hypotenuse + Cathetus } \frac{\mathbf{n}}{\mathbf{2}}$$

Such Right Triangle not only provides for the accurate geometric construction and precise fractional expression of any nth Metallic Mean ($\boldsymbol{\delta_n}$), but its every geometric feature is the prototypical form of that Metallic Mean [6], [7], [8], [9].

The characteristic geometry of such Right Triangle having its catheti **1** and $\frac{\mathbf{n}}{\mathbf{2}}$, is resplendent with the corresponding nth Metallic Mean ($\boldsymbol{\delta_n}$) embedded in its every geometric aspect.

For example, the remarkable expression of Golden Ratio in every geometric feature of 1:2:$\sqrt{5}$ triangle, including all its angles and side lengths, its 'Incenter-Excenters Orthocentric system', its Gergonne and Nagel triangles, and also the Nobbs points and the Gergonne line, various triangle centers as well as the Incircle of 1:2:$\sqrt{5}$ triangle, make this triangle the quintessential form of the Golden Ratio (φ) and also of the fourth Metallic Mean (φ^3). [7]

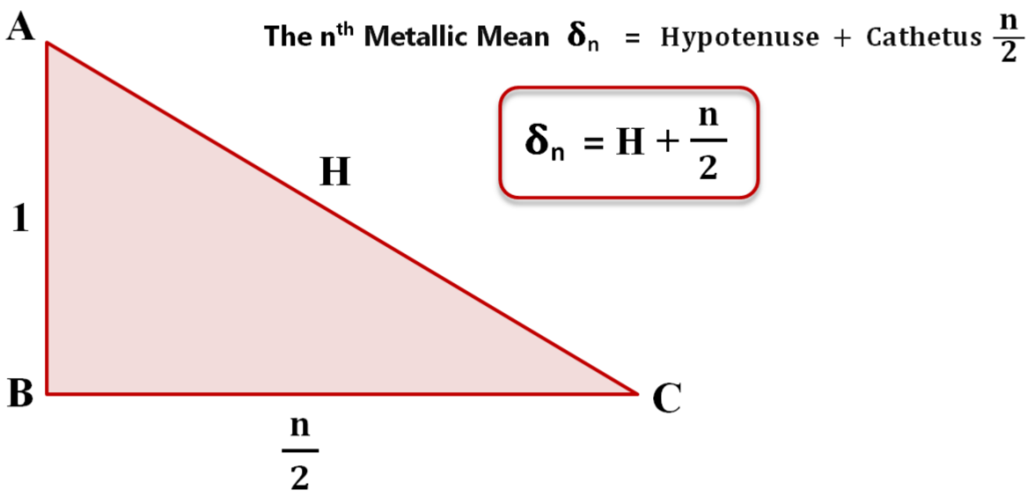

Figure 1: Generalised Right Triangle to represent any n^{th} Metallic Mean δ_n

MATHEMATICAL CORRELATIONS AMONG DIFFERENT METALLIC RATIOS :

If **K**, **m** and **n** are three positive integers such that **n** is the smallest of the three integers and $\frac{mn+4}{m-n} = k$

then, it is observed that

$\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$ where δ_k, δ_m and δ_n are the k^{th}, m^{th} and n^{th} Metallic Means respectively.

This explicit formula, among several other formulae those give the precise mathematical relations between different Metallic Means, has been recently published in the work mentioned in References [1] and [2].

The "TRIADS" Of Metallic Means :

The abovementioned explicit formula gives the "Triads" of Metallic Means as $[\delta_n, \delta_m, \delta_k]$

Where $\frac{mn+4}{m-n} = k$ and $\frac{kn+4}{k-n} = m$

hence, $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$ and also $\frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m$

Moreover,

$$\frac{km - 4}{k + m} = n \quad \text{and} \quad \frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n$$

For example, if n=6, the three integers 6, 11 and 14 satisfy the prerequisite $\frac{\mathbf{mn+4}}{\mathbf{m-n}} = \mathbf{k}$;

Hence, the three Metallic means δ_6, δ_{11} and δ_{14} form a **Triad** [δ_6, δ_{11}, δ_{14}] such that :

$$\frac{\delta_{11} \times \delta_6 + 1}{\delta_{11} - \delta_6} = \delta_{14} \quad \text{and also} \quad \frac{\delta_{14} \times \delta_6 + 1}{\delta_{14} - \delta_6} = \delta_{11} \qquad \text{Also,} \quad \frac{\delta_{14} \times \delta_{11} - 1}{\delta_{14} + \delta_{11}} = \delta_6$$

Noticeably, **n=6** forms such multiple triads:

n	6	6	6	6	6	6	6	6
m	7	8	10	11	14	16	26	46
k	46	26	16	14	11	10	8	7

: Shaded Triads have been exemplified above.

And, just like **n=6** exemplified above, every integer forms such multiple triads:

For example**, n=10**

n	10	10	10	10	10	10	10	10
m	11	12	14	18	23	36	62	114
k	114	62	36	23	18	14	12	11

Noticeably, Odd n forms Fewer Triads

n	5	5	5	5	5	5	5	5
m	6	34						
k	34	6						

It may be noticed from above Tables that every n^{th} Metallic Mean can give precise values of various Metallic Means by the formula $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$, maximum upto $(n^2 + n + 4)^{th}$ Metallic Mean:

$\mathbf{m_{max}} = \mathbf{k_{max}} = (n^2 + n + 4)$

Also noticeably, the Even Integers (Even n_s) form comparatively more Triads than the Odd n_s . Several such patterns about these Triads of Metallic Means have been discussed in detail in Reference [2]. Here, let us consider the classical correspondence of the abovementioned Formula and TRIADS with the Geometry of Metallic Ratios.

Remarkably, the abovementioned Triads of Metallic Means can be represented geometrically, as shown below.

For instance, the **Triad** [$\boldsymbol{\delta_n, \delta_m, \delta_k}$] is illustrated geometrically in following **Figure 2.**

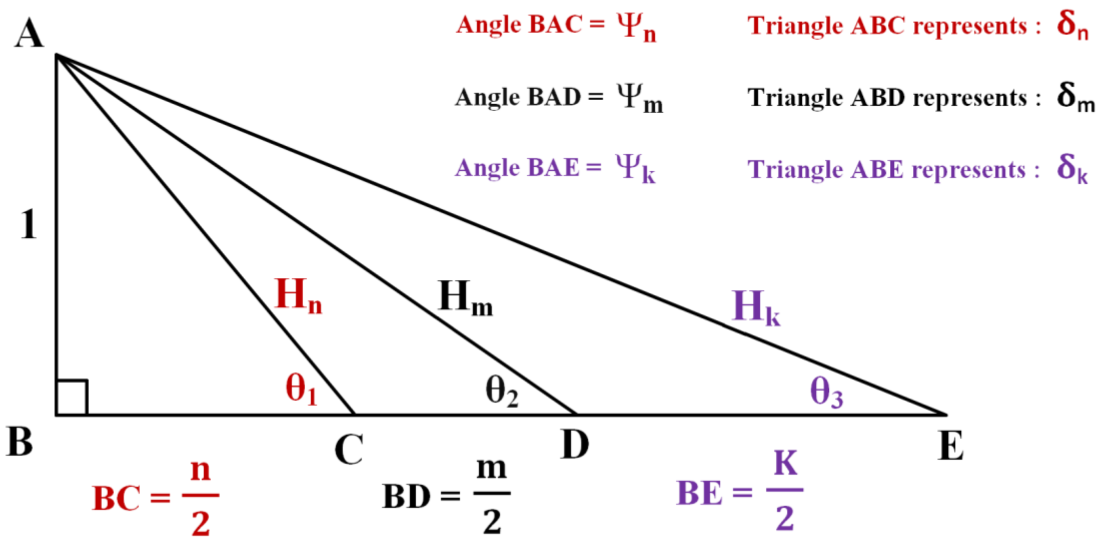

Figure 2: Three Right Triangles representing the **"Triad of Metallic Means"**

Remarkably, in above Figure 2: if the three Metallic Means $\boldsymbol{\delta_n}$, $\boldsymbol{\delta_m}$ and $\boldsymbol{\delta_k}$ constitute a **Triad** as $\frac{\mathbf{mn+4}}{\mathbf{m-n}} = \mathbf{k}$, then

$$\boldsymbol{\theta_1 = \theta_2 + \theta_3} \quad (\text{ and also } \boldsymbol{\Psi_n + 90^0 = \Psi_m + \Psi_k})$$

In other words,

$$\arctan\frac{2}{n} = \arctan\frac{2}{m} + \arctan\frac{2}{k}$$

Solving it gives the correlations: $\frac{\mathbf{mn+4}}{\mathbf{m-n}} = \mathbf{k}$ and $\frac{\mathbf{kn+4}}{\mathbf{k-n}} = \mathbf{m}$

And also $\frac{\mathbf{km\ -4}}{\mathbf{k+m}} = \mathbf{n}$

And hence satisfy the prerequisite for $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$ and $\frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m$

And also $\frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \boldsymbol{\delta_n}$

Moreover, **the Hypotenuses** of these Triad-Triangles : $\mathbf{H_n}$, $\mathbf{H_m}$ and $\mathbf{H_k}$ in **Figure 2** exhibit following relations.

$$\frac{\mathbf{H_k}}{\mathbf{H_m}} = \frac{\mathbf{2}}{\mathbf{m-n}} \times \mathbf{H_n}$$

Simplying this, we get $\frac{k^2+4}{m^2+4} = \frac{n^2+4}{(m-n)^2}$

And solving it gives $\mathbf{k} = \frac{\mathbf{mn+4}}{\mathbf{m-n}}$; which is the prerequisite for formation of a Triad $[\boldsymbol{\delta_n}, \boldsymbol{\delta_m}, \boldsymbol{\delta_k}]$

Moreover, entire geometry of such Triad-Triangles is resplendent with the precise correlation among the three Metallic Means $\boldsymbol{\delta_n}$, $\boldsymbol{\delta_m}$ and $\boldsymbol{\delta_k}$. Following intriguing relations are observed in above Figure 2. Consider the larger acute angles of the three triangles;

$$\Psi_m + \Psi_k = 2\arctan \boldsymbol{\delta}_n$$

$$\Psi_m - \Psi_n = 2\arctan \frac{1}{\delta_k}$$

$$\Psi_k - \Psi_n = 2\arctan \frac{1}{\delta_m}$$

Similarly, $\arctan\frac{1}{\delta_n} = \arctan\frac{1}{\delta_m} + \arctan\frac{1}{\delta_k}$

simplifying which we get : $\frac{\mathbf{km-4}}{\mathbf{k+m}} = \mathbf{n}$ **and** $\frac{\boldsymbol{\delta}_k \times \boldsymbol{\delta}_m - \mathbf{1}}{\boldsymbol{\delta}_k + \boldsymbol{\delta}_m} = \boldsymbol{\delta_n}$

And,

$$\arctan\frac{1}{\delta_n} + \arctan\frac{1}{\delta_m} + \arctan\frac{1}{\delta_k} = 2\arctan\frac{1}{\delta_n} = (\theta_1) = [\theta_2 + \theta_3]$$

$$= \arctan\frac{2}{n} = [180^0 - (\Psi_m + \Psi_k)]$$

= Half of the Smaller Acute Angle of the Pythagorean Triple associated with the Right Triangle for $\boldsymbol{\delta}_n$ (Described in following Section)

Similarly,

$$\arctan \delta_n + 90^0 = \arctan \delta_m + \arctan \delta_k$$

And,

$$\arctan \delta_n + \arctan \delta_m + \arctan \delta_k = \Psi_n + 180^0 = (\Psi_m + \Psi_k) + 90^0$$

However, more interesting aspect of this geometry of Metallic Means is its classical correspondence with **Primitive Pythagorean Triples** and **Pythagoream Primes**.

Right Triangle for n^{th} Metallic Mean and the Primitive Pythagorean Triples :

Each of the Right Triangles representing various Metallic Means, and hence the abovementioned Triads of Metallic Means, are associated with particular primitive Pythagorean triple, depending upon the value of **n**. For example, Right Triangle for 6^{th} Metallic Mean (δ_6), and hence, all Triads with n=6 are associated with 3-4-5 Pythagorean Triple. For instance, consider the Triad [$\delta_6, \delta_{11}, \delta_{14}$]. If this Triad is constructed geometrically, as in above **Figure 2**, then it is observed that:

$2\theta_1 = (\theta_1 + \theta_2 + \theta_3) =$ The Smaller Acute Angle of 3-4-5 Pythagorean Triangle i.e. **arctan** $\frac{3}{4}$

$2\Psi_n - 90^0 = (\Psi_n + \Psi_m + \Psi_k) - 180^0 =$ The Larger Acute Angle of 3-4-5 Pythagorean Triangle i.e. **arctan** $\frac{4}{3}$

Likewise, Right Triangle for 3^{rd} Metallic Mean (and all Triads with n=3), or Right Triangle for 10^{th} Metallic Mean (and Triads with n=10), are all related to 5-12-13 Pythagorean Triple, and so on.

Consider the fractional expression of the n^{th} Metallic Mean : $\delta_n = \frac{n + \sqrt{n^2 + 4}}{2}$

The radical $(n^2 + 4)$ in this Fractional expression of δ_n determines the Pythagorean Triple associated with the Right Triangle representing a Metallic Mean (δ_n). Any such Right Triangle representing a Metallic Mean δ_n is associated with such Primitive Pythagorean Triple whose Hypotenuse is Factor of the Radical $(n^2 + 4)$, with following observed sub-rules :

If n is Odd : Hypotenuse of associated Pythagorean Triple = $(n^2 + 4)$

If n is Even and multiple of Four (n=4x) : Hypotenuse of associated Pythagorean Triple = $nx + 1 = \frac{(n^2+4)}{4}$

If n is Even but not Multiple of Four : Hypotenuse of associated Pythagorean Triple = $\frac{(n^2+4)}{8}$ and the Smaller Cathetus of associated Pythagorean Triple = $\frac{n}{2}$

Noticeably, the doubling of the smaller acute angle of the $1 : \frac{n}{2}$ Right Triangle for any n^{th} Metallic Mean, produces the associated Pythagorean Triple; as illustrated below in **Figure 3.** For example, if the 26.565^0 angle of $1 : \frac{1}{2}$ triangle for Golden Ratio is doubled to 53.13^0 : the associated 3-4-5 Pythagorean Triple is produced.

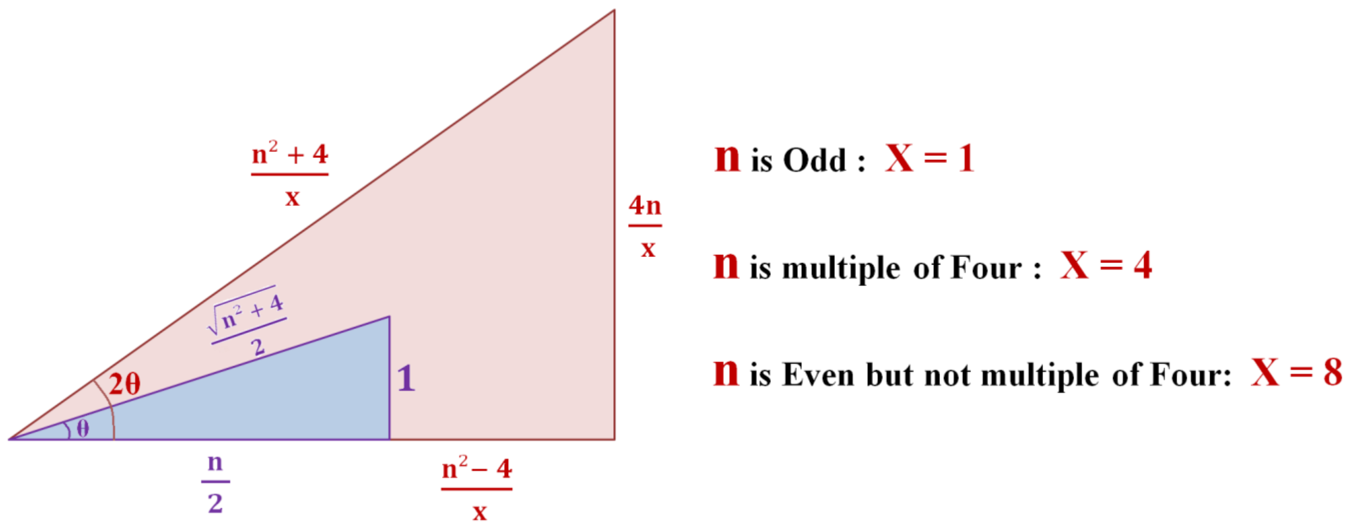

Figure 2: Doubling of the smaller acute angle of the Right Triangle for n^{th} Metallic Mean

The following **Table 1** shows associated Pythagorean Triples for first few values of n:

Table 1: Primitive Pythagorean Triples associated with the Right Triangles for Metallic Means: depending upon **n**

n	Associated Pythagorean Triple
1	3-4-5
2	None
3	5-12-13
4	3-4-5
5	20-21-29
6	3-4-5
7	28-45-53
8	8-15-17
9	36-77-85
10	5-12-13
11	44-117-125
12	12-35-37
13	52-165-173
14	7-24-25

The classical correspondence between these Right Triangles for Metallic Means and the corresponding Primitive Pythagorean Triple is manifested as follows.

Consider again the **Figure 2** representing the **Triad** of Metallic Means [$\delta_n, \delta_m, \delta_k$].

$$2\theta_1 = (\theta_1 + \theta_2 + \theta_3) = \text{The Smaller Acute Angle of associated Pythagorean Triple.}$$

Similarly, $2\Psi_n - 90^0 = (\Psi_n + \Psi_m + \Psi_k) - 180^0 =$ The Larger Acute Angle of associated Pythagorean Triple.

Special cases :

1) For **n=1 :** the right Triangle that represents Golden Ratio has its Cathetus 1 longer than its Second Cathetus $\frac{n}{2}$ Hence, for the Right Triangle representing Golden Ratio, and the Triad formed with n=1 [$\delta_1, \delta_2, \delta_6$] **:**
   $\mathbf{2\theta_1 - 90^0} = (\theta_1 + \theta_2 + \theta_3) - 90^0 =$ The Smaller Acute Angle of associated Pythagorean Triple 3-4-5.
   And, $\mathbf{2\Psi_n} = (\Psi_n + \Psi_m + \Psi_k) - 90^0 =$ The Larger Acute Angle of associated Pythagorean Triple.

2) Similarly, in case of **n=2 :** the Triangle representing Silver Ratio is an Isosceles Right Triangle. Hence, for Triad formed with n=2, like [$\delta_2, \delta_3, \delta_{10}$] or [$\delta_2, \delta_4, \delta_6$]:
   $\mathbf{2\,\theta_1} = (\theta_1 + \theta_2 + \theta_3) = \mathbf{90^0}$,
   and $\mathbf{2\,\Psi_n + 90^0} = (\Psi_n + \Psi_m + \Psi_k) = \mathbf{180^0}$
   And hence there is no associated Pythagorean Triple for n=2 .

3) For **n=3 or 4 :**
   $2\theta_1 = (\theta_1 + \theta_2 + \theta_3) =$ The **Larger** Acute Angle of associated Pythagorean Triple.
   And, $2\Psi_n - 90^0 = (\Psi_n + \Psi_m + \Psi_k) - 180^0 =$ The **Smaller** Acute Angle of associated Pythagorean Triple.

Note the Right Triangles those represent the First and Fourth Metallic Means: δ_1 (that is Golden Ratio φ) and δ_4 (which equals φ^3), are similar triangles.

In all the Right Triangles those represent 5th Mean onwards, and Triads formed with n=5 onwards, (n = 5, 6, 7….. and so on) : $\mathbf{2\theta_1} = (\theta_1 + \theta_2 + \theta_3) =$ The **Smaller** Acute Angle of associated Primitive Pythagorean Triple ;

$2\Psi_n - 90^0 = (\Psi_n + \Psi_m + \Psi_k) - 180^0 =$ The **Larger** Acute Angle of associated Primitive Pythagorean Triple.

Moreover, an intriguing relation is also observed between the Hypotenuses of the Three Triangles in **Figure 2** (viz. H_n, H_m and H_k) and the Hypotenuse of associated Pythagorean Triple **H**.

If n is Odd : $\frac{H_k}{H_m} \times (m-n) = \sqrt{H}$

If n is Even and multiple of Four : $\frac{H_k}{H_m} \times (m\text{-}n) = 2\sqrt{H}$

If n is Even but not Multiple of Four : $\frac{H_k}{H_m} \times (m\text{-}n) = 2\sqrt{2}\,\sqrt{H}$

Thus, consider the following couple of facts. **First:** the generalised geometric substantiation of all Metallic Means based upon Right Triangles, as elaborated in the work mentioned in References [6] and [7]. And **second:** the close correspondence between the Right Triangles representing Metallic Ratios and the Primitive Pythagorean Triples, as described here. These couple of facts clearly highlight the underlying proposition that the Metallic Means are more closely associated with; and more holistically represented by the "**Right Angled Triangles**", rather than Pentagon, Octagon or any other (n^2+4)-gon.

Metallic Means and Pythagorean Primes : The Prime Families of Metallic Means

From the close correspondence between Metallic Means and Pythagorean Triples described so far, it becomes obvious that various Metallic Means are also closely associated with different Pythagorean Primes.

Consider the radical $\mathbf{(n^2+4)}$ in the Fractional expression of the n^{th} Metallic Mean $(\boldsymbol{\delta_n})$. By Fermat's Theorem on Sums of Two Squares, this radical is an integer multiple of a prime of the form $p \equiv 1 \pmod 4$. The **Greatest Prime Factor** (i.e. the Largest Prime Divisor) of this radical $(n^2 + 4)$ is a Pythagorean Prime, as shown below in **Table 2.**

n	$\mathbf{n^2+4}$	**Greatest Prime Factor of $\mathbf{(n^2+4)}$: A Pythagorean Prime**
1	5	5
2	8	2
3	13	13
4	20	5
5	29	29
6	40	5
7	53	53
8	68	17
9	85	17

10	104	13
11	125	5
12	148	37
13	173	173
14	200	5
15	229	229
16	260	13

Table 2: The Greatest Prime Factors of the radical $(n^2 + 4)$

It is noticeable from above table that various values of n have common Greatest Prime Factor of $(n^2 + 4)$.

For example, for 3th, 10th and 16th Metallic Means, the common Greatest Prime Factor of the radical $(n^2 + 4)$ is **13**.

Also, for 1st, 4th, 6th, 11th Metallic Means, the common Greatest Prime Factor of the radical $(n^2 + 4)$ is **5**.

Hence, the different Metallic Means can be classified into various groups corresponding to the Greatest Prime Factors **(GPF)** of the radical $(\mathbf{n^2+4})$. This GPF is necessarily a Pythagorean Prime $(4x + 1)$, as shown below in **Table 3**.

Table 3 : Prime Families of Metallic Means

Greatest Prime Factor (GPF) of $[n^2+4]$	**n_s of the associated Metallic Means (δ_n)**
5	**1, 4, 6, 11**
13	**3, 10, 16, 29**
17	**8, 9, 26, 43**
29	**5, 24, 34**

37	**12, 25, 49**
41	**18, 23, 59**

Noticeably, as described in previous section: the Hypotenuse of associated Pythagorean Triple is a factor of (n^2+4), and the associated Pythagorean Primes, as shown in Table 3 are the Greatest Prime Factors of (n^2+4). Note: the 8th and the 9th Metallic Means both have Pythagorean Prime 17 as the GPF of their (n^2+4), however they have different associated Primitive Pythagorean Triples, as shown in Table 1.

Moreover, consider the **Triads** of Metallic Means [$\boldsymbol{\delta_n}$, $\boldsymbol{\delta_m}$, $\boldsymbol{\delta_k}$]. Noticeably Two of the Three Metallic Ratios forming such Triad, belong to same Prime Family i.e. Two of the Three Metallic Ratios have common Greatest Prime Factors of their respective (n^2+4) radicals.

For example, consider the Triad [$\boldsymbol{\delta_4}$, $\boldsymbol{\delta_5}$, $\boldsymbol{\delta_{24}}$] : $\boldsymbol{\delta}_5$, and $\boldsymbol{\delta}_{24}$ belong to the **29** prime family. Similarly, consider the Triad [$\boldsymbol{\delta_5}$, $\boldsymbol{\delta_6}$, $\boldsymbol{\delta_{34}}$] : $\boldsymbol{\delta}_5$, and $\boldsymbol{\delta}_{34}$ belong to the **29** prime family.

Consider another example, the Triad [$\boldsymbol{\delta_2}$, $\boldsymbol{\delta_3}$, $\boldsymbol{\delta_{10}}$] : $\boldsymbol{\delta}_3$, and $\boldsymbol{\delta}_{10}$ belong to the **13** prime family. Similarly, consider the Triad [$\boldsymbol{\delta_3}$, $\boldsymbol{\delta_4}$, $\boldsymbol{\delta_{16}}$] : $\boldsymbol{\delta}_3$, and $\boldsymbol{\delta}_{16}$ belong to the **13** prime family.

Moreover, beside the Greatest Prime Factors indicated in above Table, multiple Pythagorean Primes constitute the factors of various (n^2+4) radicals.

$(59)^2 + 4 = (41 \times 17 \times 5)$

$(49)^2 + 4 = (37 \times 13 \times 5)$, and so on.

Noticeably, the Greatest Prime Factor of (n^2+4) for $n=43$ is 109

$(43)^2 + 4 = 109 \times 17$

However, why it has been included in the Prime 17 Family in above Table, that will become obvious with following couple of illustrations.

Remarkably, the Metallic Means belonging to same Prime Family exhibit very distinctive relations among themselves, as illustrated below.

For instance, consider the Prime Family of Metallic Means associated with the Pythagorean Prime **13 :** $\boldsymbol{\delta}_3$, $\boldsymbol{\delta}_{10}$ and $\boldsymbol{\delta}_{16}$

Metallic Means associated with the Pythagorean Prime **13 :** $\boldsymbol{\delta}_3$, $\boldsymbol{\delta}_{10}$ and $\boldsymbol{\delta}_{16}$

Prime 13

10+3	=	13	=	16-3
$\boldsymbol{\delta}_{10}$		$\boldsymbol{\delta}_3$		$\boldsymbol{\delta}_{16}$

Hence, the Pythagorean Prime of this Family **13** = $\dfrac{\mathbf{3^2+4}}{\mathbf{1}}$: Note the Digit **1** in Denominator.

Here, the **3rd** and **(3 + 1)th** Metallic Means give the precise value of **16th** Mean : $\dfrac{\boldsymbol{\delta}_4 \times \boldsymbol{\delta}_3 + \mathbf{1}}{\boldsymbol{\delta}_4 - \boldsymbol{\delta}_3} = \boldsymbol{\delta}_{16}$

Similarly, the **3rd** and **(3 - 1)th** Metallic Means give the precise value of **10th** Mean : $\dfrac{\boldsymbol{\delta}_3 \times \boldsymbol{\delta}_2 + \mathbf{1}}{\boldsymbol{\delta}_3 - \boldsymbol{\delta}_2} = \boldsymbol{\delta}_{10}$

Moreover, $\dfrac{\mathbf{13-1}}{\mathbf{2}} = 6$:this 6 forms a new Triad with members 10 and 16 of the family: [$\boldsymbol{\delta}_6$, $\boldsymbol{\delta}_{10}$, $\boldsymbol{\delta}_{16}$]:

$$\frac{\boldsymbol{\delta}_{16} \times \boldsymbol{\delta}_6 + \mathbf{1}}{\boldsymbol{\delta}_{16} - \boldsymbol{\delta}_6} = \boldsymbol{\delta}_{10} \quad \text{and} \quad \frac{\boldsymbol{\delta}_{10} \times \boldsymbol{\delta}_6 + \mathbf{1}}{\boldsymbol{\delta}_{10} - \boldsymbol{\delta}_6} = \boldsymbol{\delta}_{16}$$

Further, beyond 3, 10 and 16, more members would be added in the family of Pythagorean Prime 17, next member in the family would be 29 which equals the sum of all smaller members in the family;

$3 + 10 + 16 = 29$ and, the Prime $13 = 29 - 16$

And,

$2 \times 16 + 3 + 1 = 36$:this 36 forms a new Triad with members 16 and 29 of the family [$\boldsymbol{\delta}_{16}$, $\boldsymbol{\delta}_{29}$, $\boldsymbol{\delta}_{36}$]:

$$\frac{\boldsymbol{\delta}_{36} \times \boldsymbol{\delta}_{16} + \mathbf{1}}{\boldsymbol{\delta}_{36} - \boldsymbol{\delta}_{16}} = \boldsymbol{\delta}_{29} \quad \text{and} \quad \frac{\boldsymbol{\delta}_{29} \times \boldsymbol{\delta}_{16} + \mathbf{1}}{\boldsymbol{\delta}_{29} - \boldsymbol{\delta}_{16}} = \boldsymbol{\delta}_{36}$$

Likewise, consider another example for illustration and comparison. the Metallic Means associated with the Pythagorean Prime **17 :** $\boldsymbol{\delta}_8$ **,** $\boldsymbol{\delta}_9$ and $\boldsymbol{\delta}_{26}$

Metallic Means associated with the Pythagorean Prime **17 :** $\boldsymbol{\delta}_8$, $\boldsymbol{\delta}_9$ and $\boldsymbol{\delta}_{26}$

Prime 17

8 + 9 = 17 = 26 - 9

$\boxed{\delta_8}$ $\boxed{\delta_9}$ $\boxed{\delta_{26}}$

And, the Prime 17 $= \dfrac{9^2 + 4}{5}$: Note the Digit **5** in Denominator.

Here, the **9th** and **(9 + 5)th** Metallic Means give the precise value of **26th** Mean : $\dfrac{\delta_9 \times \delta_{14} + 1}{\delta_{14} - \delta_9} = \delta_{26}$

Similarly, the **9th** and **(9 - 5)th** Metallic Means give the precise value of **8th** Mean : $\dfrac{\delta_9 \times \delta_4 + 1}{\delta_9 - \delta_4} = \delta_8$

Moreover, $\dfrac{17 - 5}{2} = 6$: this 6 forms a new Triad with members 8 and 26 of the family: $[\delta_6, \delta_8, \delta_{26}]$

$$\frac{\delta_{26} \times \delta_6 + 1}{\delta_{26} - \delta_6} = \delta_8 \quad \text{and} \quad \frac{\delta_8 \times \delta_6 + 1}{\delta_8 - \delta_6} = \delta_{26}$$

Further, beyond 8, 9 and 26, more members would be added in the family of Pythagorean Prime 17, like next member in the family would be 43 which equals the sum of all smaller members in the family;

8 + 9 + 26 = 43 and, the Prime 17 = 43 – 26

And,

$2 \times 26 + 9 + 5 = 66$:this 66 forms a new Triad with members 26 and 43 of the family: $[\delta_{26}, \delta_{43}, \delta_{66}]$

$$\frac{\delta_{66} \times \delta_{26} + 1}{\delta_{66} - \delta_{26}} = \delta_{43} \quad \text{and} \quad \frac{\delta_{43} \times \delta_{26} + 1}{\delta_{43} - \delta_{26}} = \delta_{66}$$

Such several distinctive correlations are observed among the Metallic Means belonging to the same Pythagorean Prime Families, and these correlations are bound to generate more such intriguing mathematical formulae, which may provide the precise relations between different Metallic Ratios.

On the last note, it is worth mentioning here that several other intriguing properties of Metallic Means and their abovementioned TRIADS are described in details in the works mentioned in the References. For instance, these TRIADS of Metallic Means are found to be closely associated with the Pascal's Triangle [11]; the geometric substantiation of Metallic Ratios and their TRIADS [6] [7] [8] [9]; and special positions of Integers 3, 6 and 9 in the realm of Metallic Means [2][12]. Further, all imperical formulae those provide the precise relations between different Metallic Means are described in the work mentioned in Reference [1].

Conclusion:

This paper brought together the generalised geometry of all Metallic Means and the mathematical formula that provides the precise correlations between different Metallic Ratios. The paper illustrated the concept of the "Triads of Metallic Means", and also the close correspondence between Metallic Ratios and the Pythagorean Triples as well as Pythagorean Primes.

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