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# Golden Ratio and other Metallic Means ; The Geometric Substantiation of all Metallic Ratios with Right Triangles

Dr. Chetansing Rajput

M.B.B.S. (Mumbai University) India, Asst. Commissioner (Govt. of Maharashtra)

Email: chetansingkrajput@gmail.com

Lecture Link: https://www.youtube.com/watch?v=raosniXwRhw

Website: https://goldenratiorajput.com/

**Abstract**

This paper introduces the concept of special right triangles those provide the accurate geometric substantiations of all Metallic Ratios. These right angled triangles not only have the precise Metallic Means embedded in all their geometric features, but they are also observed to be the quintessential forms of the corresponding Metallic Ratios. These special right triangles manifest the corresponding Metallic Ratios more holistically than the regular pentagon, octagon or tridecagon, etc.

**Keywords:** Metallic Mean, Golden Ratio, Fibonacci sequence, Pi, Phi, Pythagoras Theorem, Divine Proportion, Silver Ratio, Golden Mean, Right Triangle, Pell Numbers, Lucas Numbers, Golden Proportion, Metallic Ratio

## Introduction

An intriguing geometric relationship exists between the Fibonacci-like Integer Sequence and the corresponding Lucas Sequence, associated with each Metallic Ratio. For example, the **1:2:**$\sqrt{5}$ right triangle having its catheti equalling the 2nd and 3rd Fibonacci numbers, exhibits a classical geometric relationship with the **3-4-5** right triangle, whose catheti are the corresponding Lucas numbers. An outstanding synergy between these two right triangles provides for the phenomenal and precise expression of Golden Ratio.**[1]**

More importantly, just like the Golden Ratio, such classical geometric synergism can be generalised for all other Metallic Ratios.

The prime objective of this work is to introduce and elaborate the concept of such special right angled triangles those provide the geometric substantiation all Metallic Means. Moreover, this paper introduces and generalises the classical geometric relationships between the Integer Sequence and the corresponding Lucas Sequence, associated with any Metallic Ratio.

Further, this paper also explicates certain new geometric aspects of the Metallic Ratios. Each Metallic Ratio is observed to be accurately represented by another special right triangle, which provides the precise fractional expression of that Ratio **[2]**. This work generalises the geometric substantiation of all Metallic Ratios, on basis of such "Fractional Expression Triangles", which are the prototypical forms of the corresponding Metallic Means. Hence, the further objective of this work is to elaborate and generalise the concept of such "Fractional Expression Triangle" that provides the most accurate expression of any nth Metallic Mean ($\boldsymbol{\delta_n}$).

## 1:2:$\sqrt{5}$ Triangle and 3-4-5 Pythagorean Triple : A Classical Geometric Synergy

The **1:2:$\sqrt{5}$** triangle is found to exhibit a classical geometric relationship with the **3-4-5** right triangle, which is the first of primitive Pythagorean triples. The **1:2:$\sqrt{5}$ triangle** and **3-4-5 triple** are not only invariably formed together, but they also exhibit an outstanding synergy between all their corresponding sides and angles, that provides for the phenomenal expression of Golden Ratio.**[1]**

Remarkably, there exists a precise complementary relationship between 1:2:$\sqrt{5}$ triangle and 3-4-5 right triangles. And the classical correspondence between these two triangles culminates in some unconventional geometric outcomes. All corresponding angles and catheti of these two right triangles synergize with each other to reflect the precise Golden Ratio, as follows.

The angles of 1:2:$\sqrt{5}$ triangle and 3-4-5 triangle are found to be entangled with each other in a remarkable way, and hence they add up to very peculiar values, as illustrated in **Figure 1**.

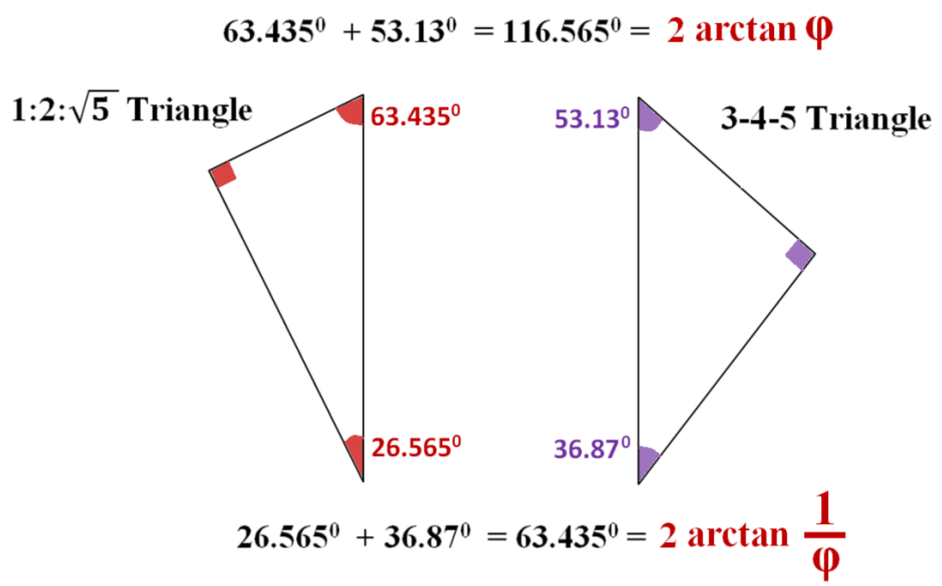


**Figure 1:** The corresponding angles of the two triangles add up, to reflect **the Golden Ratio.**

As shown in above diagram, the corresponding acute angles of the two triangles add up to reflect the precise value of **Golden Ratio**, noticeably they add up to the **angles of Golden Rhombus**, whose diagonals are perfectly in Golden Proportion.

$$53.13^0 + 63.435^0 = 116.565^0 = 2 \arctan \varphi$$

and also, $$36.87^0 + 26.565^0 = 63.435^0 = 2 \arctan \frac{1}{\varphi}$$

In other words, the average of the corresponding acute angles of these two right triangles equals the arctangent of Golden Ratio.

Remarkably, beside Golden Rhombus, the angles formed by combining the corresponding angles of these two right triangles, are also associated with those polyhedrons whose geometry is full of Golden Ratio, such as, $53.13^0 + 63.435^0 = \mathbf{116.565^0}$ is the dihedral angle of Regular Dodecahedron, Dodecadodecahedron, Small

Stellated Dodecahedron as well as Small Icosihemidodecahedron. Similarly, $36.87^0$ + $26.565^0$ **=** $\mathbf{63.435^0}$ is also the dihedral angle of Great dodecahedron and Great Stellated Dodecahedron.

Noteworthy here, the angle $\mathbf{36.87^0}$ of 3-4-5 triangle is the **complementary angle** for **twice the** $\mathbf{26.565^0}$ angle of 1:2:$\sqrt{5}$ triangle.

And, the angle $\mathbf{53.13^0}$ of 3-4-5 triangle is the **supplementary angle** for **twice the** $\mathbf{63.435^0}$ angle of 1:2:$\sqrt{5}$ triangle.

Further, two smaller angles of these two triangles, $\mathbf{26.565^0}$ & $\mathbf{36.87^0}$**,** add up to angle $\mathbf{63.435^0}$ of 1:2:$\sqrt{5}$ triangle, while the angle $\mathbf{53.13^0}$ of Pythagorean triple is twice the $\mathbf{26.565^0}$ angle of 1:2:$\sqrt{5}$ triangle. Hence, the dissection of any of these two triangles, causes the emergence of a couple of 1:2:$\sqrt{5}$ triangles along with one Pythagorean triple.

Beside angles, the classical geometric synergy between 1:2:$\sqrt{5}$ triangle and 3-4-5 triangle is naturally manifested in their side lengths also. As shown below in **Figure 2,** the 1:2:$\sqrt{5}$ triangle ABC merged with an equivalent sized the 3-4-5 triangle ADC, with their common hypotenuse AC, reveals the Golden Ratio, precisely embedded in such blend of the two triangles.

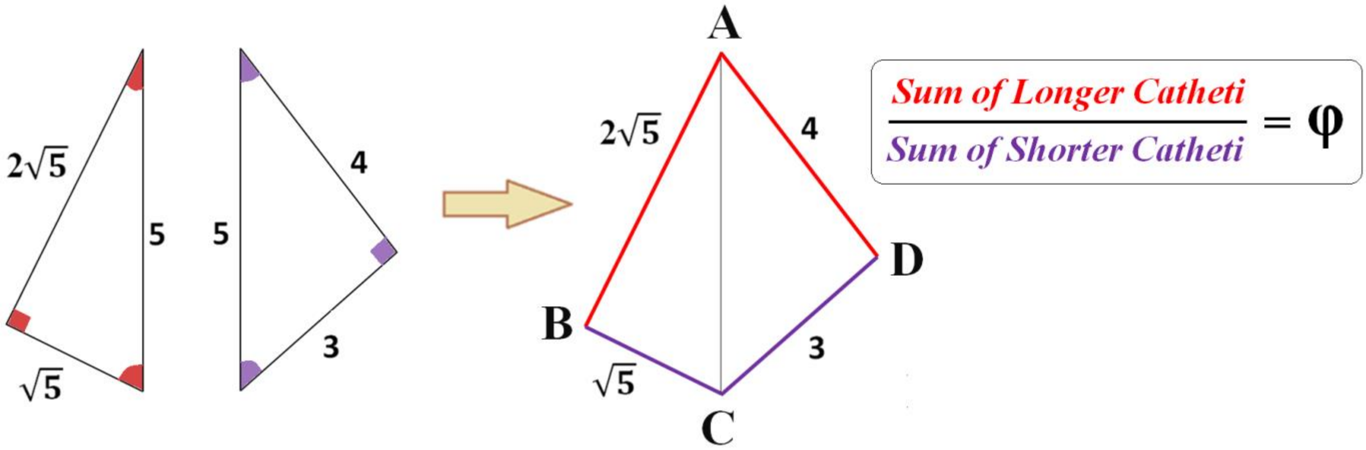


**Figure 2:** The Golden Ratio in amalgam of 1:2:$\sqrt{5}$ triangle and 3-4-5 triangle.

In these **equivalent sized** 1:2:$\sqrt{5}$ and 3-4-5 triangles, with **common hypotenuse AC = 5**, the Longer Catheti of two triangles (AB + AD) add up to $\mathbf{2\varphi^3}$, and the Shorter Catheti (CB + CD) add up to $\mathbf{2\varphi^2}$

Hence, in the quadrilateral □ABCD in above Figure 2, formed by merger of the two triangles,

$$\frac{\textbf{The Sum of the Longer Catheti of the Two Triangles}}{\textbf{Sum of their Shorter Catheti}} = \frac{\mathbf{AB+AD}}{\mathbf{CB+CD}} = \boldsymbol{\varphi}$$

Hence, to summarise the Golden Synergy illuasrated in **Figures 1 & 2**, between the 1:2:$\sqrt{5}$ and 3-4-5 triangles having common hypotenuse:

: The Average of the corresponding longer catheti $2\sqrt{5}$ & 4 is $\boldsymbol{\varphi^3}$

: The Average of the corresponding shorter catheti $\sqrt{5}$ & 3 is $\boldsymbol{\varphi^2}$

: The Average of the corresponding larger acute angles $53.13^0$ & $63.435^0$ is $\mathbf{arctan}\ \boldsymbol{\varphi}$



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: The Average of the corresponding smaller acute angles $36.87^0$ & $26.565^0$ is $\mathbf{arctan}\,\frac{\mathbf{1}}{\boldsymbol{\varphi}}$

And, **the deviation** of all acute angles from the arctangent of Golden Ratio is $\mathbf{arctan}\,\frac{\mathbf{1}}{\boldsymbol{\varphi}^5}$

In other words, the right triangle with its catheti in Golden Ratio **1:φ**, occupies the **exact mean position** between the 1:2:$\sqrt{5}$ triangle and the 3-4-5 Pythagorean triple.

Several such intriguing relations between 1:2:$\sqrt{5}$ and 3-4-5 right triangles have been discussed in detail in the work mentioned in Reference **[1].**

The noticeable aspect of these two right triangles is that their catheti precisely equal a couple of Fibonacci Numbers and corresponding Lucas Numbers; **the catheti of 1:2:$\sqrt{5}$ triangle are 2$^{nd}$ and 3$^{rd}$ Fibonacci terms, and the catheti of the 3-4-5 triangle are corresponding Lucas numbers.**

## Generalisation:

Remarkably, the abovementioned geometric relationship can be generalised for all Metallic Ratios. The couple of right triangles, having their catheti as the n$^{th}$ and (n+k)$^{th}$ terms of the Integer Sequence and the corresponding terms the Lucas Sequence associated with any Metallic Mean, exhibit the similar geometric correlations; for example the couple of right triangles having their catheti as the n$^{th}$ and (n+k)$^{th}$ terms of Fibonacci and Lucas Sequences; or Pell and Pell-Lucas Sequences, and so on.

Noticeably, such classical correspondence is also observed between two right triangles having their catheti equalling the non-consecutive terms of Integer Sequence and corresponding Lucas Sequence ($k > 1$), provided k is an odd integer.

For instance, consider any m$^{th}$ Metallic Ratio is $\boldsymbol{\delta}_{\mathbf{m}}$

And, let $G_1$, $G_2$, $G_3$ ......... $G_n$ be the Integer Sequence associated with this Metallic Mean $\boldsymbol{\delta}_{\mathbf{m}}$; and $L_1$, $L_2$, $L_3$.....$L_n$ be the corresponding Lucas Sequence.

For example: Fibonacci Sequence for Golden Ratio ($\boldsymbol{\delta}_{\mathbf{1}}$) and the corresponding Lucas Sequence 1, 3, 4, 7, 11, 18, 29........; similarly Pell Sequence for Silver Ratio ($\boldsymbol{\delta}_{\mathbf{2}}$) and the corresponding Pell-Lucas Sequence 2, 6, 14, 34, 82, 198, 478......, and so on [4],[5].

And **n** is any Positive Integer, while **k** is an **"Odd"** Positive Integer.

Now, consider the couple of right triangles shown in following **Figure 3**, having their catheti equalling $\mathbf{G_n}$ & $\mathbf{G_{n+k}}$ ; and $\mathbf{L_n}$ & $\mathbf{L_{n+k}}$ respectively, where **k** is an Odd integer, and $\boldsymbol{\theta_1}$ & $\boldsymbol{\theta_2}$, and $\boldsymbol{\Psi_1}$ & $\boldsymbol{\Psi_2}$ being their respective acute angles. Such couple of right triangles is exactly analogues to the abovementioned 1:2:$\sqrt{5}$ and 3-4-5 triangles. And hence, they exhibit the similar geometric relationship between all their angles and side lengths, that provides the accurate expression of the corresponding Metallic Mean $\boldsymbol{\delta}_{\mathbf{m}}$

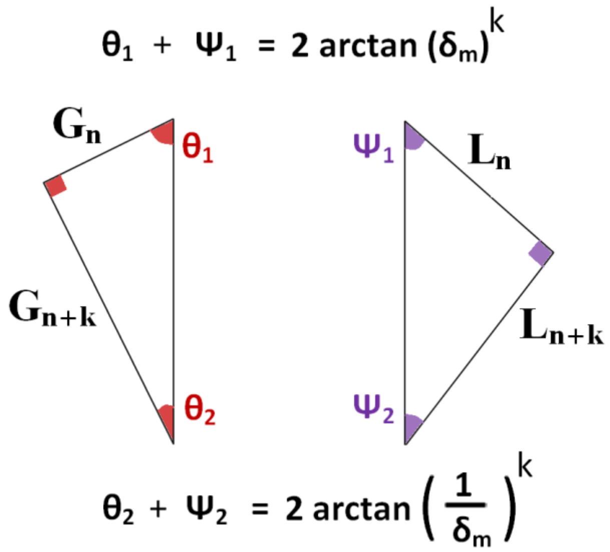


**Figure 3:** Right Triangles; generated from Integer Sequence and corresponding Lucas Sequence for $\boldsymbol{\delta_m}$

It is noticeable in above Figure 3 that the catheti as well as the angles of both right triangles are the obvious expressions of the corresponding Metallic Mean $\boldsymbol{\delta_m}$

Consider the acute angles of two right triangles:

$$\boldsymbol{\theta_1} = \arctan\left(\delta_m\right)^k + (-1)^n \times \arctan\frac{\mathbf{1}}{(\boldsymbol{\delta_m})^{2n+k}}$$

$$\boldsymbol{\theta_2} = \arctan\frac{\mathbf{1}}{(\boldsymbol{\delta_m})^{k}} - (-1)^n \times \arctan\frac{\mathbf{1}}{(\boldsymbol{\delta_m})^{2n+k}}$$

$$\boldsymbol{\Psi_1} = \arctan\left(\delta_m\right)^k - (-1)^n \times \arctan\frac{\mathbf{1}}{(\boldsymbol{\delta_m})^{2n+k}}$$

$$\boldsymbol{\Psi_2} = \arctan\frac{\mathbf{1}}{(\boldsymbol{\delta_m})^{k}} + (-1)^n \times \arctan\frac{\mathbf{1}}{(\boldsymbol{\delta_m})^{2n+k}}$$

Moreover,

$$\boldsymbol{\theta_1} + \boldsymbol{\Psi_1} = \mathbf{2} \times \arctan\left(\delta_m\right)^k$$

$$\boldsymbol{\theta_2} + \boldsymbol{\Psi_2} = \mathbf{2} \times \arctan\frac{\mathbf{1}}{(\boldsymbol{\delta_m})^{k}}$$

Hence, the average of the corresponding acute angles of such couple of right triangles is **the arctangent of the $k^{th}$ power of concerned Metallic Ratio**.

And **the deviation** of each of the four acute angles from the arctangent of the $k^{th}$ power of the Metallic Ratio is precisely the **arctangent of** $\frac{\mathbf{1}}{(\boldsymbol{\delta_m})^{2n+k}}$

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Moreover, consider the catheti of the two right triangles:

$$\mathbf{G_n} = \frac{(\delta m)^n - (-\delta m)^{-(n)}}{\sqrt{m^2+4}}$$

$$\mathbf{G_{n+k}} = \frac{(\delta m)^{n+k} - (-\delta m)^{-(n+k)}}{\sqrt{m^2+4}}$$

$$\mathbf{L_n} = (\delta_m)^n + (-\delta_m)^{-(n)}$$

$$\mathbf{L_{n+k}} = (\delta_m)^{n+k} + (-\delta_m)^{-(n+k)}$$

Now, consider the following **Figure 4**, the two right triangles ABC and ADC are made **equivalent sized**, to have their **common hypotenuse AC**.

In such equivalent sized ΔABC, the cathetus AB becomes: $G_n\sqrt{m^2+4} = (\delta_m)^n - (-\delta_m)^{-(n)}$

And, the cathetus BC becomes: $G_{n+k}\sqrt{m^2+4} = (\delta_m)^{n+k} - (-\delta_m)^{-(n+k)}$

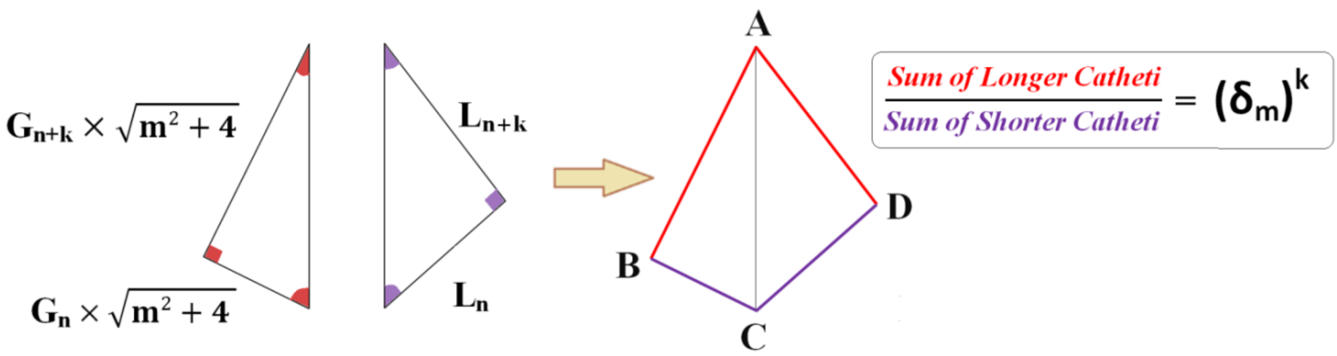



**Figure 4:** The Metallic Ratio in amalgam of the Two Right Triangles, having **Common Hypotenuse**.

Noticeably, in above Figure,

Average of the Longer Catheti of two triangles: $\dfrac{L_{n+k} + G_{n+k}\sqrt{m^2+4}}{2} = (\delta_m)^{n+k}$

Average of the Shorter Catheti of two triangles: $\dfrac{L_n + G_n\sqrt{m^2+4}}{2} = (\delta_m)^n$

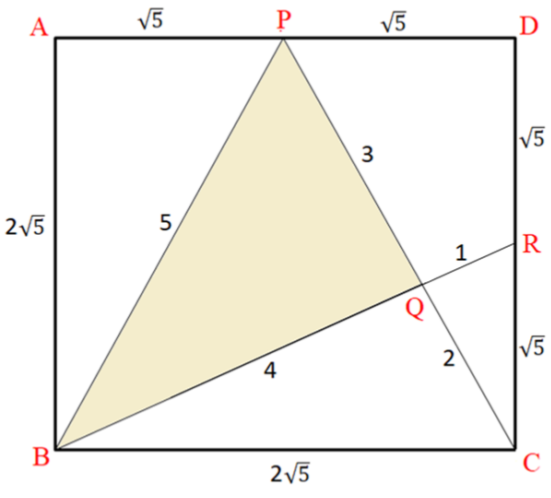
And hence,

$$\frac{\text{The Sum of the Longer Catheti of the Two Triangles}}{\text{Sum of their Shorter Catheti}} = \frac{AB+AD}{CB+CD} = (\delta_m)^k$$

For instance, consider **Figure 1**, where m =1, and $\delta_m$ = Golden Ratio($\varphi$), while n = 2 and k=1, and hence the precision values of all catheti and angles of the 1:2:$\sqrt{5}$ and 3-4-5 triangles can be derived from the abovementioned formulae; the only prerequisite for these generalised formulae is the **k** must be an Odd integer.

Moreover, another noteworthy aspect of the 1:2:$\sqrt{5}$ and 3-4-5 triangles needs to be mentioned here. As stated earlier, these two right triangles are invariably formed together. Several geometric methods those impart the concurrent formation of 1:2:$\sqrt{5}$ and 3-4-5 triangles are described in the paper mentioned in Reference **[1]**. However, the simplest and the most important method is the dissection of a square. The 3-4-5 right triangle can be geometrically produced by simple dissection of a square by three line segments, as shown below in **Figure 5**. Remarkably, such dissection of a square invariably produces multiple 1:2:$\sqrt{5}$ triangles, alongside the central 3-4-5 Pythagorean triple.

Consider the following **Figure 5**. In square ABCD, points P and R are the midpoints of side AD and DC, respectively. Connecting point P to the vertices B and C, and point R to the vertex B, produces the triangle PBQ in center; which is a 3-4-5 Pythagorean triangle. And remarkably, all other triangles formed in the figure, namely, ΔCQR, ΔCQB, ΔAPB, ΔCDP and ΔCBR; all are 1:2:$\sqrt{5}$ triangles of various sizes.




**Pythagorean Triple △PBQ is formed in Centre;**

**All other triangles formed are 1:2:$\sqrt{5}$ triangles**

**Figure 5:** Dissecting the Square; the central 3-4-5 Pythagorean triple, & 1:2:$\sqrt{5}$ triangles of various sizes.

The abovementioned method of the dissection of a square can divide any square into a 3-4-5 Pythagorean triple and multiple 1:2:$\sqrt{5}$ triangles. Beside such multiple right triangles, the square so dissected also contains a couple of irregular quadrilaterals ▫ABQP and ▫PQRD. Noticeably, the internal angles of these quadrilaterals ▫ABQP and ▫PQRD, beside two opposite right angles, are

$$\angle ABQ = \angle DPQ = \mathbf{63.435^0} = 2 \arctan \frac{1}{\varphi}$$

$$\text{And, } \angle APQ = \angle DRQ = \mathbf{116.565^0} = 2 \arctan \varphi$$

Most importantly, such simple dissected square epitomizes the geometric relationships explicated so far in this paper. Consider the quadrilateral ▫ABQP which consists of the 3-4-5 Pythagorean triangle PQB and an equivalent sized 1:2:$\sqrt{5}$ triangle PAB, having their common hypotenuse PB. Note the catheti of 1:2:$\sqrt{5}$ triangle PAB are **the 2nd and 3rd Fibonacci numbers: 1 & 2** (multiplied by $\sqrt{m^2+4} = \sqrt{5}$); and catheti of the 3-4-5 Pythagorean triangle PQB are **the corresponding Lucas Numbers, 3 & 4.**



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Hence, in confirmation with the formula: $\frac{L_n + G_n\sqrt{m^2+4}}{2} = (\delta_m)^n$ ;

BA +BQ = $2\varphi^3$

PA+ PQ = $2\varphi^2$

Similarly, in the quadrilateral ▫PQRD, the sides PD and DR are equal to **the 1st and 2nd Fibonacci numbers: 1 & 1** (multiplied by $\sqrt{m^2 + 4} = \sqrt{5}$); and the sides QR and PQ equal **the corresponding Lucas Numbers: 1 & 3.**

And hence, in confirmation with the abovementioned formula,

PQ +PD = $2\varphi^2$

DR +QR = $2\varphi$

And hence, the two Quadrilaterals ▫ABQP and ▫PQRD naturally exhibit the Golden Ratio embedded in their side lengths as:

$$\frac{\mathbf{BA+BQ}}{\mathbf{PA+PQ}} = \frac{\mathbf{PQ+PD}}{\mathbf{QR+DR}} = \varphi$$

## The Fractional Expression Triangle for Metallic Mean :

Each Metallic Mean $\delta_n$ is the root of the simple Quadratic Equation $\mathbf{X^2 - nX - 1 = 0}$, where **n** is any positive natural number. Thus, the fractional expression of the $n^{th}$ Metallic Ratio is $\delta_n = \frac{n+\sqrt{n^2+4}}{2}$

Moreover, each Metallic Ratio can be expressed as the continued fraction:

$$\delta_n = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \dots}}} \text{ ; And hence, } \delta_n = n + \frac{1}{\delta n}$$

Further, the various Metallic Ratios are mathematically related to each other. The explicit formulae those provide the precise mathematical relationships between different Metallic Means have been discussed in detail in the work mentioned in Reference **[3]**.

More importantly, each Metallic Ratio can be expressed with a special Right Angled Triangle. Any $n^{th}$ Metallic Mean can be accurately represented by the Right Triangle having its catheti **1** and $\frac{2}{n}$. Hence, the right triangle with one of its catheti = **1** may substantiate any Metallic Mean, having its second cathetus = $\frac{2}{n}$ , where n = 1 for Golden Ratio, n = 2 for Silver Ratio, n = 3 for Bronze Ratio, and so on. **[1]**,**[2]** Such Right Triangle provides the precise value of $n^{th}$ Metallic Mean by the generalised formula:

The $n^{th}$ Metallic Mean $\boldsymbol{\delta}_n = \dfrac{\text{Cathetus 1} + \text{Hypotenuse}}{\text{Second Cathetus}} = \dfrac{1 + \text{Hypotenuse}}{2/n}$

For example, the 1:2:$\sqrt{5}$ Triangle is observed to be the quintessential form of Golden Ratio ($\varphi$). This 1:2:$\sqrt{5}$ right triangle, with all its peculiar geometric features, described in the work mentioned in Reference **[1]**, turns out to be the real 'Golden Ratio Trigon' in every sense of the term. The characteristic geometry of 1:2:$\sqrt{5}$ triangle, which is resplendent with Phi ($\varphi$) =1.618….., provides the most remarkable expression of the First Metallic Ratio, viz. the Golden Ratio. And likewise, the similar Right Triangles can provide for the geometric substantiation of all Metallic Means.

As mentioned earlier, any $\mathbf{n^{th}}$ Metallic Mean can be accurately represented by the Right Triangle having its catheti **1** and $\dfrac{\mathbf{2}}{\mathbf{n}}$.

Hence, just as the 1:2:$\sqrt{5}$ Triangle provides the Fractional Expression of Golden Ratio ($\boldsymbol{\varphi}$) = $\dfrac{\mathbf{1+\sqrt{5}}}{\mathbf{2}}$ , likewise the right triangle 1:1: $\sqrt{2}$ provides geometric substantiation of the Silver Ratio $\boldsymbol{\delta_2} = \dfrac{\mathbf{1+\sqrt{2}}}{\mathbf{1}} =$ 2.41421356…., similarly the right triangle with its catheti **1** and $\dfrac{\mathbf{2}}{\mathbf{3}}$ accurately represents the Bronze Ratio, and so on **[1]**.

Hence, such right angled triangle, with its catheti **1** and $\dfrac{\mathbf{2}}{\mathbf{n}}$ ; that provides the accurate fractional expression of the $\mathbf{n^{th}}$ Metallic Ratio, can be correctly termed as the **"Fractional Expression Triangle"** for the corresponding Metallic Ratio ($\boldsymbol{\delta}_\mathbf{n}$).

Such "Fractional Expression Triangle" of any $n^{th}$ Metallic Mean exhibits certain peculiar geometric features. For instance, the angles, side lengths as well as every geometric feature of such triangle are the most accurate expressions of corresponding Metallic Ratio ($\boldsymbol{\delta}_\mathbf{n}$).

For example, consider the 1:2:$\sqrt{5}$ right triangle that represents the Golden Ratio ($\varphi$). As shown below in **Figures 6 (A):** Note, the angles of the 1:2:$\sqrt{5}$ triangle are precise expressions of the Golden Ratio., and **Figures 6 (B):** The division of the triangle sides by the touch points of the Incircle, which are also the vertices of Gergonne Triangle, is also exactly in terms of the Golden Ratio ($\boldsymbol{\varphi}$).

Remarkably, all these geometric features can be generalised for the so called "Fractional Expression Triangle" representing any $n^{th}$ Metallic Ratio ($\boldsymbol{\delta}_\mathbf{n}$), as shown in **Figure 7** which illustrates certain generalised features of such Fractional Expression Triangle ΔABC, like the accurate measures of angles in terms of corresponding Metallic Mean ($\boldsymbol{\delta}_\mathbf{n}$), the division of triangle sides by the touch-points of Incircle , etc.

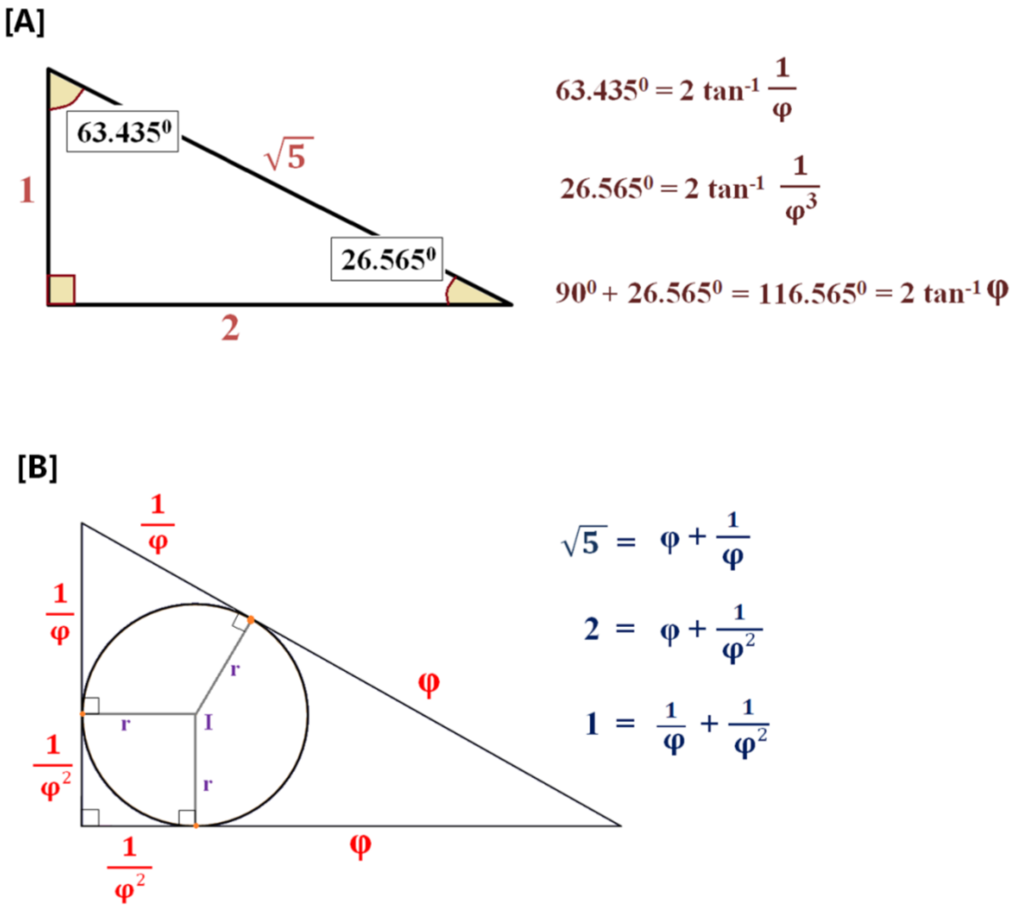


**Figure 6: (A)** The Angles of 1:2:√5 Triangle in terms of Golden Ratio, **(B)** The Side Lengths of 1:2:√5 Triangle in terms of Golden Ratio

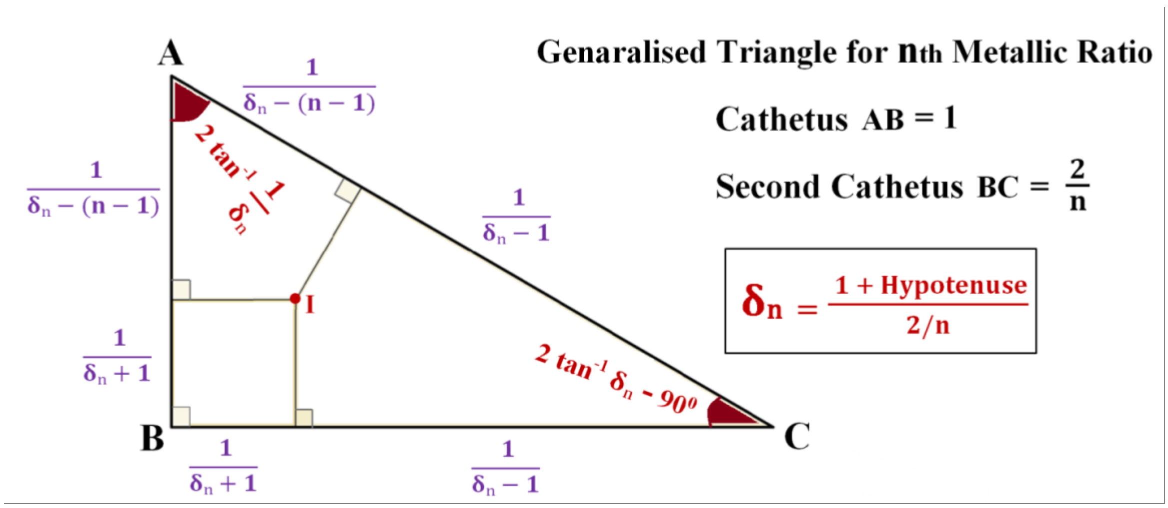


**Figure 7:** The Generalised 'Fractional Expression Triangle' for $n^{th}$ Metallic Mean.



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Moreover, several other geometric features of such "Fractional Expression Triangle" can also be generalised as follows.

Semiperimeter of this Fractional Expression Triangle (**S**) = $\frac{\boldsymbol{\delta_n}}{\boldsymbol{\delta_n - 1}}$ , for example, Semiperimeter of the 1:2:$\sqrt{5}$ right triangle representing the Golden Ratio is $\boldsymbol{\varphi^2}$ , and that of the 1:1: $\sqrt{2}$ right triangle representing the Silver ratio is $\frac{\boldsymbol{\sqrt{2}+1}}{\boldsymbol{\sqrt{2}}}$

Also, the Inradius of the generalised Fractional Expression Triangle (**r**) = $\frac{\boldsymbol{1}}{\boldsymbol{\delta_n + 1}}$

For example, just as $\frac{1}{\varphi + 1}$ i.e. $\frac{1}{\varphi^2}$ is the Inradius of the 1:2:$\sqrt{5}$ triangle, likewise $\frac{1}{\delta_2 + 1}$ is the Inradius of the 1:1: $\sqrt{2}$ triangle.

And noticeably, a fascinating relationship is observed between such Fractional Expression Triangle and its Incircle. A precise ratio is observed to exist between the Fractional Expression Triangle and its Incircle, and that ratio is an intriguing expression in terms of the corresponding Metallic Ratio $(\boldsymbol{\delta_n})$ and the **Pi** $(\boldsymbol{\pi})$.

$$\frac{\text{Area of this Fractional Expression Triangle}}{\text{Area of Its Incircle}} = \frac{(\boldsymbol{\delta n + 1})^2}{\boldsymbol{n\pi}}$$ ; where $\boldsymbol{\delta_n}$ is the $n^{th}$ Metallic Ratio.

For example, $$\frac{\text{Area of 1:2:}\sqrt{5}\text{ Triangle}}{\text{Area of Its Incircle}} = \frac{\text{Perimeter of 1:2:}\sqrt{5}\text{ Triangle}}{\text{Circumference of Its Incircle}} = \frac{\boldsymbol{\varphi}^4}{\boldsymbol{\pi}}$$

Likewise, $$\frac{\text{Area of the 1:1:}\sqrt{2}\text{ Triangle}}{\text{Area of Its Incircle}} = \frac{\text{Perimeter of 1:1:}\sqrt{2}\text{ Triangle}}{\text{Circumference of Its Incircle}} = \frac{(\textbf{Silvr Ratio})^2}{\boldsymbol{\pi}}$$

Moreover, such Fractional Expression Triangle is also the Limiting Triangle for the Pythagorean Triples formed with the Hypotenuses those equal the alternate terms of the Integer Sequence associated with that Metallic Mean. For example, the Pythagorean triples derived from Fibonacci series, approach the 1:2:$\sqrt{5}$ triangle's proportions, as the series advances. Pythagorean triples can be formed with the alternate Fibonacci numbers as the hypotenuses, like **5**-4-3, **13**-12-5, **34**-30-16, **89**-80-39, **233**-208-105, **610**-546-272 and so on. And, as such series of **Fibonacci-Pythagorean Triples** advances, the triples so formed, invariably approach 1:2:$\sqrt{5}$ triangle proportions, exactly in the same manner as the ratio between consecutive Fibonacci numbers approaches the Golden Ratio: $\lim_{n\to\infty} \frac{\mathbf{Fn}}{\mathbf{Fn-1}} \cong \boldsymbol{\varphi}$. And hence, it clearly endorses: while $\boldsymbol{\varphi}$ is the Golden Ratio in nature, the 1:2:$\sqrt{5}$ triangle is truly the Golden Trigon in geometry, in every sense of the term.

Likewise, the Pythagorean triples derived from **Pell Numbers series**, approach the 1:1:$\sqrt{2}$ triangle's proportions, as the series advances. The Pythagorean triples can be formed with the alternate Pell numbers as the hypotenuses, like **5**-4-3, **29**-21-20, **169**-120-119, **985**-697-696, **5741**-4060-4059 and so on. And, as such series of **Pell-Pythagorean Triples** advances, the triples so formed invariably approach **1:1:**$\boldsymbol{\sqrt{2}}$ limiting triangle proportions.


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More importantly, the so called Fractional Expression Triangle for any **n^th^** Metallic Mean is found to be closely related to another right triangle, which is associated with that **n^th^** Metallic Mean ($\boldsymbol{\delta}_n$). Consider the right triangle having its catheti in proportion $\mathbf{1{:}\delta_n}$. Such $\mathbf{1{:}\delta_n}$ right triangle has an interesting geometric relationship with the abovementioned "Fractional Expression Triangle" for that Metallic Mean ($\boldsymbol{\delta}_n$).

For instance, consider the $\mathbf{1{:}2{:}\sqrt{5}}$ triangle which is the Fractional Expression Triangle for the Golden Ratio, and another right triangle having its catheti in proportion $\mathbf{1{:}\ \varphi}$, as shown below in **Figure 8**.

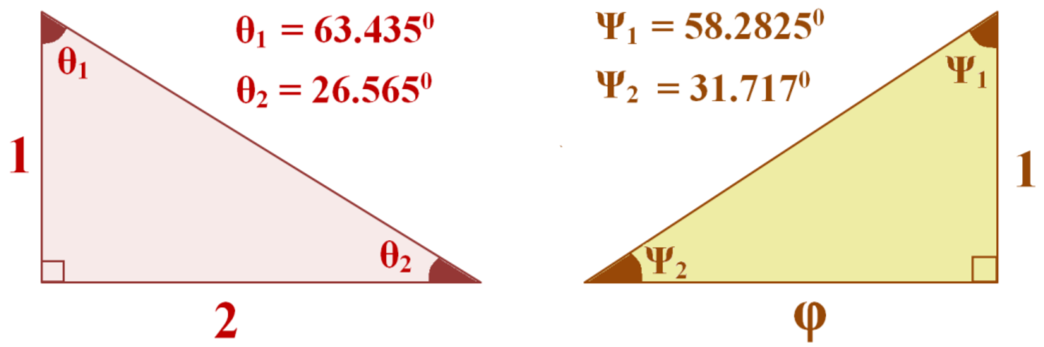


**Figure 8:** The Right Angled Triangles corresponding to the Golden Ratio

Noticeably, a close correspondence is observed between the angles of these two right triangles associated with the Golden Ratio, for instance the angle $\mathbf{26.565^0}$ of $1{:}2{:}\sqrt{5}$ triangle is the **complementary angle** for **twice the** $\mathbf{31.717^0}$ angle of the $\mathbf{1{:}\ \varphi}$ right triangle.

And, the angle $\mathbf{63.435^0}$ of $1{:}2{:}\sqrt{5}$ triangle is the **supplementary angle** for **twice the** $\mathbf{58.2825^0}$ angle of the $\mathbf{1{:}\ \varphi}$ right triangle.

Further, two smaller acute angles of the two right triangles, $\mathbf{26.565^0}$ & $\mathbf{31.717^0}$**,** add up to angle $\mathbf{58.2825^0}$ of the $\mathbf{1{:}\ \varphi}$ right triangle, while the $\mathbf{63.435^0}$ angle of $1{:}2{:}\sqrt{5}$ triangle is twice the angle $\mathbf{31.717^0}$ of the $\mathbf{1{:}\ \varphi}$ right triangle.

And remarkably, such geometric relationship between the Fractional Expression Triangle for $\boldsymbol{\delta}_n$, and the right triangle having its catheti in proportion $\mathbf{1{:}\ \delta_n}$ can be generalised as follows.

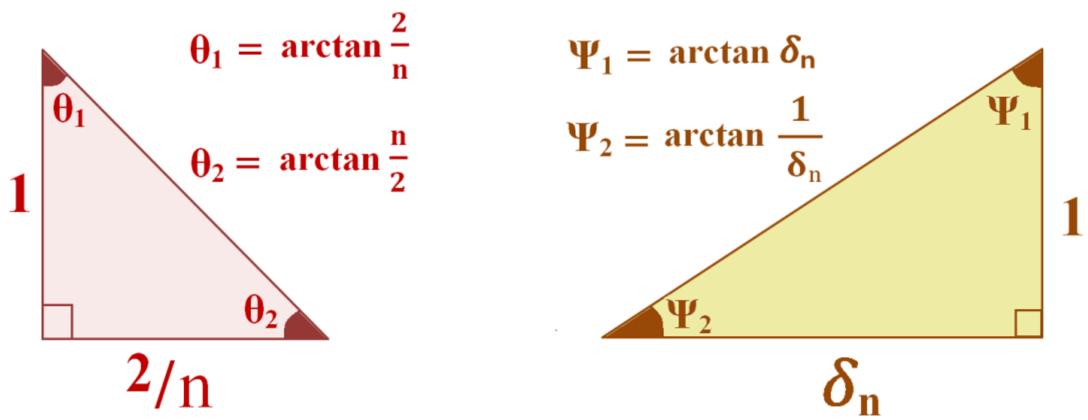


**Figure 9:** Couple of Right Angled Triangles corresponding to the n^th^ Metallic Mean $\boldsymbol{\delta}_n$



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In above **Figure 9**, the two right angled triangles associated with the $n^{th}$ Metallic Mean ($\boldsymbol{\delta_n}$) exhibit the generalised relationships as follows;

$$\arctan\frac{n}{2} \text{ is the } \textbf{Complimentary Angle} \text{ of } 2\arctan\frac{1}{\delta_n}$$

$$\arctan\frac{2}{n} \text{ is the } \textbf{Supplementary Angle} \text{ of } 2\arctan\delta_n$$

$$2\arctan\frac{1}{\delta_n} = \arctan\frac{2}{n}$$

$$\arctan\frac{1}{\delta_n} + \arctan\frac{n}{2} = \arctan\delta_n$$

And an important geometric aspect of the **1:2:$\sqrt{5}$ triangle** is worth mentioning here. The 1:2:$\sqrt{5}$ right triangle is the Fractional Expression Triangle for Golden Ratio, and hence it exhibits the abovementioned peculiar relationship with **1: $\boldsymbol{\varphi}$** right triangle. However, noticeably the catheti of 1:2:$\sqrt{5}$ triangle are equal to the $2^{nd}$ and $3^{rd}$ Fibonacci numbers, and remarkably this triangle exhibits the similar but inverse geometric relationship with **3-4-5 Pythagorean Triple**. The angle **$36.87^0$** of 3-4-5 triangle is the **complementary angle** for **twice the $26.565^0$** angle of 1:2:$\sqrt{5}$ triangle, while the angle **$53.13^0$** of 3-4-5 triangle is the **supplementary angle** for **twice the $63.435^0$** angle of 1:2:$\sqrt{5}$ triangle. Further, two smaller acute angles of the two triangles, **$26.565^0$** & **$36.87^0$,** add up to angle **$63.435^0$** of 1:2:$\sqrt{5}$ triangle, while the angle **$53.13^0$** of Pythagorean triple is twice the **$26.565^0$** angle of 1:2:$\sqrt{5}$ triangle. The classical geometric relationship between these 1:2:$\sqrt{5}$ and 3-4-5 right triangles has been described in detail in the work mentioned in Reference **[1].**

**Conclusion:**

This paper illustrated and generalised the geometric relationship between the Integer Sequence and the corresponding Lucas Sequence, associated with a Metallic Ratio. The couple of right triangles, having their catheti equalling certain terms of the Integer Sequence and the corresponding terms the Lucas Sequence, exhibit an intriguing geometric relationship, that substantiates the corresponding Metallic Mean.

Moreover, this work also elaborated the concept of the "Fractional Expression Triangle" that provides the accurate fractional expression of any $n^{th}$ Metallic Mean ($\boldsymbol{\delta_n}$). Various geometric features of such "Fractional Expression Triangle" are explicated here, and all those features are found to be the obvious manifestations of the corresponding Metallic Ratio.

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