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Metallic Means and Right Triangles
The Geometric Substantiation of all Metallic Ratios

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Abstract

This paper introduces certain new geometric aspects of the Metallic Ratios. Each Metallic Ratio is observed to be closely associated with a special right triangle, which provides the precise fractional expression of that Metallic Ratio. This work explicates the geometric substantiation of each Metallic Mean, on basis of the right angled triangle which is the quintessential form of that particular ratio. Every single feature of such special right triangle is the geometric expression of corresponding Metallic Mean.

Keywords: Metallic Mean, Golden Ratio, Fibonacci sequence, Pi, Phi, Pythagoras Theorem, Divine Proportion, Silver Ratio, Golden Mean, Right Triangle, Pell Numbers, Lucas Numbers, Golden Proportion, Metallic Ratio

Introduction

The prime objective of this work is to introduce and elaborate the geometry of Metallic Mean, on basis of the special right angled triangle, which provides the precise fractional expression of that Metallic Ratio.

Each Metallic Mean $\boldsymbol{\delta_n}$ is the root of the simple Quadratic Equation $\mathbf{X^2 - nX - 1 = 0}$, where $\mathbf{n}$ is any positive natural number. Thus, the fractional expression of the n^{th} Metallic Ratio is $\boldsymbol{\delta_n} = \dfrac{\mathbf{n + \sqrt{n^2 + 4}}}{\mathbf{2}}$

Moreover, each Metallic Ratio can be expressed as the continued fraction:

$$\boldsymbol{\delta_n} = \mathbf{n} + \cfrac{\mathbf{1}}{\mathbf{n} + \cfrac{\mathbf{1}}{\mathbf{n} + \cfrac{\mathbf{1}}{\mathbf{n} + \dots}}} \quad ; \quad \text{And hence, } \boldsymbol{\delta_n} = \mathbf{n} + \frac{\mathbf{1}}{\boldsymbol{\delta}\mathbf{n}}$$

Further, the various Metallic Ratios are mathematically related to each other. The explicit formulae those provide the precise mathematical relationships between different Metallic Means have been discussed in detail in the work mentioned in Reference **[3]**.

More importantly, each Metallic Ratio can be accurately expressed with a special Right Angled Triangle. Any n^{th} Metallic Mean can be represented by the Right Triangle having its catheti **1** and $\frac{\mathbf{2}}{\mathbf{n}}$. Hence, the right triangle with one of its catheti = **1** may substantiate any Metallic Mean, having its second cathetus = $\frac{\mathbf{2}}{\mathbf{n}}$, where n = 1 for Golden Ratio, n = 2 for Silver Ratio, n = 3 for Bronze Ratio, and so on **[1]**.

Such Right Triangle provides the precise value of n^{th} Metallic Mean by the generalised formula:

$$\text{The } n^{th} \text{ Metallic Mean } \boldsymbol{\delta}_n = \frac{\text{Cathetus 1 } + \text{ Hypotenuse}}{\text{Second Cathetus}} = \frac{1 + \text{Hypotenuse}}{2/n}$$

The main aim of this work is to put forward the concept of such "Fractional Expression Triangle" that not only provides the accurate fractional expression of any n^{th} Metallic Mean ($\boldsymbol{\delta}_\mathbf{n}$), but also has its every geometric feature as the prototypical form of that Metallic Mean.

The Fractional Expression Triangle :

The 1:2:$\sqrt{5}$ Triangle is observed to be the quintessential form of Golden Ratio (φ). This 1:2:$\sqrt{5}$ right triangle, with all its peculiar geometric features, described in the work mentioned in Reference **[1]**, turns out to be the real 'Golden Ratio Trigon' in every sense of the term. The characteristic geometry of 1:2:$\sqrt{5}$ triangle, which is resplendent with Phi (φ)=1.618....., provides the most remarkable expression of the First Metallic Ratio viz. the Golden Ratio. And likewise, the similar Right Triangles can provide for the geometric substantiation of all Metallic Means. As mentioned earlier, any $\mathbf{n^{th}}$ Metallic Mean can be accurately represented by the Right Triangle having its catheti **1** and $\frac{\mathbf{2}}{\mathbf{n}}$.

Hence, just as the 1:2:$\sqrt{5}$ Triangle provides the Fractional Expression of Golden Ratio $(\boldsymbol{\varphi}) = \frac{\mathbf{1+\sqrt{5}}}{\mathbf{2}}$, likewise the right triangle 1:1: $\sqrt{2}$ provides geometric substantiation of the Silver Ratio $\boldsymbol{\delta_2} = \frac{\mathbf{1+\sqrt{2}}}{\mathbf{1}} =$ 2.41421356...., similarly the right triangle with its catheti **1** and $\frac{\mathbf{2}}{\mathbf{3}}$ accurately represents the Bronze Ratio, and so on **[1]**,**[2]**.

Hence, such right angled triangle, with its catheti **1** and $\frac{\mathbf{2}}{\mathbf{n}}$; that provides the fractional expression of the $\mathbf{n^{th}}$ Metallic Mean, can be correctly termed as the **"Fractional Expression Triangle"** for the corresponding Metallic Ratio ($\boldsymbol{\delta}_\mathbf{n}$).

Such "Fractional Expression Triangle" of any n^{th} Metallic Mean exhibits certain peculiar geometric features. For instance, the angles, side lengths as well as every geometric feature of such triangle are the precise expressions of that Metallic Ratio ($\boldsymbol{\delta}_\mathbf{n}$).

For example, consider the 1:2:$\sqrt{5}$ right triangle that represents the Golden Ratio (φ). As shown below in **Figures 1 (A):** Note, the angles of the 1:2:$\sqrt{5}$ triangle are precisely expressed in terms of the Golden Ratio.

And **Figures 1 (B):** The division of the sides of :2:$\sqrt{5}$ triangle by the touch points of the Incircle, which are also the vertices of Gergonne Triangle, is also exactly in terms of the Golden Ratio (**φ**).

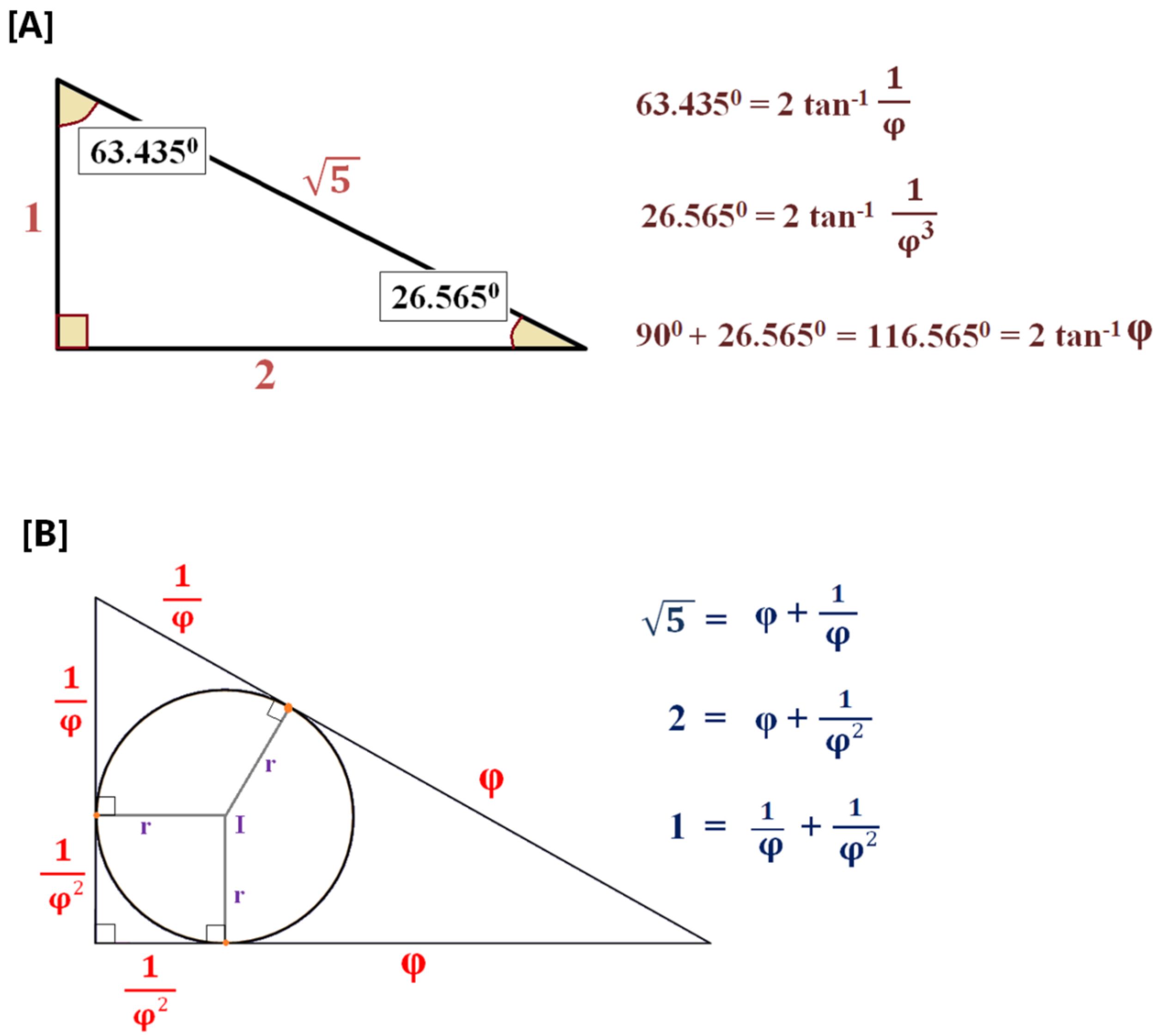

Figure 1: (A) The Angles of 1:2:$\sqrt{5}$ Triangle in terms of Golden Ratio, **(B)** The Side Lengths of 1:2:$\sqrt{5}$ Triangle in terms of Golden Ratio

Remarkably, all these geometric features can be generalised for the "Fractional Expression Triangle" representing any n[th] Metallic Ratio ($\boldsymbol{\delta_n}$), as illustrated below in **Figure 2.**

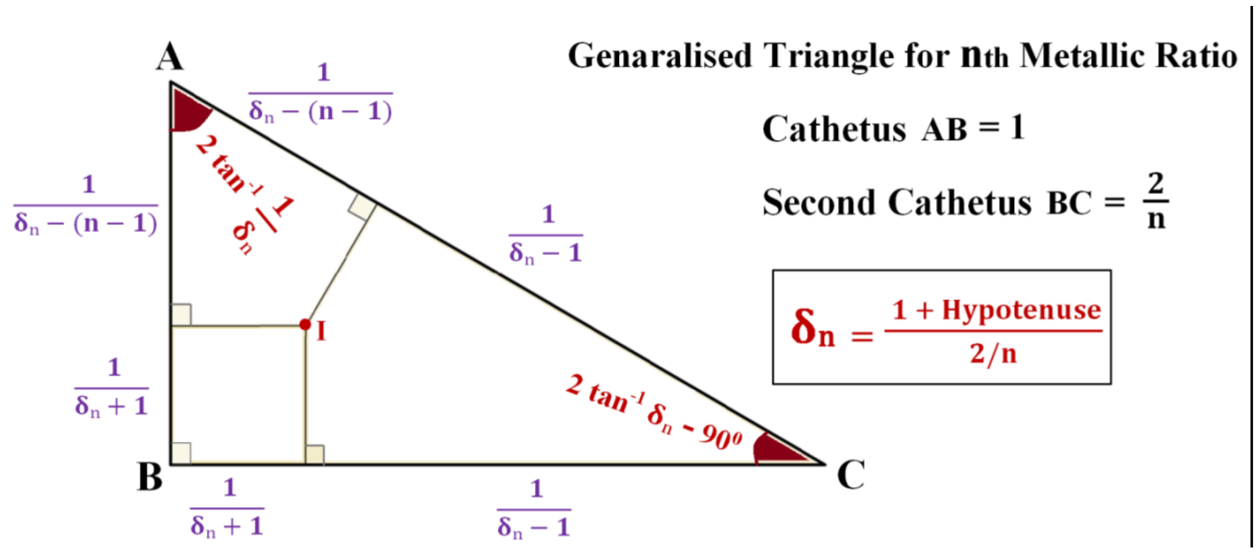

Figure 2: The Generalised 'Fractional Expression Triangle' for n^{th} Metallic Mean.

Moreover, several other geometric features of such "Fractional Expression Triangle" can also be generalised as follows.

Semiperimeter of this Fractional Expression Triangle = $\frac{\delta_n}{\delta_n - 1}$, for example, Semiperimeter of the 1:2:$\sqrt{5}$ right triangle representing the Golden Ratio is φ^2 and that of the 1:1: $\sqrt{2}$ right triangle representing the Silver ratio is $\frac{\sqrt{2}+1}{\sqrt{2}}$

Also, the Inradius of the generalised Fractional Expression Triangle = $\frac{1}{\delta_n + 1}$

For example, just as $\frac{1}{\varphi + 1}$ i.e. $\frac{1}{\varphi^2}$ is the Inradius of the 1:2:$\sqrt{5}$ triangle, $\frac{1}{\delta_2 + 1}$ is the Inradius of the 1:1: $\sqrt{2}$ triangle.

And noticeably, a fascinating relationship is observed between such Fractional Expression Triangle and its Incircle. A precise ratio is observed to exist between the Fractional Expression Triangle and its Incircle, and that ratio is an intriguing expression in terms of Pi $(\boldsymbol{\pi})$ and the corresponding Metallic Ratio $(\boldsymbol{\delta_n})$.

$$\frac{\text{Area of this Fractional Expression Triangle}}{\text{Area of Its Incircle}} = \frac{(\boldsymbol{\delta n} + \mathbf{1})^2}{\mathbf{n\pi}}$$; where $\boldsymbol{\delta_n}$ is the n^{th} Metallic Ratio.

For example, $$\frac{\text{Area of 1:2:}\sqrt{5}\text{ Triangle}}{\text{Area of Its Incircle}} = \frac{\text{Perimeter of 1:2:}\sqrt{5}\text{ Triangle}}{\text{Circumference of Its Incircle}} = \frac{\boldsymbol{\varphi}^4}{\boldsymbol{\pi}}$$

Likewise, $$\frac{\text{Area of the 1:1:}\sqrt{2}\text{ Triangle}}{\text{Area of Its Incircle}} = \frac{\text{Perimeter of 1:1:}\sqrt{2}\text{ Triangle}}{\text{Circumference of Its Incircle}} = \frac{(\textbf{Silvr Ratio})^2}{\boldsymbol{\pi}}$$

Moreover, such Fractional Expression Triangle is also the Limiting Triangle for the Pythagorean Triples formed with the Hypotenuses those equal the alternate terms of the Integer Sequence associated with that Metallic Mean. For example, the Pythagorean triples derived from Fibonacci series, approach the 1:2:$\sqrt{5}$ triangle's proportions, as the series advances. Pythagorean triples can be formed with the alternate Fibonacci numbers as the hypotenuses, like **5**-4-3, **13**-12-5, **34**-30-16, **89**-80-39, **233**-208-105, **610**-546-272 and so on. And, as such series of **Fibonacci-Pythagorean Triples** advances, the triples so formed, invariably approach 1:2:$\sqrt{5}$ triangle proportions, exactly in the same manner as the ratio between consecutive Fibonacci numbers approaches the Golden Ratio: $\lim_{n\to\infty}\frac{\mathrm{Fn}}{\mathrm{Fn-1}} \cong \varphi$. And hence, it clearly endorses: while φ is the Golden Ratio in nature, the 1:2:$\sqrt{5}$ triangle is truly the Golden Trigon in geometry, in every sense of the term **[1]**.

Likewise, the Pythagorean triples derived from **Pell Numbers series**, approach the 1:1:$\sqrt{2}$ triangle's proportions, as the series advances. The Pythagorean triples can be formed with the alternate Pell numbers as the hypotenuses, like **5**-4-3, **29**-21-20, **169**-120-119, **985**-697-696, **5741**-4060-4059 and so on. And, as such series of **Pell-Pythagorean Triples** advances, the triples so formed invariably approach **1:1:$\sqrt{2}$** limiting triangle proportions.

More importantly, the so called Fractional Expression Triangle for any **nth** Metallic Mean is found to be closely related to another right triangle, which is associated with that **nth** Metallic Mean ($\boldsymbol{\delta_n}$). Consider the right triangle having its catheti in proportion $\mathbf{1{:}\delta_n}$. Such $\mathbf{1{:}\delta_n}$ right triangle has an interesting geometric relationship with the abovementioned Fractional Expression Triangle.

For instance, consider the couple of right triangles associated with Golden Ratio (φ): the **1:2:$\sqrt{5}$** triangle which is the Fractional Expression Triangle for the Golden Ratio, and the right triangle having its catheti in proportion $\mathbf{1{:}\ \varphi}$, as shown below in **Figure 3**.

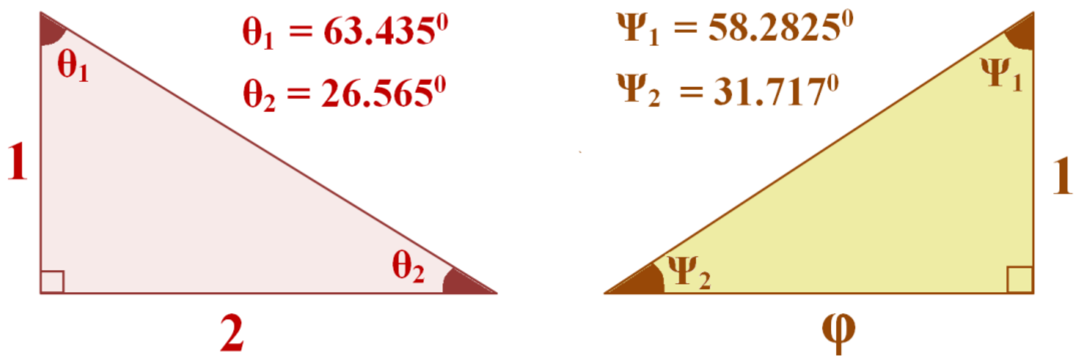

Figure 3: The Right Angled Triangles corresponding to the Golden Ratio

Remarkably, a close correspondence is observed between the angles of these two right triangles associated with Golden Ratio, for instance the angle **26.565^0** of 1:2:$\sqrt{5}$ triangle is the **complementary angle** for **twice the 31.717^0** angle of the $\mathbf{1{:}\ \varphi}$ right triangle.

And, the angle **63.435^0** of 1:2:$\sqrt{5}$ triangle is the **supplementary angle** for **twice the 58.2825^0** angle of the $\mathbf{1{:}\ \varphi}$ right triangle.

Further, two smaller acute angles of these two right triangles, **26.565⁰** & **31.717⁰,** add up to angle **58.2825⁰** of the **1: φ** right triangle, while the **63.435⁰** angle of 1:2:$\sqrt{5}$ triangle is twice the angle **31.717⁰** of the **1: φ** right triangle.

Similarly, consider **1:1:$\sqrt{2}$** triangle which is the Fractional Expression Triangle for Silver Ratio, and the right triangle having its catheti in proportion **1**: $\boldsymbol{\delta_2}$, as shown below in **Figure 4**.

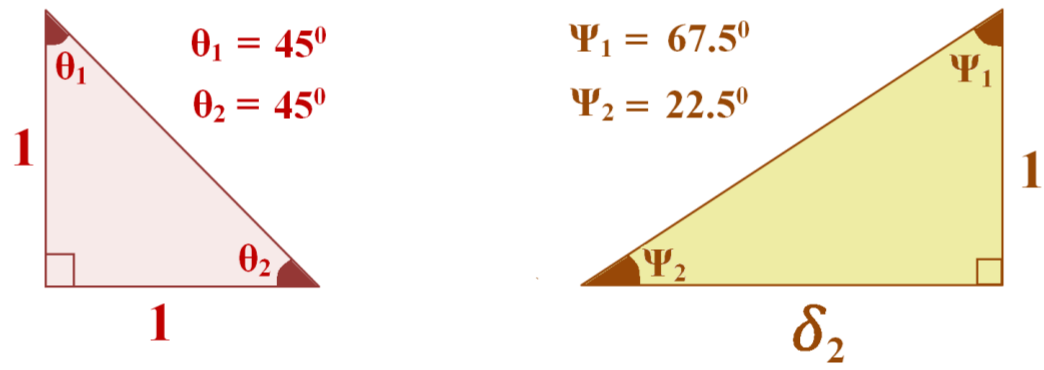

Figure 4: The Right Angled Triangles corresponding to the Silver Ratio

Noteworthy here, the angle $\boldsymbol{\theta_2}$ **= 45⁰** of 1:1:$\sqrt{2}$ triangle is the **complementary angle** for **twice the 22.5⁰** angle of the **1:** $\boldsymbol{\delta_2}$ right triangle.

And, the angle $\boldsymbol{\theta_1}$ **= 45⁰** of 1:1:$\sqrt{2}$ triangle is the **supplementary angle** for **twice the 67.5⁰** angle of the **1:** $\boldsymbol{\delta_2}$ right triangle.

Further, two smaller acute angles of the two right triangles, **45⁰** & **22.5⁰,** add up to angle **67.5⁰** of the **1:** $\boldsymbol{\delta_2}$ right triangle, while the **45⁰** angle of 1:1:$\sqrt{2}$ triangle is twice the angle **22.5⁰** of the **1:** $\boldsymbol{\delta_2}$ right triangle.

Hence, the relationship between the Fractional Expression Triangle for $\boldsymbol{\delta_n}$, and the right triangle having its catheti in proportion **1:** $\boldsymbol{\delta_n}$ can be generalised as follows.

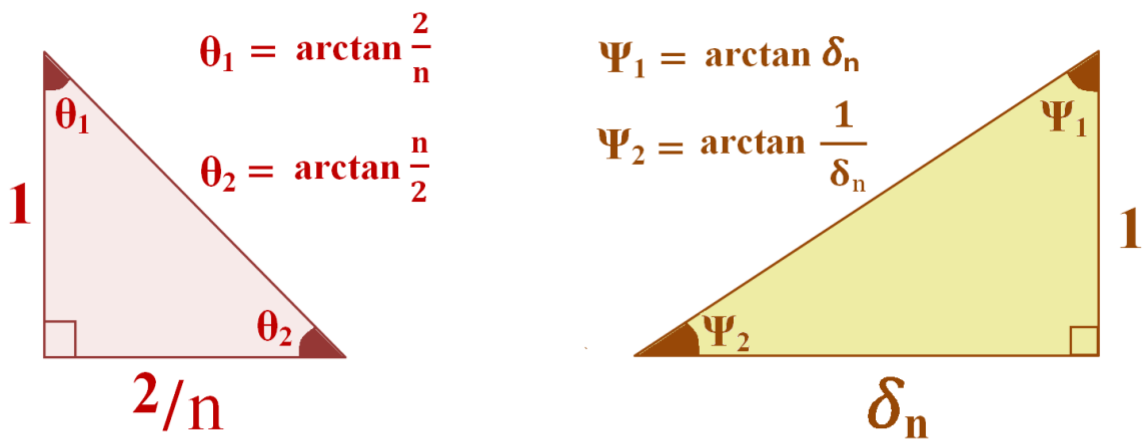

Figure5: Couple of Right Angled Triangles corresponding to the n^{th} Metallic Mean δ_n

In above **Figure 5**, the two right angled triangles corresponding to the n^{th} Metallic Mean $\boldsymbol{\delta_n}$ exhibit the generalised relationships as follows;

$\mathbf{arctan\,\frac{n}{2}}$ is the **Complimentary Angle** of $\mathbf{2\ arctan\,\frac{1}{\delta_n}}$

$\mathbf{arctan\,\frac{2}{n}}$ is the **Supplementary Angle** of $\mathbf{2\ arctan\,\delta_n}$

$$\mathbf{2\ arctan\,\frac{1}{\delta_n} = arctan\,\frac{2}{n}}$$

$$\mathbf{arctan\,\frac{1}{\delta_n} + arctan\,\frac{n}{2} = arctan\,\delta_n}$$

Noticeably, the **1:2:$\sqrt{5}$ triangle**, which is the Fractional Expression Triangle for Golden Ratio, exhibits the similar but inverse relationship with **3-4-5 Pythagorean Triple**. The angle **36.87^0** of 3-4-5 triangle is the **complementary angle** for **twice the 26.565^0** angle of 1:2:$\sqrt{5}$ triangle, while the angle **53.13^0** of 3-4-5 triangle is the **supplementary angle** for **twice the 63.435^0** angle of 1:2:$\sqrt{5}$ triangle. Further, two smaller acute angles of the two triangles, **26.565^0** & **36.87^0,** add up to angle **63.435^0** of 1:2:$\sqrt{5}$ triangle, while the angle **53.13^0** of Pythagorean triple is twice the **26.565^0** angle of 1:2:$\sqrt{5}$ triangle. The classical geometric relationship between these two right triangles has been described in detail in the work mentioned in Reference **[1].**

Conclusion:

This paper introduced and elaborated the concept of a special right triangle, called Fractional Expression Triangle, which provides the precise fractional expression of a Metallic Mean. The geometry of such Fractional Expression Triangle is observed to be respledent with the corresponding Metallic Mean. The geometric features of such right triangle were explicated and generalised for any n^{th} Metallic Ratio. Moreover, the paper also discussed the intriguing relationship between such Fractional Expression Triangle for the $\mathbf{n^{th}}$ Metallic Mean ($\boldsymbol{\delta}_n$) and the right triangle having its catheti in proportion **1:$\boldsymbol{\delta_n}$**

References:

[1] Rajput, Chetansing (2021). Golden Ratio. *JOURNAL OF ADVANCES IN MATHEMATICS*, *20*, 19–42. **https://doi.org/10.24297/jam.v20i.8945**

[2] Rajput, Chetansing (2021). Golden Ratio and other Metallic Means: The Geometric Substantiation of all Metallic Ratios with Right Triangles. *JOURNAL OF ADVANCES IN MATHEMATICS*, *20*, 174-187. **https://doi.org/10.24297/jam.v20i.9034**

[3] Rajput, Chetansing (2021). Metallic Ratios : Beyond the Golden Ratio; The Mathematical Relationships between different Metallic Means. *JOURNAL OF ADVANCES IN MATHEMATICS*, *20*, 158-166. **https://doi.org/10.24297/jam.v20i.9023**