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Boundedness of the gradient of a solution for the fourth order equation in general domains

Wei Liu, Gao Jia, Honghong Qi
University of Shanghai for Science and Technology, Shanghai 200093, China
luvet@126.com, gaojia89@163.com, 535684285@qq.com

ABSTRACT

Based on new integral estimate, we establish boundedness of the gradient of a solution for a fourth order equation in an arbitrary three-dimensional domain

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1 Introduction

Higher-order elliptic boundary problems have abundant applications in physics and engineering [10] and have also been studied in many areas of mathematics, including conformal geometry (Paneitz operator, Q-curvature [2, 3]) and non-linear elasticity [4].

Unfortunately, we know little about fundamental properties of the solutions to general higher order PDEs, such as boundedness, continuity and regularity near a boundary point. Their investigation brought challenging hypotheses and surprising counterexamples. For instance, Hadamard's 1908 conjecture regarding positivity of the biharmonic Green function [6] was actually refuted in 1949 (see [5]). In the case of higher order equations, the maximum principle has been established only in relatively nice domains. In 1960 the maximum principle has been established only in relatively nice domains. In 1960 the maximum principle has been extended to higher order elliptic equations on smooth domains, and later, in the beginning of 90's, to three-dimensional domains diffeomorphic to a polyhedron [8] or having a Lipschitz boundary [11]. In particular, it ensures that in such domains a biharmonic function satisfies the weak maximum principle

$$\|\nabla u\|_{L^\infty(\overline{\Omega})} \le C\|\nabla u\|_{L^\infty(\partial\Omega)}. \qquad (1.1)$$

Since without direct analogues of (1.1) for higher order equations in general domains, the properties of the solutions become more involved. To be more specially, let $\Omega \subset R^n$ be a bounded domain and consider the boundary value problem

$$\Delta^2 u = f(x) \;\; in \;\; \Omega, \; u \; \in \; W_0^{2,2}(\Omega), \qquad (1.2)$$

where $W_0^{2,2}(\Omega)$ is a completion of $C_0^\infty(\Omega)$ in the norm of the Sobolev space $W^{2,2}(\Omega)$, and f is a reasonably nice function (e.g.$C_0^\infty(\Omega)$). Motivated by (1.1), we naturally ask if the gradient of a solution to (1.2) is bounded in an arbitrary domain $\Omega \subset R^n$. It turns out that this property may fail when $n \ge 4$ (see the counterexamples in [9]). In dimension three the boundedness of the gradient of a solution was an open problem.

Recently, S. Mayboroda and V. Maz'ya [12] solved the open problem. They state the boundedness of the gradient of the solution to (1.2) under no restrictions on the underlying domain. It is a sharp property in the sense that the function u satisfying (1.2) generally does not exhibit more regularity. In paper [13], they expand the biharmonic operator Δ^2 to the general polyharmonic operator $(-\Delta)^m$, i.e., the following equaiton

$$(-\Delta)^m u = f(x) \;\; in \;\; \Omega, \; u \; \in \; W_0^{2,2}(\Omega), \qquad (1.3)$$

where $m \in \mathbb{N}$ and f is a reasonably nice function. They establish boundedness of derivatives $[m - \frac{n}{2} + \frac{1}{2}]$ for the solutions to (1.3) without any restrictions on the geometry of the underlying domain but in $2 \le n \le 2m+1$. It is shown that this result is sharp and cannot be improved in general domains.

In this paper, our main result is

Theorem 1.1 Let Ω be an arbitrary bounded domain in $\mathbb{R}^3$, and

$$\Delta^2 u - a_0 \Delta u + a_1 u = f(x) \;\; \text{in } \Omega, \; u \in W_0^{2,2}(\Omega), \qquad (P)$$

where a_0, a_1 are non-negative constants and $f(x) \in C_0^\infty(\Omega)$. Then the solution to the boundary value problem (P) satisfies

$$|\nabla u| \in L^\infty(\Omega). \qquad (1.4)$$

The present paper establishes pointwise estimates on variational solutions to (P) in an arbitrary three- dimensional bounded domain. It is shown that the boundedness of the gradient of a solution to (P) is a sharp property and can not be improved (see the counterexamples in [12] and [13]).

The paper is organized as follows. In Section 2, we give some notations and main integral global estimate. In Section 3, we obtain local L^2 estimate and accomplish the proof of Theorem 1.1.

2 Notations and Integral global estimate

First, we give some notations: $\mathbb{S}^2$: the unit sphere in $\mathbb{R}^3$; δ_ω: the Laplace- Beltrami operator on $\mathbb{S}^2$; ∇_ω: the gradient on $\mathbb{S}^2$; C, C_i : various positive constants, the exact values of which are not important; $B_r(Q)$: the ball with radius r centered at Q; B_r: the ball with radius r centered at the origin; $S_r(Q)$: the sphere with radius r centered at Q; S_r: the sphere with radius r centered at the origin; $C_{r,R}(Q) = B_R(Q) \backslash \overline{B_r(Q)}$; $C_{r,R} = B_R \backslash \overline{B_r}$; $T_r(Q) = B_r(Q) \cap \Omega$; $T_{r,R}(Q) = C_{r,R}(Q) \cap \Omega$; $I_r(Q) = B_r(Q) \cap \partial\Omega$; $d(x)$: the distance from x to $\partial\Omega$; $A \approx B$: $C^{-1}A \le B \le CA$ for some $C > 0$.

Let (r,ω) be sphere coordinates in $\mathbb{R}^3$, i.e. $r=|x|\in(0,\infty)$ and $\omega=\frac{x}{|x|}$ is a point of $\mathbb{S}^2$. we usual write the sphere coordinates as (r,θ,ϕ), where $\theta\in[0,2\pi)$, and $\phi\in[0,\pi]$, thus

$$\omega=\frac{x}{|x|}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta).$$

Since it is more convenient that we use $t=\log|x|^{-1}$, we denote the mappings by χ

$$\mathbb{R}^3\ni x\xrightarrow{\chi}(t,\omega)\in\mathbb{R}\times\mathbb{S}^2.$$

Lemma 2.1 Let Ω be an open set in $\mathbb{R}^3$, a_0,a_1 are non-negative constants, $u\in C_0^\infty(\Omega),\ v_1=e^t(u\circ\chi^{-1}),v_2=u\circ\chi^{-1}$ and $v_3=e^{-t}u\circ\chi^{-1}$. Then

$$\begin{aligned}&\int_{\mathbb{R}^3}(\Delta^2u(x)-a_0\Delta u(x)+a_1u)u(x)|x|^{-1}g\left(\log|x|^{-1}\right)dx\\&=\int_{\mathbb{R}}\int_{\mathbb{S}^2}[(\delta_\omega v_1)^2g+2(\partial_t\nabla_\omega v_1)^2g+(\partial_t^2v_1)^2g-(\nabla_\omega v_1)^2(\partial_t^2g+\partial_tg+2g)\\&+a_0|\nabla_\omega v_2|^2g-(\partial_tv_1)^2(2\partial_t^2g+3\partial_tg-g)+a_0(\partial_tv_2)^2g+a_1v_3^2g\\&+\frac{1}{2}v_1^2(\partial_t^4g+2\partial_t^3g-\partial_t^2g-2\partial_tg)-\frac{a_0}{2}v_2^2(\partial_t^2g+\partial_tg)]d\omega dt,\end{aligned}\tag{2.1}$$

for every function g on $\mathbb{R}$ such that both side of (2.1) are well-defined.

Proof. It is well-known that the Laplace operator in three dimension can be written by

$$\Delta=e^{2t}(\partial_t^2-\partial_t+\delta_w).$$

Let us start the spherical coordinates $\mathbb{R}^3\ni x\xrightarrow{\chi}(t,\omega)\in\mathbb{R}\times\mathbb{S}^2$. Then

$$\begin{aligned}&\int_{\mathbb{R}^3}(\Delta^2u(x)-a_0\Delta u(x)+a_1u)u(x)|x|^{-1}g\left(\log|x|^{-1}\right)dx\\&=\int_{\mathbb{R}^3}\Delta u(x)\Delta(u(x)|x|^{-1}g\left(\log|x|^{-1}\right))dx\\&-a_0\int_{\mathbb{R}^3}\Delta u(x)u(x)|x|^{-1}g\left(\log|x|^{-1}\right)dx+a_1\int_{\mathbb{R}^3}u(x)u(x)|x|^{-1}g\left(\log|x|^{-1}\right)dx\\&=\int_{\mathbb{R}}\int_{\mathbb{S}^2}[(\partial_t^2-3\partial_t+2+\delta_w)v_1(t,w)(\partial_t^2-\partial_t+\delta_w)(v_1(t,w)g(t))]dwdt\\&-a_0\int_{\mathbb{R}}\int_{\mathbb{S}^2}[(\partial_t^2-\partial_t+\delta_w)v_2(t,w)\cdot v_2(t,w)g(t))]dwdt+a_1\int_{\mathbb{R}}\int_{\mathbb{S}^2}v_3^2(t,w)g(t))dwdt\\&=I_1-I_2+I_3.\end{aligned}\tag{2.2}$$

First,

$$\begin{aligned}I_1=&\int_{\mathbb{R}}\int_{\mathbb{S}^2}[(\delta_\omega v_1)^2g+2(\partial_t\nabla_\omega v_1)^2g+(\partial_t^2v_1)^2g-(\nabla_\omega v_1)^2(\partial_t^2g+\partial_tg+2g)\\&-(\partial_tv_1)^2(2\partial_t^2g+3\partial_tg-g)+\frac{1}{2}v_1^2(\partial_t^4g+2\partial_t^3g-\partial_t^2g-2\partial_tg)]dwdt.\end{aligned}\tag{2.3}$$

(see the reference [12]).

Next, we calculate I_2. Since

$$\int_{\mathbb{R}}\int_{\mathbb{S}^2} \partial_t^2 v_2(t,w)\cdot v_2(t,w)g(t)d\omega dt$$
$$= -\int_{\mathbb{R}}\int_{\mathbb{S}^2} (\partial_t v_2(t,w))^2 g(t)dwdt + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{S}^2} v_2^2(t,w)\partial_t^2 g(t)dwdt,$$
$$\int_{\mathbb{R}}\int_{\mathbb{S}^2} \partial_t v_2(t,w)\cdot v_2(t,w)g(t)dwdt = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{S}^2} v_2^2(t,w)\partial_t g(t)dwdt,$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{S}^2} \delta_w v_2(t,w)\cdot v_2(t,w)g(t)dwdt = -\int_{\mathbb{R}}\int_{\mathbb{S}^2} |\nabla_w v_2(t,w)|^2 g(t)dwdt.$$

Thus

$$I_2 = a_0\int_{\mathbb{R}}\int_{\mathbb{S}^2} (-\partial_t v_2(t,w))^2 g(t) - |\nabla_w v_2(t,w)|^2 g(t) + \frac{1}{2}v_2^2(t,w)(\partial_t^2 g(t) + \partial_t g(t))dwdt. \quad (2.4)$$

Thus (2.1) holds.

Lemma 2.2 Consider the following two ordinary differential equations

$$\frac{d^4 g_1(t)}{dt^4} + 2\frac{d^3 g_1(t)}{dt^3} - \frac{d^2 g_1(t)}{dt^2} - 2\frac{dg_1(t)}{dt} = \delta, \quad (2.5)$$

and

$$\frac{d^2 g_2(t)}{dt^2} + \frac{dg_2(t)}{dt} = \delta, \quad (2.6)$$

where δ is the Dirac delta function. The solutions to (2.5) and (2.6) which are bounded and vanish at $+\infty$ are given by

$$g_1(t) = \begin{cases} -\frac{1}{6}e^t + \frac{1}{2}, & \text{if } t \le 0, \\ -\frac{1}{6}e^{-2t} + \frac{1}{2}e^{-t}, & \text{if } t > 0, \end{cases} \quad (2.7)$$

and

$$g_2(t) = \begin{cases} -1, & \text{if } t \le 0, \\ -e^{-t}, & \text{if } t > 0, \end{cases} \quad (2.8)$$

respectively.

The proof is basic. We omit here.

Lemma 2.3 Let Ω be a bounded domain in $\mathbb{R}^3$, a_0, a_1 be non-negative constants, $O \in \mathbb{R}^3 \setminus \Omega$, $u \in C_0^\infty(\Omega)$ and $v_1 = e^t(u\circ\chi^{-1}), v_2 = u\circ\chi^{-1}, v_3 = e^{-t}u\circ\chi^{-1}$. Then for every $\xi \in \Omega$ and $\tau = \log|\xi|^{-1}$, we have

$$\int_{\mathbb{R}^3} (\Delta^2 u(x) - a_0\Delta u(x) + a_1 u(x))u(x)|x|^{-1}\overline{g}(\log(\frac{|\xi|}{|x|}))dx$$
$$\ge \frac{1}{2}\int_{\mathbb{S}^2} (v_1^2(\tau,\omega) + a_0 v_2^2(\tau,\omega))d\omega,$$

where

$$\overline{g}(t) = \begin{cases} -\frac{1}{6}e^t + \frac{3}{2}, & \text{if } t \le 0, \\ -\frac{1}{6}e^{-2t} + \frac{3}{2}e^{-t}, & \text{if } t > 0. \end{cases} \quad (2.9)$$

Proof. Let

$$\overline{g}(t) = g_1(t) - g_2(t), \tag{2.10}$$

where $g_1(t)$ and $g_2(t)$ are defined in (2.7) and (2.8), respectively. Thus, we have

$$\overline{g}(t) = \begin{cases} -\frac{1}{6}e^{t} + \frac{3}{2}, & \text{if } t \le 0, \\ -\frac{1}{6}e^{-2t} + \frac{3}{2}e^{-t}, & \text{if } t > 0. \end{cases} \tag{2.11}$$

Let us start with the expansion of v_1 by means of spherical harmonic and the eigenvalues of the Laplace-Beltrami operator on the unit sphere in three dimension are $p(p+1),\ (p = 0, 1, 2, ...)$ and we have the inequality

$$\int\limits_{S^2} |\delta_w v_1|^2 dw \ge 2 \int\limits_{S^2} |\nabla_w v_1|^2 dw. \tag{2.12}$$

Now, we replace g (in Lemma 2.1) by $\overline{g}(t-\tau)(t \in \mathbb{R})$. From (2.1), (2.5), (2.6), (2.10) and (2.12), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} (\Delta^2 u(x) - a_0 \Delta u(x)) u(x) |x|^{-1} \overline{g}\left(\log \frac{|\xi|}{|x|}\right) dx \\ &\ge \int\limits_{\mathbb{R}} \int\limits_{S^2} -(\nabla_\omega v_1(t,w))^2 (\partial_t^2 \overline{g}(t-\tau) + \partial_t \overline{g}(t-\tau)) - (\partial_t v_1(t,w))^2 (2\partial_t^2 \overline{g}(t-\tau) \\ &+ 3\partial_t \overline{g}(t-\tau) - \overline{g}(t-\tau)) dw dt \\ &+ \int\limits_{\mathbb{R}} \int\limits_{S^2} -\frac{1}{2} v_1^2(t,w) (\partial_t^4 g_2(t-\tau) + 2\partial_t^3 g_2(t-\tau) - \partial_t^2 g_2(t-\tau) - 2\partial_t g_2(t-\tau)) dw dt \\ &+ \int\limits_{\mathbb{R}} \int\limits_{S^2} -\frac{a_0}{2} v_2^2(t,w) (\partial_t^2 g_1(t-\tau) + \partial_t g_1(t-\tau)) dw dt + \frac{1}{2} \int\limits_{S^2} (v_1^2(\tau,\omega) + a_0 v_2^2(\tau,\omega)) d\omega. \end{aligned} \tag{2.13}$$

In order to prove Lemma 2.3, our goal is to show that the follow inequalities

$$\begin{cases} \partial_t^2 \overline{g}(t-\tau) + \partial_t \overline{g}(t-\tau) \le 0, \\ 2\partial_t^2 \overline{g}(t-\tau) + 3\partial_t \overline{g}(t-\tau) - \overline{g}(t-\tau) \le 0, \\ \partial_t^4 g_2(t-\tau) + 2\partial_t^3 g_2(t-\tau) - \partial_t^2 g_2(t-\tau) - 2\partial_t g_2(t-\tau) \le 0. \\ \partial_t^2 g_1(t-\tau) + \partial_t g_1(t-\tau) \le 0. \end{cases}$$

First, we compute $\overline{g}(t-\tau)$ and get

$$\partial_t \overline{g}(t-\tau) = \begin{cases} -\frac{1}{6}e^{(t-\tau)}, & \text{if } t \le \tau, \\ \frac{1}{3}e^{-2(t-\tau)} - \frac{3}{2}e^{-(t-\tau)}, & \text{if } t > \tau, \end{cases} \tag{2.14}$$

and

$$\partial_t^2 \overline{g}(t-\tau) = \begin{cases} -\frac{1}{6}e^{(t-\tau)}, & \text{if } t \le \tau, \\ -\frac{2}{3}e^{-2(t-\tau)} + \frac{3}{2}e^{-(t-\tau)}, & \text{if } t > \tau. \end{cases} \tag{2.15}$$

(2.14) and (2.15) give

$$\partial_t^2 \overline{g}(t-\tau) + \partial_t \overline{g}(t-\tau) = \begin{cases} -\frac{1}{3}e^{(t-\tau)}, & \text{if } t \le \tau, \\ -\frac{1}{3}e^{-2(t-\tau)}, & \text{if } t > \tau, \end{cases} \tag{2.16}$$

and

$$2\partial_t^2 \overline{g}(t-\tau) + 3\partial_t \overline{g}(t-\tau) - \overline{g}(t-\tau) = \begin{cases} -\frac{2}{3}e^{(t-\tau)} - \frac{3}{2}, & \text{if } t \le \tau, \\ -\frac{1}{6}e^{-2(t-\tau)} - 3e^{-(t-\tau)}, & \text{if } t > \tau. \end{cases} \tag{2.17}$$

Obviously, the functions (2.16) and (2.17) are non-positive.

Next, we know the fact

$$\partial_t^4 g_2(t-\tau) + 2\partial_t^3 g_2(t-\tau) - \partial_t^2 g_2(t-\tau) - 2\partial_t g_2(t-\tau) = 0. \tag{2.18}$$

Finally, we compute $g_1(t-\tau)$ and obtain

$$\partial_t g_1(t-\tau) = \begin{cases} -\frac{1}{6}e^{(t-\tau)}, & \text{if } t \le \tau, \\ \frac{1}{3}e^{-2(t-\tau)} - \frac{1}{2}e^{-(t-\tau)}, & \text{if } t > \tau, \end{cases} \tag{2.19}$$

and

$$\partial_t{}^2 g_1(t-\tau) = \begin{cases} -\frac{1}{6}e^{(t-\tau)}, & \text{if } t \le \tau, \\ -\frac{2}{3}e^{-2(t-\tau)} + \frac{1}{2}e^{-(t-\tau)}, & \text{if } t > \tau, \end{cases} \tag{2.20}$$

which give

$$\partial_t^2 g_1(t-\tau) + \partial_t^2 g_1(t-\tau) = \begin{cases} -\frac{1}{3}e^{(t-\tau)}, & \text{if } t \le \tau, \\ -\frac{1}{3}e^{-2(t-\tau)}, & \text{if } t > \tau. \end{cases} \tag{2.21}$$

The function (2.21) is non-positive.

3 Local L^2 estimate

To start, we need to establish the following energy estimates for the solutions of the elliptic equations.

Lemma 3.1 Let Ω be an arbitrary domain in $\mathbb{R}^n$, $u \in W^{2,2}(T_{4R}(Q))$ for some $Q \in \mathbb{R}^n \backslash \Omega$ and $R > 0$. Suppose that u satisfies $\Delta^2 u - a_0 \Delta u + a_1 u = 0$ in $T_{4R}(Q)$ for the real constants a_0, a_1 and $u = 0, \nabla u = 0$ on $I_{4R}(Q)$. Then

$$\frac{1}{r^2}\int_{T_r(Q)} |\nabla u|^2 dx + \int_{T_r(Q)} |\nabla^2 u|^2 dx \le C\left(1 + \frac{1}{r^2} + \frac{1}{r^4}\right)\int_{T_{r,2r}(Q)} |u|^2 dx, \tag{3.1}$$

where $0 < r < 2R$ and C is a positive constant only depending on a_0.

Proof. Let $\eta \in C_0^\infty(B_{2r}(Q))$ such that

$0 \le \eta \le 1$ in $B_{2r}(Q)$, $\eta = 1$ in $B_r(Q)$ and $|\nabla^k \eta| \le Cr^{-k}$, for $0 \le k \le 4$.

Since $u \in W^{2,2}(T_{4R}(Q))$ and $u = 0, \nabla u = 0$ on $I_{4R}(Q)$, we have $u\eta^2 \in W_0^{2,2}(T_{4R}(Q))$. We will show that for any $\varepsilon > 0$,

$$\int_\Omega |\nabla^2(u\eta^2)|^2 dx \le \varepsilon \int_\Omega |\nabla^2(u\eta^2)|^2 dx + C(\varepsilon, a_0)\left(1 + \frac{1}{r^2} + \frac{1}{r^4}\right)\int_{T_{r,2r}(Q)} |u|^2 dx. \tag{3.2}$$

This, together with the Poincaré inequality

$$\int_{T_{2r}(Q)} |\nabla(u\eta^2)|^2 dx \le Cr^2 \int_{T_{2r}(Q)} |\nabla^2(u\eta^2)|^2 dx, \tag{3.3}$$

yields the estimate (3.1).

To prove (3.2), we use integration by parts and $\Delta^2 u - a_0 \Delta u + a_1 u = 0$ in $T_{2r}(Q)$ to obtain

$$\int_\Omega |\nabla^2(u\eta^2)|^2 dx = \int_\Omega |\Delta(u\eta^2)|^2 dx = \int_\Omega \{|\Delta(u\eta^2)|^2 - \Delta u \Delta(u\eta^4) + a_0 \Delta u (u\eta^4)\} - a_1 u u \eta^4 dx. \tag{3.4}$$

A direct computation shows that

$$\int_\Omega [\Delta(u\eta^2)\Delta(u\eta^2) - \Delta u\Delta(u\eta^4) + a_0\Delta u(u\eta^4) - a_1 uu\eta^4]dx$$
$$= \int_\Omega [u\Delta(u\eta^2)\Delta(\eta^2) + 4\left|\nabla u\nabla(\eta^2)\right|^2 + 2u(\nabla u\nabla(\eta^2))\Delta(\eta^2)$$
$$- u\Delta u(2\left|\nabla(\eta^2)\right|^2 + \eta^2\Delta(\eta^2) - a_0\eta^4) - a_1 u^2\eta^4]dx. \tag{3.5}$$

By the Hölder inequality, the first term in the right side of (3.5) reduces

$$\int_\Omega u\Delta(u\eta^2)\Delta(\eta^2)dx \le \varepsilon \int_\Omega \left|\nabla^2(u\eta^2)\right|^2 dx + \frac{C_\varepsilon}{r^4}\int_{T_{r,2r}(Q)} |u|^2 dx. \tag{3.6}$$

Since

$$u\frac{\partial u}{\partial x_i}\varphi = \frac{1}{2}\frac{\partial}{\partial x_i}\left(|u|^2\varphi\right) - \frac{1}{2}|u|^2\frac{\partial\varphi}{\partial x_i}, \tag{3.7}$$

which gives estimate of the third term in the right side of (3.5)

$$\int_\Omega u(\nabla u\nabla(\eta^2))\Delta(\eta^2) \le \frac{C}{r^4}\int_{T_{r,2r}(Q)} |u|^2 dx. \tag{3.8}$$

Meanwhile,

$$\eta^2\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\varphi = \frac{\partial}{\partial x_i}\left(\frac{\partial\,(u\eta^2)}{\partial x_j}u\varphi\right) - \frac{\partial^2\,(u\eta^2)}{\partial x_i\partial x_j}u\varphi - u\frac{\partial u}{\partial x_j}\frac{\partial\varphi}{\partial x_i}\eta^2$$
$$- u\frac{\partial u}{\partial x_i}\frac{\partial\eta^2}{\partial x_j}\varphi - u^2\frac{\partial\eta^2}{\partial x_j}\frac{\partial\varphi}{\partial x_i}, \tag{3.9}$$

by (3.8) and (3.9), the second term in the right side of (3.5) has

$$\int_\Omega \left|\nabla u\nabla(\eta^2)\right|^2 \le \varepsilon\int_\Omega \left|\nabla^2(u\eta^2)\right|^2 dx + \frac{C_\varepsilon}{r^4}\int_{T_{r,2r}(Q)} |u|^2 dx. \tag{3.10}$$

For the term $\eta^2 u\Delta u$, we note that

$$\eta^2 u\Delta u\varphi = \frac{\partial}{\partial x_i}(\eta^2 u\frac{\partial u}{\partial x_i}\varphi) - u\frac{\partial u}{\partial x_i}\frac{\partial\,(\eta^2\varphi)}{\partial x_i} - \eta^2|\nabla u|^2\varphi. \tag{3.11}$$

By (3.7), (3.9) and (3.11), the last term in the right side of (3.5) is

$$\int_\Omega -u\Delta u(\left|\nabla(\eta^2)\right|^2 + \eta^2\Delta(\eta^2) + a_0\eta^4 - a_1u^2\eta^4)$$
$$\le \varepsilon\int_\Omega \left|\nabla^2(u\eta^2)\right|^2 dx + C_\varepsilon(|a_0|(1+\frac{1}{r^2}) + \frac{1}{r^4})\int_{T_{r,2r}(Q)} |u|^2 dx. \tag{3.12}$$

Thus, (3.6), (3.8), (3.10) and (3.12) imply that (3.2) holds.

The following Lemma reflects the rate of growth of solutions near a boundary point based on Lemma 2.3.

Lemma 3.2 Let Ω be a bounded domain in $\mathbb{R}^3$, $Q \in \mathbb{R}^3\backslash\Omega$ and $R > 0$. Suppose

$$\Delta^2 u - a_0\Delta u + a_1 u = f(x) \;\text{ in } \Omega,\; u \in W_0^{2,2}(\Omega), \tag{3.13}$$

where a_0, a_1 are non-negative constants and $f(x) \in C_0^\infty(\Omega \setminus B_{4R}(Q))$. Then

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$$\frac{1}{\rho^4}\int\limits_{S_\rho(Q)\cap\Omega}|u|^2d\sigma_x \le C\left(\frac{1}{R^5}+R\right)\int\limits_{T_{R,4R}(Q)}|u|^2dx \quad \text{for every} \quad \rho<R,$$

where *C* is a positive constant depending on a_0.

Proof. Without loss of generality, we consider $Q=O$. Let us approximate Ω by a sequence of domains $\{\Omega_n\}_{n=1}^{\infty}$ with smooth boundaries satisfying

$$\bigcup_{n=1}^{\infty}\Omega_n=\Omega \quad \text{and} \quad \overline{\Omega}_n\subset\Omega_{n+1} \quad \text{for every} \quad n\in N.$$

Choose $n_0\in N$ such that supp$f\subset\Omega_n$ for every $n\ge n_0$ and denote by u_n the solution of the Dirichlet problem

$$\Delta^2u_n-a_0\Delta u_n=f(x) \ \text{ in } \Omega_n, \quad u_n\in W_0^{2,2}(\Omega_n), \quad n\ge n_0.$$

The sequence $\{u_n\}_{n=n_0}^{\infty}$ converges to u in $W_0^{-2,2}(\Omega)$ (see [14]).

Next, let smooth function $\eta\in C_0^\infty(B_{2R})$ such that

$0\le\eta\le1$ in B_{2R}, $\eta=1$ in B_R and $|\nabla^k\eta|\le CR^{-k}$, $\quad k\le4.$

Also, fix $\tau=\log\rho^{-1}$ and let $\overline{g}$ be the function in (2.13).

In particular,

$$\left|\nabla_x^k\overline{g}(\log\frac{\rho}{|x|})\right|\le CR^{-k}, \quad 0\le k\le4, \quad x\in C_{R,2R}, \quad \rho<R. \tag{3.14}$$

Now, Consider the difference

$$\begin{aligned}&\int_{R^3}(\Delta^2(u_n(x)\eta(x))-a_0\Delta(u_n(x)\eta(x))+a_1u_n\eta(x))\cdot(\eta(x)u_n(x)|x|^{-1}\overline{g}(\log\frac{\rho}{|x|}))dx\\&-\int_{R^3}(\Delta^2u_n(x)-a_0\Delta u_n(x)+a_1u_n(x))(u_n(x)|x|^{-1}\overline{g}(\log\frac{\rho}{|x|})\eta^2(x))dx.\end{aligned} \tag{3.15}$$

We view (3.15) as

$$\int_{R^3}([\Delta^2-a_0\Delta+a_1,\eta]u_n(x))(\eta(x)u_n(x)|x|^{-1}\overline{g}(\log\frac{\rho}{|x|})). \tag{3.16}$$

The integral in (3.15) and (3.16) are understood in the sense of pairing between $W_0^{2,2}(\Omega_n)$ and its dual space. Obviously, the support of (3.16) is a subset of $supp\nabla\eta\subset T_{R,2R}$.

By (3.14), Lemma 3.1 and the Cauchy inequality, we obtain

$$\begin{aligned}&\int_{R^3}([\Delta^2-a_0\Delta+a_1,\eta]u_n(x))(\eta(x)u_n(x)|x|^{-1}\overline{g}(\log\frac{\rho}{|x|}))dx\\&\le C\sum_{k=0}^{2}\frac{1}{R^{5-2k}}\int_{T_{R,2R}}|\nabla^ku_n(x)|^2dx+C\sum_{k=0}^{1}\frac{1}{R^{3-2k}}\int_{T_{R,2R}}|\nabla^ku_n(x)|^2dx\\&\le C(\frac{1}{R^5}+R)\int_{T_{R,4R}}|u_n(x)|^2dx.\end{aligned} \tag{3.17}$$

On the other hand, since $\Delta^2u-a_0\Delta u=0$ in $B_{4R}(Q)\bigcap\Omega_n$ and η is supported in B_{2R}, hence the integral in (3.15) (the second term in (3.15) is equal to 0) is equal to

$$\int_{R^3}(\Delta^2(u_n(x)\eta(x))-a_0\Delta(u_n(x)\eta(x))+a_1u_n(x)\eta(x)\cdot(\eta(x)u_n(x)|x|^{-1}\overline{g}(\log\frac{\rho}{|x|}))dx. \tag{3.18}$$

To estimate (3.18), we employ Lemma 2.3 with $u=\eta u_n$. Then (3.18) is bounded from below by

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$$\frac{1}{2}\int_{\mathbb{S}^2} v_1^2(\tau,\omega)d\omega \geq \frac{C}{\rho^4}\int_{S_\rho\bigcap\Omega}|u_n(x)|^2 d\sigma_x. \tag{3.19}$$

Hence for every $\rho < R$, by (3.17)- (3.19), we have

$$\frac{1}{\rho^4}\int_{S_\rho\bigcap\Omega}|u_n(x)|^2 d\sigma_x \leq C(\frac{1}{R^5}+R)\int_{T_{R,4R}}|u_n(x)|^2 dx. \tag{3.20}$$

Finally, it can be finished by taking the limit as $n\to\infty$.

The following proposition is devoted to the proof of Theorem 1.1. In addition, we will establish sharp local estimates for the solutions in a neighborhood of a boundary point.

Proposition 3.3 Let Ω be an arbitrary bounded domain in $\mathbb{R}^3$, $Q\in\mathbb{R}^3\backslash\Omega$ and $R>0$. Suppose

$$\Delta^2 u - a_0\Delta u + a_1 u = f(x) \ \text{ in } \Omega,\ u\in W_0^{2,2}(\Omega), \tag{3.21}$$

where a_0, a_1 are non-negative constants and $f(x)\in C_0^\infty(\Omega\setminus B_{4R}(Q))$. Then for every $x\in T_{R/4}(Q)$,

$$|\nabla u(x)|^2 \leq C(1+|x-Q|^4)(\frac{1}{R^5}+R)\int_{T_{R/4,4R}(Q)}|u(y)|^2 dy, \tag{3.22}$$

and

$$|u(x)|^2 \leq C|x-Q|^2(\frac{1}{R^5}+R)\int_{T_{R/4,4R}(Q)}|u(y)|^2 dy, \tag{3.23}$$

where C is a positive constant depending on a_0.

Proof. Since $\Delta^2 u - a_0\Delta u + a_1 u = 0$ in $T_{4R}(Q)$, by an interior estimate for solutions of the elliptic equations (see [1,7])

$$|\nabla u(x)|^2 \leq \frac{C}{d(x)^3}\int_{B_{d(x)/2}(x)}|\nabla u(y)|^2 dy, \tag{3.24}$$

for $\mathrm{B}_{d(x)/2}(x)\subset \mathrm{B}_{4R}(Q)$. Let x_0 be a point on the boundary of Ω such that $d(x)=|x-x_0|$. Since $x\in T_{R/4}(Q)$ and $d(x)\leq|x-Q|$, we have $x\in B_{R/4}(x_0)$. By Lemma 3.1 and $B_{d(x)/2}(x)\subset B_{2d(x)}(x_0)$,

$$\begin{aligned}\frac{C}{d(x)^3}\int_{B_{d(x)/2}(x)}|\nabla u(y)|^2 dy &\leq C(\frac{1}{d(x)}+\frac{1}{d(x)^5})\int_{B_{2d(x)}(x_0)}|u(y)|^2 dy\\ &\leq C\left(1+\frac{1}{d(x)^4}\right)\int_{\mathrm{S}_{2d(x)}(x_0)}|u(y)|^2 d\sigma_y.\end{aligned} \tag{3.25}$$

Next, we analyse the upper estimate of the right side of (3.25) by Lemma 3.2. Since $d(x)\leq R/4$, thus $2d(x)\leq 3R/4$. By the condition $\Delta^2 u - a_0\Delta u + a_1 u = 0$ in $T_{4R}(Q)$ and

$$|Q-x_0|\leq|Q-x|+|x-x_0|\leq R/2, \tag{3.26}$$

we have $\Delta^2 u - a_0\Delta u + a_1 u = 0$ in $T_{3R}(x_0)$. Therefore Lemma 3.2 holds with x_0 in place of $Q, 3R/4$ in place of R and $\rho = 2d(x)$, i.e.,

$$\begin{aligned}\int_{\mathrm{S}_{2d(x)}(x_0)}|u(y)|^2 dy &\leq Cd(x)^4(\frac{1}{R^5}+R)\int_{T_{3R/4,3R}(x_0)}|u(y)|^2 dy\\ &\leq Cd(x)^4(\frac{1}{R^5}+R)\int_{T_{R/4,4R}(Q)}|u(y)|^2 dy.\end{aligned} \tag{3.27}$$

By (3.24), (3.25) and (3.27), we obtain

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$$|\nabla u(x)|^2 \le C(1+d(x)^4)(\frac{1}{R^5}+R)\int_{T_{R/4,4R}(Q)} |u(y)|^2 dy. \qquad (3.28)$$

Clearly, $d(x) \le |x-Q|$, so that (3.28) implies (3.21).

Based on the interior estimate for solutions of the elliptic equations

$$|u(x)|^2 \le \frac{C}{d(x)^3}\int_{B_{d(x)/2}(x)} |u(y)|^2 dy, \qquad (3.29)$$

the process of the proof (3.23) is the similar as the estimate of $|\nabla u(x)|$.

The proof of theorem 1.1.

By Proposition 3.3, it's also known that the gradients of solutions in the neighborhood of all boundary points of Ω are bounded. Thus we complete the proof of theorem 1.1.

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