ISSN 2347-1921

Numerical Solution of Double Integral of Singular Derivatives Using Trapezoidal Method with Romberg Acceleration

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ABSTRACT

The goal of this paper is to evaluate numerically a double integral of partial Derivatives Using *RTRT* Method. For Trapezoidal method (one of Newton-Cotes formula) which will be based on two dimensions x and y. In addition to that Romberg acceleration rule will be used to get more accurate results together with less time (faster convergence) and number of subintervals which are involved. We shall refer to this method by *RTRT*, where *R* stands for Romberg acceleration and *T* for Trapezoidal rule.

Indexing terms/Keywords

Double Integral; Singular Derivatives; Trapezoidal Method; Romberg Acceleration

Academic Discipline And Sub-Disciplines

Applied Mathematics, Numerical Analysis

Council for Innovative Research
Peer Review Research Publishing System
Journal: JOURNAL OF ADVANCES IN MATHEMATICS
Vol .9, No 9
www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

INTRODUCTION

There are many research discussed the solution of single integrals numerically [5,9], but numerical evaluation of double integral is, in general, more complicated than that of single integral, since the integrand depends on two variables and we will deal with the regions or surfaces in the first and not with intervals as is the case in the second [6].

Numerical evaluation of this kind of integral is generally very complicated, since the surface areas should be found together with their moment of inertia of plan surfaces and the volumes under the double integral.

Mohammed et. el. [7] introduced an approach to find numerical solution of double integrals of improper integrand using three different methods: RM (RS), RT (RS) and RS (RS), which depend respectively on middle point rule plus Trapezoidal rule, Simpson rule on the dimension y, and Simpson rule on the dimension x using Romberg acceleration rule on both dimensions x and y, without ignoring the impropriety in both dimensions. They deduced that RM (RS) is the best based on the accuracy and rapidity of convergence.

Dheya'e [1], used four methods based on combination of Romberg acceleration rule with Simpson rule, and Romberg acceleration rule with middle point principle to solve double integrals, their integrands are continuous, but they are with improper derivatives or just improper. These methods are RM (RS), RM (RM), RS (RM) and RS (RS), which gave good results. For the improper integrals in a point or more, she deduced that RM (RM) and RM (RS) have superiority on the other two methods, since these later methods are slow in providing accuracy in too large number of subintervals on the both dimensions x and y , as give correct solution for very small number of decimal places. By introducing many arbitrary examples she showed that the improper integrals, RM (RS) is the best if the impropriety can be removed, while RM (RM) is preferable in case that impropriety cannot be removed. As far as the integrand with improper derivative, she shoed that RM (RS) is the best with respect to rapid convergence to the analytical values with quite less number of subintervals.

1.The RTRT Method

It is a combination of Trapezoid rule with Romberg acceleration rule on both *x* and *y* dimensions. The aim is evaluation the approximate value of the double integral

$$I = \int_c^d \int_a^b f(x\ y)\,dx\,dy \qquad \ldots (1)$$

The integral *I* can be rewritten in the form

$$I = \int_c^d F(y)\,dy \qquad \ldots (2)$$

where

$$F(y) = \int_a^b f(x,y)\,dx \qquad \ldots (3)$$

the Trapezoid rule gives an approximate value for the integral in (2) (on the dimension y), which is

$$I = \frac{h}{2}\left[F(y_\circ) + 2\sum_{i=1}^{m-1} F(y_i) + F(y_m)\right] + E(h) \qquad \ldots (4)$$

where $y_i = c + ih$, $(i = 1, 2, \cdots, m)$, and *m* is number of subintervals to which the interval $[c, d]$ is partitioned, and $h = \dfrac{d-c}{m}$, $E(h)$ stands for correction terms on the dimension y . To evaluate $F(y_i)$ approximately y_i will be substituted in (3) to have

$$F(y_i) = \int_a^b f(x,y)\,dx \qquad \ldots (5)$$

Applying Trapezoid rule too on the integral in (5) yields the following expression

$$F(y_i) = \frac{\bar{h}}{2}\left[F(y_\circ) + 2\sum_{j=1}^{n-1} F(x_j, y_i) + F(x_n, y_i)\right] + \bar{E}(\bar{h}) \qquad \ldots (6)$$

where $x_j = c + j\bar{h}$, $(j = 1, 2, \cdots, n)$, and n is number of subintervals to which the interval $[a, b]$ is partitioned and $\bar{h} = \dfrac{b-a}{n}$, $\bar{E}(\bar{h})$ represents the correc-tion terms on the dimension x .

Romberg acceleration rule will be applied to improve values of $F(y_i)$. These resulting values are nothing but the approximate values of the integral in (3) and substituting $F(y_i)$ for each $i = 1, 2, \cdots, m$ in the expression (4), and thus values of the integral in (2) will be obtained by Romberg acceleration and Trapezoid rules and give approximate value of the integral in (1).

When Romberg acceleration rule is applied on x dimension, then the error will depend on the behavior of the integrand $f(x, y_i)$. If this function has improper partial derivatives in one or both limits of the interval $[a, b]$ then the impropriety will be eliminated on the dimension x and the correction terms on the x is

$$\bar{E}_T(\bar{h}) = A_T \bar{h}^2 + B_T \bar{h}^4 + C_T \bar{h}^6 + \cdots \qquad \text{... (7)}$$

which is provided by Fox [2], where $A_T, B_T, C_T, \cdots$ are constants depend on the derivatives of the function $f(x, y_i)$ at the two limits $x = a$ and $x = b$, and $\bar{E}_T(\bar{h})$ are correction terms of Trapezoid rule on x.

The error (on y) depends on the behavior of the function $F(y)$ on the interval $[c, d]$.

If the $F(y)$ is (or its derivatives are) improper at one of the endpoints of the interval $[c, d]$, then error on y will be (Fox [2])

$$E_T(h) = a_1 g_1(x) + a_2 g_2(x) + \cdots + A_T h^2 + B_T h^4 + C_T h^6 + \cdots \qquad \text{... (8)}$$

where $E_T(h)$ represents correction terms for Trapezoid rule on y .

As far as partitioning concern, the two intervals $[a, b]$ and $[c, d]$ are respectively partitioned into n and m subintervals, and we will choose

$n = 1, 2, 4, 8, \cdots$ and $m = 1, 2, 4, 8, \cdots$

In Trapezoid rule we let $n = 1$, then the value of the integral in (2) will be calculated by Romberg acceleration method on values of the Trapezoid rule, next we take $n = 2$ following the same procedure which mentioned above, and so on until the absolute error becomes less than or equal to some fixed number *EPS*1 (on y) (Sastry [4] and Saxena [3]).

Evaluating the integral (2) needs finding $F(y_1)$ by Romberg acceleration rule together with Trapezoid rule on the integral in (5), next we choose $m = 1$ in the expression (4), then we have to calculate $F(y_1)$ from the expression (6) where $n = 1$. the resulting vales will be tabulated with corresponding values of n and m .

If $m = 2$, then $F(y_1)$ and $F(y_2)$ should be calculated for (4) by applying (6) for $n = 1$, $n = 2$ and so on. Assuming that the error in $F(y_1)$ is less than or equal to *EPS* for $n = 16$ and the absolute error in $F(y_2)$ is less than or equal to *EPS* when $n = 64$, then the approximate value of the integral which is the solution of (1) when $m = 2$ will be fixed in the table with the largest value $n = 64$, and so on for $m > 2$.

2- Romberg Integral

This method referred to the German scientist Romberg in 1955 based on triangular arrangement of numerical appro-ximated values of specific integral with successive application on one of Newton-Cotes (Nasser [8]), which takes the form

$$I - T(h) = A_T h^2 + B_T h^4 + C_T h^6 + \cdots \qquad \text{... (9)}$$

Choosing two different values for h, namely h_1 and h_2 give

$$I - T(h_1) = A_T h_1^2 + B_T h_1^4 + C_T h_1^{16} + \cdots \qquad \text{... (10)}$$

$$I - T(h_2) = A_T h_2^2 + B_T h_2^4 + C_T h_2^{16} + \cdots \qquad \text{... (11)}$$

Substituting $h_2 = h_1/2$ in (11) and solving the resulting formula with (10) to obtain A_T after elimination of terms containing $h^4, h^6, \cdots$ we will get

$$I \cong \frac{2^2 T\left(\frac{h}{2}\right) - T(h)}{2^2 - 1} \quad \text{... (12)}$$

here $h = h_1$. Formula (12) does not give the actual value of the integral, but its approximate value which is closer to the value of the integral than $T(h)$ or $T\left(\frac{h}{2}\right)$. Let us define $T\left(h, \frac{h}{2}\right)$ as

$$T\left(h, \frac{h}{2}\right) = \frac{2^2 T\left(\frac{h}{2}\right) - T(h)}{2^2 - 1} \quad \text{... (13)}$$

Thus,

$$1 - T\left(h, \frac{h}{2}\right) = A'_T h^4 + B'_T h^6 + \cdots \quad \text{... (14)}$$

where $A'_T, B'_T, \cdots$ are constants. In a analogous way closer value of the integral can be obtained using $T\left(h, \frac{h}{2}\right)$. Hence, we have a table of values called Romberg table.

In general, values of this table can be obtained using Trapezoid rule and the formula $T = \dfrac{2^k \alpha\left(\frac{h}{2}\right) - \alpha(h)}{2^k - 1}$ ... (15)

4- Examples: To evaluate the integral $I = \int_0^1 \int_0^1 \sqrt{x^2 + y^2}\, dx\, dy$ numerically, bearing in mind that its actual value is 0.7651957165 rounded to 10 decimal places. The integrand has improper partial derivatives at the point $(x, y) = (0, 0)$. the type of impropriety is square root. Depending on (3) it is possible to find $F(y)$ by analytical integration

$$F(y) = \frac{1}{2}\sqrt{y^2 + 1} + \frac{1}{2} y^2 \ln\left(\sqrt{y^2 + 1} + 1\right) - \frac{1}{2} y^2 \ln y$$

It is clear that $F(y)$ has square root impropriety at $y = 0$, hence the error from Trapezoid rule on y will be [7],

$$E_T(h) = a_1 h^3 + a_2 h^3 \ln h + a_3 h^5 + a_4 h^7 + \cdots + A_T h^2 + B_T h^4 + \cdots$$

$$= A_T h^2 + a_1 h^3 + a_2 h^3 \ln h + B_T h^4 + a_3 h^5 + a_4 h^7 + C_T h^6 + \cdots$$

where $a_i, A_T, B_T, \cdots$ are constants, $i = 1, 2, 3, \cdots$

Using *RTRT* method with $EPS = 10^{12}$ and $EPS1 = 10^{13}$ yields the results in Table1. which are correct to ten decimal places, and we get value equal to the analytical one when $n = 32$ and $m = 256$ (2^{11} subintervals):

Table1. Solution of the double integral $I = \int_0^1\int_0^1 \sqrt{x^2 + y^2}\, dx\, dy$

N	RTRT	M
32	1.3977935747	1
64	0.5763142858	2
32	0.7925329247	4
32	0.7612909740	8
32	0.7654559971	16
32	0.7651873201	32
32	0.7651958498	64
32	0.7651957151	128
32	0.7651957165	256

Now, to find numerical solution of the double integral $I = \int_0^1\int_0^1 \sqrt{x + y}\, dx\, dy$, [8], the integrand has improper partial derivatives at the point $(x, y) = (0, 0)$, the type of impropriety is again square root. The analytical value of this integral is 0.975161133, which is rounded to nine decimal places. Integrating (3) analytically gives us

$$F(y) = \frac{3}{2}(1+y)^{3/2} - \frac{3}{2}y^{3/2}$$

Since the $F(y)$ has improper derivative at $y = 0$, with square root impro-priety, hence the error from Trapezoid rule on y will be

$$E_T(h) = a_1h^2 + a_2h^{3/2} + a_3h^3 + \cdots + A_Th^2 + B_Th^4 + \cdots$$

$$= a'_1h^2 + a'_2h^{3/2} + a'_3h^3 + \cdots$$

where $a'_i, a_i, A_T, B_T, \cdots$ are constants, $i = 1, 2, 3, \cdots$

Using the tolerances $EPS = 10^{-12}$ and $EPS1 = 10^{-13}$ we obtained the results which appears in table2. Note that correct value of the integral is that corresponding to $n = 32$ and $m = 256$ (2^{13} subintervals).

Table2. Solution of the double integral $I = \int_0^1\int_0^1 \sqrt{x + y}\, dx\, dy$

N	RTRT	m
1	1.3977935747	32
2	0.5763142858	64
4	0.7925329247	32
8	0.7612909740	32
16	0.7654559971	32
32	0.7651873201	32
64	0.7651958498	32
128	0.7651957151	32
256	0.7651957165	32

ISSN 2347-1921

As a third example, let us take the double integral $I=\int_0^1\int_0^1\sqrt{x^4+y^4}\,dx\,dy$, which has improper partial derivatives at $(x,y)=(0,0)$, and the type of impropriety is again the square root. The actual value of this integral is yet unknown. In this case $F(y)$ cannot be obtained by analytical integration, thus we expect that the error of Trapezoid rule $E_T(h)$ on the dimension *y* will take the form

$$E_T(h)=A_Th^2+B_Th^4+C_Th^6+\cdots$$

as $A_T, B_T, C_T,\cdots$ stand for constants.

We fixed $EPS=10^{-12}$ and $EPS1=10^{-13}$, because that the actual value of the double integral is unknown, from table3. we realize that the correct solution for 8 decimal places is reached when $n=64$ and $m=32$ (2^{11} sub-intervals), for 9 decimal places will correspond $n=64$ and $m=64$ (2^{12} sub-intervals), and for ten decimal places when $n=64$ and $m=128$ (2^{13} sub-intervals).

Table3. Solution of the double integral $I=\int_0^1\int_0^1\sqrt{x^4+y^4}\,dx\,dy$

N	*RTRT*	*m*
1	1.256096080	64
2	0.360012597	64
4	0.556843506	64
8	0.544522204	64
16	0.544715462	64
32	0.544714706	64
64	0.544714707	64
128	0.544714707	64

3- Discussion

Three double integrals are introduced as examples, all are of improper partial derivatives. To solve any one of them we use method of *RTRT* on the dimensions x and y, and we got results equal to the analytical values up to 10 or 9 decimal places using 2^{13} subintervals which is a relatively small number. Hence we recommend this method of *RTRT* to solve double integral numerically.

ACKNOWLEDGMENTS

Our thanks to the experts who have contributed towards development of the template.

REFERENCES

1. Dheya'e A.M. 2009 "Some of Numerical Methods to Evaluation Singular, Double and
2. Triple Integrations Using Math lab", Msc, thesis, University of Kufa, College of
3. Education for Women, Department of Mathematics.
4. Fox L. 1967 " Romberg Integration for a Class of Singular Integrands", comput. J.10,
5. pp. 87-93.
6. [3] H .C. Saxena, 2000 "Finite Differences and Numerical Analysis", pp. 120-137.
7. S.S. Sastry, 2008 "Introductory Methods of Numerical Analysis", pp. 197-211.

8. Maturi D.A ,Bajamal A.Z. ,ALGethami B.M. 2014. " Numerical Solutions of Volterra
9. Integral Equation of Second Kind Using Implicit Trapezoidal",Journal of Advances
10. in Mathematic Vol 8, No 2 .
11. Mohammed A. H. 2002 "Evaluation of Double Integrations ", comput J. Vol.7, No.3,
12. pp. 21-28.
13. Mohammed A. H., Hayder A. K. and Hassen A. F. 2009 "On The Numerical Integration",
14. a paper accepted by scientific conference of Morocco.
15. Naser R.H. 2011 "A Comparison Between the Two Accelerations Methods Aitken's and
16. Romberg's in Computing Integrals Numerically",Msc ,thesis , University of Kufa,
17. College of Education for Women, Department of Mathematics.
18. P.Kandasamy ,K. Thilagavathy ,K.Gunavathi, 2004 " Numerical Methods ", pp. 299-316,