ISSN 2347-1921

ON THE FRACTIONAL INTEGRAL OPERATORS FOR REAL POSITIVE DEFINITE SYMMETRIC MATRIX

Yashwant Singh
Department of Mathematics,
Government College, Kaladera, Jaipur, Rajasthan, India
E-mail: dryashu23@yahoo.in
And
Nandavasanthnadiger Kulkarni
Department of Mathematics,
Dayananda Sagar college of arts, science and commerce,
Kumaraswamy layout, Bangalore, Karnataka, India
E-mail: nandavnadiger@gmail.com

Abstract

In the present paper fractional integral operators associated with $\overline{H}$-function for real positive symmetric definite matrix have been discussed. These operators have a wide range of applications in the field of Mathematical Physics and Linear differential equations. A number of special cases of our operators have been mentioned.

Key words: Fractional Integral Operators; $\overline{H}$-function; H-function; Matrix transform; Symmetric matrix

(2000 Mathematical Subject Classification: 33C99)

Council for Innovative Research
Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol. 9, No. 6

www.ciriam.com . editoriam@gmail.com

ISSN 2347-1921

Introduction

Fractional integration is an immediate generalization of repeated integration. Fractional integral operators occur in the solutions of linear differential equations, partial differential equations and in the integral representations of hypergeometric functions of one or more variables. Riesz [9] and Garding [3] respectively introduced Riemann-Liouville integral of vector and matrix variables and applied them in the solution of differential equation associated with Cauchy's problem.

(a) $\overline{H}$ -function with matrix argument

Let X is a $p\times p$ real symmetric positive definite matrix of functionally independent variables. Let the $\overline{H}$ -function of X be denoted by

$$\overline{H}_{P,Q}^{M,N}\left[z\right]=\overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \tag{1.1}$$

where
$$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,...,p)$ and $b_j(j=1,...,Q)$ are complex parameters, $\alpha_j\geq 0(j=1,...,P),\beta_j\geq 0(j=1,...,Q)$ (not all zero simultaneously) and exponents $A_j(j=1,...,N)$ and $B_j(j=N+1,...,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$ -function given by equation (1.1) have been given by (Buschman and Srivastava[1]).

$$\Omega\equiv\sum_{j=1}^{M}\left|\beta_j\right|+\sum_{j=1}^{N}\left|A_j\alpha_j\right|-\sum_{j=M+1}^{Q}\left|\beta_jB_j\right|-\sum_{j=N+1}^{P}\left|\alpha_j\right|>0 \tag{1.3}$$

and $\left|\arg(z)\right|<\frac{1}{2}\pi\Omega$ (1.4)

The behavior of the $\overline{H}$ -function for small values of $|z|$ follows easily from a result recently given by (Rathie [8],p.306,eq.(6.9)).

We have

$$\overline{H}_{P,Q}^{M,N}\left[z\right]=0\left(\left|z\right|^{\gamma}\right),\gamma=\min_{1\leq j\leq N}\left[\operatorname{Re}\left(b_j/\beta_j\right)\right],|z|\to 0 \tag{1.5}$$

If we take $A_j=1(j=1,...,N)$, $B_j=1(j=M+1,...,Q)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}$ reduces to the Fox's H-function [2].

It is assumed that $\overline{H}(XY)=\overline{H}(YX)$ for real symmetric $m\times m$ positive definite matrices X and Y, $\overline{H}(X)$ is defined by the following integral equation:

$$\int_{X>0}|X|^{\rho-\frac{m+1}{2}}\overline{H}(X)dX=\frac{\prod_{j=1}^{M}\Gamma_m(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma_m(\frac{m+1}{2}-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma_m(\frac{m+1}{2}-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma_m(a_j-\alpha_j\xi)} \tag{1.6}$$

Sethi [10] discussed the following fractional integral operators involving H -function of matrix arguments:

$$R[f(X)] = R\left[{}^{(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}}_{\sigma,\rho,\gamma}; f(X)\right]$$

$$= \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} H^{p,q}_{r,s}\left[\gamma(I-UX^{-1})\Big|^{(a_j,\alpha_j)_{1,r}}_{(b_j,\beta_j)_{1,s}}\right] f(U)dU \qquad (1.7)$$

$$K[f(X)] = K\left[{}^{(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}}_{\delta,\rho,\gamma}; f(X)\right] =$$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} H^{p,q}_{r,s}\left[\gamma(I-XU^{-1})\Big|^{(a_j,\alpha_j)_{1,r}}_{(b_j,\beta_j)_{1,s}}\right] f(U)dU \qquad (1.8)$$

Where $f(X) = f(x_{11},...,x_{1m},x_{21},...,x_{mm})$ be a real bounded function of a complex parameter.

(b) Matrix transform

A generalized matrix transform or M-transform of a function $f(X)$ of a $m \times m$ real symmetric positive definite or strictly negative definite matrix X is defined as follows:

$$M_f(s) = \int_{X>0} |X|^{s-\frac{m+1}{2}} f(X)dX \qquad (X>0) \qquad (1.9)$$

Whenever $M_f(s)$ exists. Also $f(X)$ is assumed to be a symmetric function i.e. $f(BX) = f(XB) = f\left(B^{1/2}XB^{1/2}\right)$ for $B = B' > 0$. When $X < 0$ replace X by $-X$ in M -transform.

(c) Integral operators involving $\overline{H}$ -function

$$Y[f(X)] = Y\left[f(X)\Big|\sigma,\rho,\gamma;{}^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}\right] =$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} \overline{H}^{M,N}_{P,Q}\left[\gamma(1-UX^{-1})\Big|^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}\right] f(U)dU \qquad (1.10)$$

$$N[f(X)] = N\left[f(X)\Big|\delta,\rho,\gamma;{}^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}\right] =$$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} \overline{H}^{M,N}_{P,Q}\left[\gamma(I-XU^{-1})\Big|^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}\right] f(U)dU \qquad (1.11)$$

The above defined operators exists under the following conditions:

(i) $P \geq 1, Q < \infty, \frac{1}{P} + \frac{1}{Q} = 1, |\arg(I-a)| < \pi$

(ii) ($\mathrm{Re}(\sigma) > \frac{1}{Q}, \mathrm{Re}(\delta) > \frac{1}{P}, \mathrm{Re}(\rho) > \frac{m+1}{2}$

(iii) $\mathrm{Re}(\alpha + \min_{1\leq j\leq M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}$

(iv) $f(X) \in L_P(0,\infty)$.

The last condition ensures that $Y[f(X)]$ and $N[f(X)]$ both exist and also both belong to $L_P(0,\infty)$.

ISSN 2347-1921

2. Main Results

The following theorems of the operators defined by (1.10) and (1.11) have been established in the expression of matrix transform:

Theorem 1: If $f(X) \in L_P(0,\infty)\ 1 \le P \le 2[or f(X) \in M_P(0,\infty)\ and\ P > 2]$ where $\mathrm{Re}(\alpha + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}$,

$\mathrm{Re}(\sigma) > -\frac{1}{Q}, \mathrm{Re}(t) > \frac{m+1}{2}, \mathrm{Re}(\sigma - t + 1) > \frac{m+1}{2}$ and $|\arg(I-a)| < \pi$ then

$$M\{Y[f(X)]\} = \frac{\Gamma_m\left(\sigma - t + \frac{m+1}{2}\right)}{\Gamma_m(\rho)}$$

$$\overline{H}^{M,N+1}_{P+1,Q+1}\left[\gamma I \left| \begin{matrix} \left(\frac{m+1}{2},1;1\right),(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\left(\frac{m+1}{2}-t-\sigma-\rho,1;1\right) \end{matrix} \right.\right] M[f(U)] \qquad (2.1)$$

Where I is $m \times m$ identity matrix.

Proof: Taking the matrix transform of equation (1.10), we get

$$M\{Y[f(X)]\} = \int_{X>0} |X|^{t-\frac{m+1}{2}} \left[\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\right.$$

$$\left.\int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} \overline{H}^{M,N}_{P,Q}\left[\gamma(I-UX^{-1}) \left| \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right] f(U)dU\right] dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\{Y[f(X)]\} = \frac{1}{\Gamma_m(\rho)} \int_{X>0} |U|^{\sigma} f(U)dU$$

$$\int_{0<U<X} |X|^{t-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} \overline{H}^{M,N}_{P,Q}\left[\gamma(I-UX^{-1}) \left| \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right] dX$$

On evaluating X -integral with the help of the result given by Mathai and Saxena [7],

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\frac{m+1}{2}} H^{p,q}_{r,s}\left[|XZ| \left| \begin{matrix} (a_j,\alpha_j)_{1,r} \\ (b_j,\beta_j)_{1,s} \end{matrix} \right.\right] dX =$$

$$\Gamma_m(\rho) H^{p,q+1}_{r+1,s+1}\left[|Z| \left| \begin{matrix} \left(\frac{m+1}{2}-\delta,1\right),(a_j,\alpha_j)_{1,r} \\ (b_j,\beta_j)_{1,s},\left(\frac{m+1}{2}-\delta-\rho,1\right) \end{matrix} \right.\right] \qquad (1.11)$$

Where $\mathrm{Re}(\alpha + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}$ and $\mathrm{Re}(\rho) > \frac{m+1}{2} - 1$.

We obtain the required result.

Theorem 2: If $f(X) \in L_P(0,\infty)\ 1 \le P \le 2[or f(X) \in M_P(0,\infty)\ and\ P > 2]$ where

$\mathrm{Re}(\delta + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}$, $\mathrm{Re}(\delta) > -\frac{1}{Q}, \mathrm{Re}(t) > \frac{m+1}{2}, \mathrm{Re}(\delta + t) > \frac{m+1}{2}$ and $|\arg(I-a)| < \pi$ then

ISSN 2347-1921

$$M\left\{N[f(X)]\right\} = \frac{\Gamma_m(\delta+1)}{\Gamma_m(\rho)}$$

$$\overline{H}^{M,N+1}_{P+1,Q+1}\left[\gamma I\left|\begin{matrix}\left(\frac{m+1}{2}-\delta,1;1\right),(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\left(\frac{m+1}{2}-\delta-\rho,1;1\right)\end{matrix}\right.\right]M[f(U)] \qquad (2.2)$$

Where I is $m\times m$ identity matrix.

Proof: Taking the matrix transform of equation (1.11), we get

$$M\left\{N[f(X)]\right\} = \int_{X>0} |X|^{\delta-\frac{m+1}{2}}\left[\frac{|X|^{\delta}}{\Gamma_m(\rho)}\right.$$

$$\left.\int_{U>X} |U|^{-\sigma-\rho}|U-X|^{\rho-\frac{m+1}{2}}\,\overline{H}^{M,N}_{P,Q}\left[\gamma(I-XU^{-1})\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]f(U)dU\right]dX$$

And changing the order of integration and evaluation X -integral with the help of (1.11), we obtain the required result.

Theorem 3: If $f(X)\in L_P(0,\infty), g(X)\in L_P(0,\infty)$ where $\mathrm{Re}(\delta+\min_{1\le j\le M}\frac{b_j}{\beta_j})>\frac{m+1}{2}$,

$\mathrm{Re}(\delta)>-\frac{1}{Q}, \mathrm{Re}(\rho)>\frac{m+1}{2}, \mathrm{Re}(\sigma)>\max\left(\frac{1}{P}+\frac{1}{Q}\right)$ and $|\arg(I-a)|<\pi$ then

$$\int_{X>0} f(X)Y\left[g(X)\middle|\sigma,\rho,\gamma;{}^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}\right]dX =$$

$$\int_{X>0} g(X)N\left[f(X)\middle|\sigma,\rho,\gamma;{}^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}\right]dX \qquad (2.3)$$

Proof: Equation (2.3) immediately follows on interpreting it with the help of equations (1.10) and (1.11).

Special Cases

(i) If we put $M=1, N=1, P=2, Q=2, A_j=1=B_j, \gamma=1$,then the operators (1.10) and (1.11) reduce to their Mellin transforms in the following form:

$$Y[f(X)] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}H^{1,2}_{2,2}\left[(I-UX^{-1})\right]f(U)dU$$

Here

$$Y[f(X)] = Y\left[f(X)\middle|\sigma,\rho,1;{}^{(a_1;\alpha_1;1),(a_2;\alpha_2)_{1,2}}_{(b_1,\beta_1),(b_2,\beta_2;1)_{1,2}}\right]$$

And

$$H^{1,2}_{2,2}\left[(I-UX^{-1})\right] = H^{1,2}_{2,2}\left[(I-UX^{-1})\left|\begin{matrix}(a_1,\alpha_1;1),(a_2,\alpha_2)\\ (b_1,\beta_1),(b_2,\beta_2;1)\end{matrix}\right.\right]$$

Then

$$Y[f(X)] = \frac{\Gamma_m(\chi_1)|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|I-UX^{-1}|^{\rho-\beta_1-\frac{m+1}{2}}\,{}_2F_1\left[-;(I-UX^{-1})\right]f(U)dU$$

Where

$$\Gamma(\chi_1)=\frac{\Gamma_m\left(\frac{m+1}{2}-\alpha_1+\beta_1\right)\Gamma_m\left(\frac{m+1}{2}-\alpha_2+\beta_2\right)}{\Gamma_m\left(\frac{m+1}{2}-\alpha_1-\alpha_2+\beta_1+\beta_2\right)}$$

$${}_2F_1\left[-;\left(I-UX^{-1}\right)\right]=$$

$${}_2F_1\left[\frac{m+1}{2}-\alpha_1-\beta_1,\frac{m+1}{2}-\alpha_2-\beta_2;\frac{m+1}{2}-\beta_1-\beta_2;-(I-UX^{-1})\right]$$

By virtue of the result [11].

Taking M -transform on both sides, we get

$$M\{Y(f(X)\}=\frac{\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)}\,{}_3F_2(-;I)M[f(U)]$$

Where $\Gamma(\chi_2)=\dfrac{\Gamma_m\left(\frac{m+1}{2}+\sigma\right)\Gamma_m\left(\rho+\beta_1\right)}{\Gamma_m\left(\sigma+\rho+\beta_1+\frac{m+1}{2}\right)}$

And

$${}_3F_2(-;I)={}_3F_2\left(\frac{m+1}{2}-\alpha_1+\beta_1,\frac{m+1}{2}-\alpha_2+\beta_2,\sigma+\frac{m+1}{2};\right.$$
$$\left.\frac{m+1}{2}-\beta_2+\beta_1,\sigma+\rho+\frac{m+1}{2}+\beta_1;I\right)$$

$$N[f(X)]=\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{\sigma}|U-X|^{\rho-\frac{m+1}{2}}H_{2,2}^{1,2}\left[(I-XU^{-1})\right]f(U)dU$$

Where

$$N[f(X)]=N\left[f(X)\middle|\sigma,\rho,1;^{(a_1;\alpha_1;1),(a_2;\alpha_2)_{1,2}}_{(b_1,\beta_1),(b_2,\beta_2;1)_{1,2}}\right]$$

And

$$H_{2,2}^{1,2}\left[(I-XU^{-1})\right]=H_{2,2}^{1,2}\left[(I-XU^{-1})\middle|^{(a_1,\alpha_1;1),(a_2,\alpha_2)}_{(b_1,\beta_1),(b_2,\beta_2;1)}\right]$$

Then

$$N[f(X)]=\frac{\Gamma_m(\chi_1)|X|^{\delta+\rho-\frac{m+1}{2}}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{-\delta-\rho}|I-XU^{-1}|^{\rho+\beta_1-\frac{m+1}{2}}\,{}_2F_1\left[-;(I-XU^{-1})\right]f(U)dU$$

Where

$${}_2F_1\left[-;\left(I-XU^{-1}\right)\right]=$$

$${}_2F_1\left[\frac{m+1}{2}-\alpha_1+\beta_1,\frac{m+1}{2}-\alpha_2+\beta_2;\frac{m+1}{2}+\beta_1-\beta_2;-(I-XU^{-1})\right]$$

Taking M -transform on both sides, we get

ISSN 2347-1921

$$M\{N(f(X)\} = \frac{(-1)^{\rho-\frac{m+1}{2}}\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)}\,{}_2F_1(-;I)M[f(U)]$$

(ii) Putting $P=0, Q=1, M=1, N=0, \gamma=1, A_j=1=B_j$, then operators (1.10) and (1.11) reduce to their Mellin transform in the following forms:

$$Y[f(X)] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} H_{0,1}^{1,0}\left[(I-UX^{-1})\Big|_{(\beta,1)}^{-}\right] f(U)dU$$

Where

$$Y[f(X)] = Y\left[f(X)\Big|\sigma,\rho,1;_{(\beta,1)}^{-}\right]$$

And

$$H_{0,1}^{1,0}\left[(I-UX^{-1})\Big|_{(\beta,1)}^{-}\right] = |I-UX^{-1}|^{\beta} e^{-tr(i-UX^{-1})}$$

$$= \frac{|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |I-UX^{-1}|^{\rho+\beta-\frac{m+1}{2}} e^{-tr(1-UX^{-1})} f(U)dU$$

By virtue of the result [11]. Taking M -transform on both sides, we get

$$M\{Y(f(X)\} = \frac{\Gamma_m(\rho+\beta)}{\Gamma_m(\rho)} M[f(U)]$$

Also

$$N[f(X)] = \frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right] f(U)dU$$

Where

$$N[f(X)] = N\left[f(X)\Big|\delta,\rho,1;_{(\beta,1)}^{-}\right]$$

And

$$H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right] = |I-XU^{-1}|^{\beta} e^{-tr(i-XU^{-1})}$$

$$= \frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\frac{m+1}{2}} |I-XU^{-1}|^{\rho+\beta-\frac{m+1}{2}} e^{-tr(1-XU^{-1})} f(U)dU$$

By virtue of the result [5]. Taking M -transform on both sides, we get

$$M\{N[f(X)]\} = \frac{\Gamma_m\left(\rho-\delta+\beta-\frac{m+1}{2}\right)}{\Gamma_m(\rho)} M[f(U)]$$

If we put $\alpha_j = \beta_j = 1; (j=1,\dots,P; j=1,\dots,Q)$ the operators reduce to G -function given by Vyas [12].

Theorem4. If $f(X) \in L_P(0,\infty)\ 1 \le P \le 2[or f(X) \in M_P(0,\infty)\ and\ P > 2]$ where $\mathrm{Re}(\alpha) > \frac{m+1}{2}$,

$\mathrm{Re}(\sigma) > -\frac{1}{Q}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\alpha-\sigma) > \frac{m+1}{2}$ and $|\arg(I-a)| < \pi$ then

$$M\left\{R[^{\alpha}_{\sigma,\rho,a}; f(X)]\right\} = \frac{\Gamma_m\left(\sigma - s + \frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\sigma - s + \alpha + \rho + \frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$$_1F_1\left[\rho+\alpha; \sigma - s + \alpha + \rho + \frac{m+1}{2}; I\right] M[f(U)] \quad (2.4)$$

Proof: Using the Mellin transform of

$$R[f(X)] = R\left[^{(a_r),(b_r)}_{\sigma,\rho,a}; f(X)\right] =$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} G^{p,q}_{r,s}\left[a(I-UX^{-1})\Big|^{(a_r)}_{(b_r)}\right] f(U)dU \quad (2.5)$$

We get

$$M\left\{R[^{\alpha}_{\sigma,\rho,a}; f(X)]\right\} =$$

$$\int_{X>0} \frac{|X|^{s-\frac{m+1}{2}} |X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \left[\int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{-}_{(b_1)}\right] f(U)dU\right] dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$\int_{0<U<X} |X|^{s-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-UX^{-1})\right]\Bigg] dX = \frac{1}{\Gamma_m(\rho)} \int_{U>X} |U|^{\sigma} f(U)dU$$

$$\int_{X>U} |X|^{S-\sigma-\rho-\alpha-\frac{m+1}{2}} |X-U|^{\rho+\alpha-\frac{m+1}{2}} e^{-tr(1-UX^{-1})} dX$$

On evaluating X -integral with the help of result given by Mathai [5]

$$\int_0^1 e^{-tr(XZ)} |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\delta-\frac{m+1}{2}} dX$$

$$= \frac{\Gamma_m(\delta)\Gamma_m(\rho-\delta)}{\Gamma_m(\rho)}\, {}_1F_1[\delta; \rho; -Z] \quad (2.6)$$

For $\mathrm{Re}(\delta) > \frac{m+1}{2}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\rho-\delta) > \frac{m+1}{2}$

We obtain the required result.

Theorem5. If $f(X) \in L_P(0,\infty)\ 1 \le P \le 2[or f(X) \in M_P(0,\infty)\ and\ P > 2]$ where $\mathrm{Re}(\alpha) > \frac{m+1}{2}$,

$\mathrm{Re}(\delta) > -\frac{1}{P}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\alpha-\rho) > \frac{m+1}{2}, \frac{1}{P} + \frac{1}{Q} = 1$ and $|\arg(I-a)| < \pi$ then

$$M\left\{K[^{\alpha}_{\delta,\rho,a};f(X)]\right\}=\frac{\Gamma_m\left(\delta+s+\frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\delta+s+\alpha+\rho+\frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$${}_1F_1\left[\rho+\alpha;\delta+s+\alpha+\rho+\frac{m+1}{2};I\right]M[f(U)] \qquad (2.7)$$

Proof: Using the Mellin transform of

$K[f(X)]=K\left[^{(a_r),(b_r)}_{\sigma,\rho,a};f(X)\right]=$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{\delta}|X-U|^{\rho-\frac{m+1}{2}}G^{p,q}_{r,s}\left[a(I-XU^{-1})\Big|^{(a_r)}_{(b_r)}\right]f(U)dU \qquad (2.8)$$

We get

$M\left\{K[^{\alpha}_{\delta,\rho,a};f(X)]\right\}=$

$$\int_{X>0}\frac{|X|^{s-\frac{m+1}{2}}|X|^{\delta}}{\Gamma_m(\rho)}\left[\int_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}G^{1,0}_{0,1}\left[a(I-XU^{-1})\Big|^{-}_{(b_1)}\right]f(U)dU\right]dX \qquad (2.9)$$

Changing the order of integration and evaluating X -integral with the help of (2.5), we obtain the required result.

When $M=1,N=0,P=0,Q=1,a=1$ in (2.5) and (2.8) reduces to the following from of operators:

$R[f(X)]=R\left[^{\alpha,\beta}_{\sigma,\rho,1};f(X)\right]=$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right]f(U)dU \qquad (2.10)$$

And

$K[f(X)]=K\left[^{\alpha,\beta}_{\sigma,\rho,1};f(X)\right]=$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{U>X}|U|^{-\delta-\sigma}|U-X|^{\rho-\frac{m+1}{2}}G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right]f(U)dU \qquad (2.11)$$

Theorem6. If $f(X)\in L_P(0,\infty)\ 1\le P\le 2[or f(X)\in M_P(0,\infty)\ and\ P>2]$ where $\mathrm{Re}(\alpha)>\frac{m+1}{2}$,

$\mathrm{Re}(\sigma)>-\frac{1}{Q},\mathrm{Re}(\rho)>\frac{m+1}{2},\mathrm{Re}(\alpha-\sigma)>\frac{m+1}{2},\frac{1}{P}+\frac{1}{Q}=1$ and $|\arg(I-a)|<\pi$ then

$$M\left\{R[^{\alpha}_{\sigma,\rho,1};f(X)]\right\}=\frac{\Gamma_m(\sigma+\alpha-\beta-s)\Gamma_m(\rho+\beta)}{\Gamma_m(\sigma-s+\alpha+\rho)\Gamma_m(\rho)\Gamma_m(\alpha-\beta)}M[f(U)] \qquad (2.12)$$

Proof: Using the Mellin transform of (2.10), we get

$M\left\{R[^{\alpha}_{\sigma,\rho,1};f(X)]\right\}=$

$$\int_{X>0} \frac{|X|^{-\sigma-\rho}|X|^{s-\frac{m+1}{2}}}{\Gamma_m(\rho)} \left[\int_{0<U<X} |U|^{\alpha}|X-U|^{\rho-\frac{m+1}{2}} G_{1,1}^{1,0}\left[a(I-UX^{-1})\Big|_{\beta}^{\alpha}\right] f(U)dU \right] dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\left\{R[_{\sigma,\rho,1}^{\alpha}; f(X)]\right\} = \frac{1}{\Gamma_m(\rho)} \int_{U>0} |U|^{\sigma} f(U)dU$$

$$\int G_{1,1}^{1,0}\left[(I-XU^{-1})\Big|_{\beta}^{\alpha}\right] |X|^{s-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} dX$$

Using the result given by Mathai [5].

$$G_{1,1}^{1,0}\left[X\Big|_{\beta}^{\alpha}\right] = \frac{1}{\Gamma_m(\alpha-\beta)} |X|^{\beta}|I-U|^{\alpha-\beta-\frac{m+1}{2}} \tag{2.13}$$

Provided

$$0 < X < I, \mathrm{Re}(\alpha-\beta) > \frac{m+1}{2}$$

We get

$$M\left\{R[_{\sigma,\rho,1}^{\alpha,\beta}; f(X)]\right\} = \frac{1}{\Gamma_m(\rho)} \int_{U>0} |U|^{\sigma-\alpha-\beta-\frac{m+1}{2}} f(U)dU$$

$$\frac{1}{\Gamma_m(\alpha-\beta)} \int_{X>U} |X|^{s-\sigma-\alpha}|X-U|^{\beta+\rho-\frac{m+1}{2}} dX$$

On evaluating X -integral with the help of the following result

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}}|I-X|^{\rho-\frac{m+1}{2}} dX = \frac{\Gamma_m(\delta)\Gamma_m(\rho)}{\Gamma_m(\rho+\delta)} \tag{2.14}$$

For $\mathrm{Re}(\delta) > 0, \mathrm{Re}(\rho) > \frac{m+1}{2}$

We arrive at the required result.

References

1. Buschman, R.G. and Srivastava , H.M. ; The $\overline{H}-$function associated with a certain class of Feynman integrals, J.Phys.A:Math. Gen. 23, (1990),4707-4710.

2. Fox,C.; A formal solution of certain dual integral equations, Trans. Amer. Math. Soc. 119, (1965), 389-395.

3. Garding, L.; The solution of Cauchy's problem for two totally hyperbolic linear differential equation by means of Reisz integrals, Ann. Of Math. 48(1947), 785-826.

4. Inayat-Hussain, A.A.; New properties of hypergeometric series derivable from Feynman integrals :II A generalization of the H -function, J. Phys. A. Math. Gen. 20(1987).

5. Mathai, A.M.; Special function of matrix argument II, Proc. Nat. Acad. Sci. India 35, Sect. A(1995), 367-393.

6. Mathai, A.M. and Saxena , R.K.; Gneralized Hypergeometric Functions With Applications in Statistics and Physical Sciences, Spring-verleg, Lecture NotesNo. 348, Heidelberg (1973).

7. Mathai, A.M. and Saxena, R.K.; A generalized probability distribution, Univ. Nac. Tucuman Rev. Ser. A21, (1978), 193-202.

8. Rathi, A.K. ; A new generalization of generalized hypergeometric functions. *Le Mathematiche Fasc*. II, 52(1997),297-310.

9. Riesz, M.; L integrale de Riemann-Liouville et. le problem de Cauchy pour 'L' equations desondens, Societe Mathematique de France, Conference de La Re'Union international des Mathematiciens tenue a Paris on Juillet, Paris (1949).

10. Sethi, P.L. and Mahmoud, A.A. Altayeb; Fractional integral operators involving Fox's H -function for real positive definite symmetric matrix, J. Raj. Acad. Phy. Sci., vol.2 (2), (2003), 139-152.

11. Srivastava,H.M.,Gupta, K.C. and Goyal,S.P.; The H-Functions of One and Two Varibale with Applications,South Asian Publishers, New Delhi and Madras, (1982).

12. Vyas, A.K.; A study of Zonal polynomials and special functions of the real positive definite symmetric matrices, Ph. D. Thesis, J.N.V.Univ., Jodhpur (1996).