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On a relation of distribution with series in L2 and logarithmic averages in the case of symmetric jump behavior

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ABSTRACT

Distribution theory has an important role in applied mathematics, that generalizes the classical notion of functions in mathematical analysis. Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense.

Firstly, in the introduction part of this paper we will give some general notations, definitions and results in distribution theory, as analytic representation of distribution, distributional jump behavior, distributional symmetric jump behavior, tempered distributions, formulas for the jump of distributions in terms of Fourier series, tempered derivative and integral. Then in final part we will state two results, the first one has to do on relation of analytic functions in the upper half-plane with some logarithmic averages in the case of symmetric jump behavior and the second one is related to decomposition of tempered distribution to series.

Keywords:

Distributions; distributional jump behavior; point value; tempered derivative; Holder inequality; Schwartz space

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Introduction

1.1 Definitions

Firstly, let us give some introductory concepts. With P^n we denote the subset of R^n, which elements have nonnegative integers coordinates. For a function f, $f:\Omega\to \mathrm{C}^n, \Omega\subseteq \mathrm{R}^n$, $k=(k_1,k_2, \ ,k_n)$, $k_j\in \mathrm{N}\cup\{0\}$, $x\in\Omega$, with $f^{(k)}(x)$ we denote the differential operator

$$f^{(k)}(x)=\frac{\partial^{|k|}}{\partial_{x_1}^{k_1}...\partial_{x_n}^{k_n}}f(x)=\partial^k f(x), \qquad |k|=k_1+k_2+...+k_n.$$

With $C^\infty(\mathrm{R}^n)$ is denoted the space of all complex valued infinitely differentiable functions on R^n and $C_0^\infty(\mathrm{R}^n)$ denotes the subspace of $C^\infty(\mathrm{R}^n)$ that consists of those functions of $C^\infty(\mathrm{R}^n)$ which have compact support.

Definition 1. The support of f is the closure of the set $x\in\Omega$, of points for which f is different from zero $(f(x)\neq 0)$, and is denoted by sup. f.

With D we denote the space of $C_0^\infty(\mathrm{R}^n)$ functions, called the set of test functions in which convergence is defined in the following way : a sequence $\{\phi_\nu\}$ of functions $\phi_\nu\in D$ converges to $\phi\in D$ in D as $\nu\to\nu_0$ if and only if there is a compact set $K\subset \mathrm{R}^n$ such that supp $(\phi_\nu)\subseteq K$ for each ν, supp $(\nu)\subseteq K$ and for every *n*-tiple *k* of nonnegative integers the sequence $\{f^{(k)}\phi_\nu(t)\}$ converges to $f^{(k)}\phi(t)$ uniformly on K as $\nu\to\nu_0$.

Distributions (or generalized functions) are objects that generalize the classical notion of functions in mathematical analysis. Distributions make it possible to differentiatefunctions whose derivatives do not exist in the classical sense.

Definition 2. A distribution T is continuous linear functional on D. Instead of writing $T(\phi)$, it is conventional to write $<T,\phi>$ for the value of T acting on a test function ϕ. The space of all distributions is denoted by D'.

Schwartz space is the vector space

$$S\left(\mathrm{R}^n\right)=\left\{\phi:\mathrm{R}^n\to \mathrm{C},\ \phi\in \mathrm{C}^\infty,\ \sup_{x\in\mathrm{R}^n}\left|x^\alpha\partial^\beta\phi(x)\right|<\infty,\ \alpha,\beta\in P^n\right\}.$$

S' is the space of all continuous linear functionals on S, called the space of of tempered distributions.

Definition 3. The value of distribution f at point x_0 is defined as the limit

$$f(x_0)=\lim_{h\to 0}f(x_0+hx),$$

if the limit exists in D', that is, if

$$\lim_{h\to 0}<f(x_0+hx),\phi(x)>=f(x_0)\int_{-\infty}^{\infty}\phi(x)dx$$

for each $\phi\in D(\mathrm{R})$.

The Heaviside function is defined by

$$H(x)=\begin{cases}1, & x\geq 0\\ 0, & x<0\end{cases}$$

A distribution $f\in D'(\mathrm{R})$ is said to have a distributional jump behavior (or jump behavior) at $x=x_0\in\mathrm{R}$ if itstatisfies the distributional asymptotic relation

$$f(x_0+hx)=c\_H(-x)+c_+H(x)+o(1),$$

as $h \to 0^+$ in D', $c_\pm$ are constants and H is the Heaviside function.

The jump of f at $x = x_0$ is defined as the number $[f]_{x=x_0} = c_+ - c_-$.

A distribution $f \in D'(\mathrm{R})$ is said to have a distributional symmetric jump behavior at $x = x_0 \in \mathrm{R}$ if the jump distribution $\psi_{x_0}(x) = f(x_0 + x) - f(x_0 - x)$ has a jump behavior at $x = 0$. In such a case, we define the jump of f at $x = x_0$ as $[f]_{x=x_0} = [\psi_{x_0}]_{x=0} / 2.$

The (complex) Fourier series for the distribution T is the series

$$T \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{with} \quad c_n \overset{def}{=} \frac{1}{2\pi} < T, e^{-inx} >, \ \forall n \in \mathrm{Z}.$$

Let us define the subset of upper half-plane $\Delta_\theta^+(x_0)$ as the set of z such that $\theta \le \arg(z - x_0) \le \pi - \theta$, where $0 < \theta \le \pi / 2$. Similarly we define the lower half-plane $\Delta_\theta^-(x_0)$.

We may see f as hyperfunction, that is $f(x) = F(x + i0) - F(x - i0)$, where F is analytic for $Imz \ne 0$ or in the sense of distributions it means $f(x) = \lim_{y \to 0^+} (F(x + iy) - F(x - iy))$, then we say F is analytic representation of f.

We say that *U*(*z*), harmonic on $Imz > 0$, is a harmonic representation of $f \in D'(\mathrm{R})$ if

$$\lim_{y \to 0^+} U(x + iy) = f(x)$$

in $D'(\mathrm{R})$.

Definition 4. Tempered derivate D^k of degree $k \in P^n$ for a distribution $k \in D'(\mathrm{R}^n)$ is defined as

$$D^k f = e^{-\frac{x^2}{4}} (e^{\frac{x^2}{4}} f(x))^{(k)}$$

where

$$\frac{x^2}{4} = \frac{x_1^2 + \ldots + x_n^2}{4}.$$

Also the relation

$$D^{e_j} f(x) = f^{(e_j)}(x) + \frac{1}{2} x_j f(x), \quad D^{e_j}(x_j f(x)) = x_j D^{(e_j)}(x) f(x) + f(x) \quad (4.1)$$

is valid.

Definition 5. Tempered integral S^k, of order $k \in P^n$ of locally integrable function *F* is defined as

$$S^k F(x) = e^{-x^2|4} \int_0^x e^{t^2|4} F(t) dt^k,$$

and $k = (k_1, \ldots, k_n) \in P^n$.

We put

$$\hat{x}^r = (1 + |x_1|)^{r_1} \ldots (1 + |x_n|)^{r_n}, \ r = (r_1, \ldots, r_n) \in \square^n,$$

$$\hat{x} = \hat{x}^{(1,\ldots,1)}.$$

1.2 Auxiliary Facts

Proposition 1. A sequence $\{\phi_\nu\}$ of functions $\phi_\nu \in S$ converges to $\phi \in S$ in S as $\nu \to \nu_0$ if and only if

$$\lim_{\nu \to \nu_0} \sup_{x \in \mathrm{R}^n} \left| x^\alpha f^{(k)} \left[\phi_\nu(x) - \phi(x) \right] \right| = 0.$$

Let ϕ be an element of one of the above function spaces D or S, and f be a function for which

$$< T_f, \phi >= \int_{\mathbb{R}^n} f(t)\phi(t)dt, \quad \phi \in D(\phi \in S)$$

exists and is finite. Then T_f is regular distribution on D (or S) generated by f.

Note 1. Fourier transform $\hat{f}$ (see [4]) is continuous linear function from $S(\mathrm{R}^n)$ to $S(\mathrm{R}^n)$.

Proposition 2. A trigonometric series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

converges in D', that is, $\lim_{k \to \infty} \sum_{n=-k}^{k} c_n e^{inx}$ exists as a distribution, if and only if there are constants B and β such that

$$|c_n| \le B |n|^\beta, \ \forall n \ne 0.$$

For every distribution T the Fourier series converges in S' to T.

Note 2. Jump behavior implies symmetric jump behavior, but the converse is not true as shown by *Dirac delta function* (see [4]).

Proposition 3. Let $f \in D'$ have a distributional jump behavior. Suppose that F is analytic representation of f. Then for any $0 < \theta \le \pi / 2$

$$\lim_{z \to z_0,\, z \in \Delta\theta^{\pm}(x_0)} \frac{F(z)}{\log|z - x_0|} = -\frac{[f]_{x = x_0}}{2\pi i}. \qquad (3.1)$$

Proof. Note that if the above relation holds for one analytic representation, then it holds for any analytic representation of f. In fact by the very well known edge of the wedge theorem, any two such analytic representations differ by an entire function, and for entire function (3.1) gives zero.Next, we see that we may assume that. Indeed we can decompose $f = f_1 + f_2$ where f_2 is zero in a neighborhood of x_0 and $f_1 \in S'$. Let F_1 and F_2 be analytic representation of f_1 and f_2, respectively; then F_2 can be continued across a neighborhood of x_0 (edge of wedge theorem once again), hence $F_2(z) = F_2(x_0) + o(|z - x_0|) = o(|\log|z - x_0||)$ as $z \to x_0$. Additionally, f_1 has the same jump behavior as. Thus, we assume that $f \in S'$. Let $\hat{f} = \hat{f}_+ + \hat{f}_-$ be a decomposition such that supp $\hat{f}_- \subseteq (-\infty, 0]$ and supp $\hat{f}_+ \subseteq (-\infty, 0]$. Then,

$$F(z) = \begin{cases} \frac{1}{2\pi} < \hat{f}_+(t),\, e^{izt} >, & \operatorname{Im} z > 0 \\ -\frac{1}{2\pi} < \hat{f}_-(t),\, e^{izt} >, & \operatorname{Im} z < 0 \end{cases}$$

is an analytic representation of f. If we keep the number m on a compact set and $\lambda > 0$, then

$$F(x_0+\frac{m}{\lambda},\frac{\pm 1}{\lambda}) = \pm\frac{1}{2\pi} < \lambda e^{i\lambda x_0 x}\overset{\Lambda}{f}_{\pm}(\lambda x),\ e^{i(m+i)x} >= \frac{[f]_{x=x_0}}{2\pi i}\log\lambda + o(\log\lambda) \text{ as } \lambda\to\infty,$$

where we have used

$$e^{i\lambda x_0 x}\overset{\Lambda}{f}_{\pm}(\lambda x) = \pm[f]_{x=x_0}\frac{\log\lambda}{\lambda i}\delta(x) + o(\frac{\log\lambda}{\lambda}) \text{ as } \lambda\to\infty.$$

Proposition 4. $f\in D'$ and $U(z)$ is harmonic representation of f than

$$\lim_{z\to x_0,\ z\in l_\theta} U(z) = d_1 + \frac{\theta}{\pi}[f]_{x=x_0}$$

where l_θ is a ray in the upper half-plane starting at x_0 and making an angle θ with the ray x_0.

Prposition 5. Set of function $\{h_k;\ k\in P^n\}$ where

$$h_k(x) = h_{k_1}(x_1)\ldots h_{k_n}(x_n)$$
$$h_{k_i}(x_i) = (-1)^{k_i}(\sqrt{2\pi}k_i!)^{-1|2}e^{x_i^2|4}(e^{-x_i^2|2})^{(k_i)},\ i\in\{1,2,\ldots,n\}$$

is complete orthonormal basis for $L^2(\mathrm{R}^n)$ spaces. This means that if $f\in L^2(\mathrm{R}^n)$, series

$$\sum_{k\in P^n} c_k h_k, \qquad c_k = \int_{-\infty}^{\infty} f(x)h_k(x)dx$$

converges to f in $L^2(\mathrm{R}^n)$ and the relation

$$\sum_{k\in P^n}|c_k|^2 < \infty$$

is valid.

The converse, if for complex numbers c_k, $k\in P^n$ is true $\sum_{k\in P^n}|c_k|^2<\infty$, than series

$$\sum_{k\in P^n} c_k h_k$$

converges in L^2 norms to any function of $L^2(\mathrm{R}^n)$.

Proposition 6. It is valid

$$D^r h_k = \sqrt{\frac{k!}{(k+r)!}}h_{k-r} \text{ for } r\le k,\ D^r h_k = 0 \text{ for } r\ne k, \quad (6.1)$$

in which $r\le k$ means $r_i\le k_i$, $i=1,\ldots,n$ and $r\ne k$ means that exists $1\le j\le n$ for which $r_j > k_j$.

Proposition 7. If $F\in L^\infty(\mathrm{R}^n)$, than $SF\in L^2(\mathrm{R}^n)$.

1. Main results

Theorem 1. Let F be analytic in the upper half plane, with distributional boundary values $f(x) = F(x+i0)$. Suppose f has a distributional symmetric jump behavior at $x = x_0$. Then, for any $0 < \theta \le \pi/2$

$$F(z) \sim \frac{i}{\pi}[f]_{x=x_0} \log(z - x_0) \quad \text{as } z \in \Delta\theta^{+}(x_0) \to x_0 .$$

Proof **.** Let ψ_{x_0} be the jump distribution at $x = x_0$. Then ψ_{x_0} has a jump behavior at $x = 0$ and $\left[\psi_{x_0}\right]_{x=0} = 2[f]_{x=x_0}$.

Observe that $U(z) = F(x_0 + z) - F(x_0 - \bar{z})$ is harmonic representation of ψ_{x_0} and $V(z) = -i(F(x_0 + z) + F(x_0 - \bar{z}))$ is harmonic conjugate. Let us show if U is harmonic representation of f in the upper half-plane. Then

$$\lim_{z \to x_0,\, z \in \Delta\theta^{+}(x_0)} \frac{V(z)}{\log|z - x_0|} = \frac{[f]_{x=x_0}}{\pi} \qquad (1.1)$$

for each $0 < \theta \le \pi/2$.

We now show that we may work with any harmonic representation $U(z)$ of f. Suppose that U_1 and U_2 are two harmonic representations of f, then $U = U_1 - U_2$ represents the zerodistribution. Then by applying the reflection principle to the real and imaginary parts of U, we have that U admits a harmonic extension to a neighborhood of x_0.

Consequently, if V_1 and V_2 are harmonic conjugate to U_1 and U_2, we have that $V = V_1 - V_2$ is harmonic conjugate to U, and thus it admits a harmonic extension to a neighborhood of x_0 as well. Therefore $V(z) = O(1) = o(-\log|z - x_0|)$ shows that V_1 satisfies (1.1) if and only if V_2 does. Let F be analytic representation of f on $\operatorname{Im} z \ne 0$.

We can assume then that $U(z) = F(z) - F(\bar{z}),\ \operatorname{Im} z > 0$. We have that $V(z) = -i(F(z) + F(\bar{z})),\ \operatorname{Im} z > 0$, is harmonic conjugate to U. Therefore, an application of proposition 3 is valid (1.1).

Hence, we can apply this result and proposition 4 to U and V and obtain that $F(x_0 - \bar{z}) = F(x_0 + z) + O(1)$ and

$$F(x_0 + z) + F(x_0 - \bar{z}) = \frac{2i}{\pi}[f]_{x=x_0} \log|z| + o(|\log|z||) \quad \text{as } z \in \Delta_\theta^{+}(0) \to 0$$

and therefore follows necessary result.

Theorem2. If for any $r \in P^n$ the following relation is valid

$$\sum \tilde{k}^{-r} |a_k|^2 < \infty \text{ where } \tilde{k} \text{ is vector in } P^n \text{, with components } \tilde{k}_i = \max\{1, k_i\} \text{ and}$$

$\tilde{k}^{-r} = (\tilde{k}_1^{-r_1} \ldots \tilde{k}_n^{-r_n})$, than exists distribution $f \in S'(\mathrm{R}^n)$ such that $f = \sum_{k \in P^n}^{t} a_k h_k$. With *t* is denoted that the series $\sum_{k \in P^n} a_k h_k$ converges in the sence of converges in S' to f.

It is true the vice versa of theorem. Including and the relation $a_k =< f, h_k >,\ k \in P^n$.

Proof. Let f is any element of S'. Let us prove that exists $F \in L^2(\mathrm{R}^n)$ and $k \in P^n$ such that

$$f = D^k$$

is valid.

From the conditions of the theorem, exists $m \in P^n$ and $r \in P^n$, continues function $G(x)$, constant $C > 0$ such that

$$f = G^{(m)} \text{ and } |G(x)| \le C \hat{x}^{r} , \ x \in \mathrm{R}^n .$$

From the proposition 7, exists continues function F and constant C such that

$$D^{m+r+1}F = G^{(m)} \text{ and } |F(x)| \le C \hat{x}^{-1}$$

and $(1 = (1,1,...,1))$.

This means that

$$f = D^{m+r+1}F \text{ where } F \in L^2(\mathrm{R}^n) .$$

Respectivly

$$f = D^r F = D^r (\sum_{k \in P^n} c_k h_k) .$$

Let us show that sequence of tempered distribution (f_ν) converges in S' to $f \in S'(\mathrm{R}^n)$ than exists the sequence of functions $(F_\nu) \in L^2(\mathrm{R}^n)$ and function $F \in L^2(\mathrm{R}^n), \ k \in P^n$ such that

$$f_\nu = D^k F_r , \quad f = D^k F, \ F_r \to F$$

in L^2 norm.

If we put

$$F_{1\nu} = S\,F_\nu \text{ and } F_1 = S\,F$$

while multiplying with a polynomial and differentiation are continues operations in S' from (4.1) the necessary condition is fullfiled.

From Holder inequality we obtain

$$| F_{1r}(x) - F_1(x) | \le e^{-x^2|2} \int_0^x e^{t^2|2} |F_1(t) - F(t)|\, dt \le$$

$$e^{-x^2|2} (\int_0^x e^{t^2|2})^{1|2} (\int_0^x |F_1(t) - F(t)|^2 dt)^{1|2} \le |x|^{1|2} \varepsilon_\nu$$

where $\varepsilon_\nu = \int_{-\infty}^{\infty} |F_\nu(t) - F(t)|^2 dt)^{1|2} \to 0 \text{ when } \nu \to \infty.$

From the way offor functions $F_{1\nu}(x), \ \ F_1(x) \text{ in } S'$, we have that

$$F_{1\nu}(x) \to F_1(x) \text{ in } S' .$$

Respectively

$$f = \sum_{k \in P^n}^{t} a_k D^r h_k \ .$$

From (6.1) we have

$$f \stackrel{t}{=} \sum_{k \in P^n} a_k h_k$$

where

$$a_k = \sqrt{\frac{(k+r)!}{k!}} c_{k+r}, \; k \in P^n$$

While $|c_k| \leq M$ we have that $\sum |c_k|^2 < \infty$.

From inequality

$$\tilde{k}^{r} \leq \frac{(k+r)!}{k!} \leq (1+r)^r \tilde{k}$$

we obtain the following

$$\sum_{k \in P^n} \tilde{k}^{-r} |a_k|^2 < \infty .$$

Since

$$< f, h_s >= \sum_{k \in P^n} < a_k h_k, h_s > = a_s, \; ,$$

while

$$< h_k, h_s >= 0, \text{ for } k \neq s \text{ and } < h_s, h_s >= 1$$

we obtain that coeficients are unique. Let we show the converse.

If $\sum_{k \in P^n} \tilde{k}^{-r} |a_k|^2 < \infty$ from inequality

$$\tilde{k}^{r} \leq \frac{(k+r)!}{k!} \leq (1+r)^r \tilde{k}$$

we have

$$\sum_{k \in P^n} \frac{k!}{(k+r)!} |a_k|^2 \leq \sum_{k \in P^n} \tilde{n}^{-r} |a_k|^2 < \infty$$

respectively

$$F = \sum_{k \in P^n} \sqrt{\frac{k!}{(k+r)!}} a_k h_{k+r}$$

is element of $L^2(\mathrm{R}^n)$.

From (6.1) we obtain

$$D^r F = f \stackrel{t}{=} \sum_{k \in P^n} a_k h_k .$$

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