DOI https://doi.org/10.24297/jam.v20i.8989

Coefficient Bounds and Fekete-Szegö inequality for a Certain Families of Bi-Prestarlike Functions Defined by (M,N)-Lucas Polynomials

Najah Ali Jiben Al-Ziadi[1] and Abbas Kareem Wanas[2]
[1]Department of Mathematics, College of Education
University of Al-Qadisiyah, Al Diwaniyah, Al-Qadisiyah, Iraq
najah.ali@qu.edu.iq
[2]Department of Mathematics, College of Science
University of Al-Qadisiyah, Al Diwaniyah, Al-Qadisiyah, Iraq
abbas.kareem.w@qu.edu.iq

Abstract:
In the current work, we use the (M,N)-Lucas Polynomials to introduce a new families of holomorphic and bi-Prestarlike functions defined in the unit disk $\mathfrak{D}$ and establish upper bounds for the second and third coefficients of the Taylor-Maclaurin series expansions of functions belonging to these families. Also, we debate Fekete-Szegö problem for these families. Further, we point out several certain special cases for our results.

Keywords: Bi-Univalent function, Bi-Prestarlike function, (M,N)-Lucas Polynomials, Coefficient bounds, Fekete-Szegö problem, Subordination.

2010 Mathematical Subject Classification: 30C45, 30C50.

1 Introduction

Indicate by $\mathfrak{A}$ the collection of functions $\mathfrak{U}$ that are holomorphic in the unit disk $\mathfrak{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ that have the shape:

$$\mathfrak{U}(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n. \tag{1.1}$$

Further, let $\mathfrak{S}$ stands for the subfamily of the collection $\mathfrak{A}$ consisting of functions in $\mathfrak{D}$ satisfying (1.1) that are univalent in $\mathfrak{D}$. According to "the Koebe one-quarter theorem" (see [12]), each univalent function of this kind has an inverse $\mathfrak{U}^{-1}$ that fulfills

$$\mathfrak{U}^{-1}(\mathfrak{U}(\xi)) = \xi \qquad (\xi \in \mathfrak{D})$$

and

$$\mathfrak{U}(\mathfrak{U}^{-1}(\zeta)) = \zeta, \quad (|\zeta| < r_0(\mathfrak{U}), r_0(\mathfrak{U}) \geq \frac{1}{4}),$$

where

$$\mathfrak{U}^{-1}(\zeta) = \zeta - a_2\zeta^2 + \left(2a_2^2 - a_3\right)\zeta^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\zeta^4 + \cdots . \tag{1.2}$$

A function $\mathfrak{U} \in \mathfrak{A}$ is said to be bi-univalent in $\mathfrak{D}$ if both $\mathfrak{U}$ and $\mathfrak{U}^{-1}$ are univalent in $\mathfrak{D}$, let we name by the notation $\mathfrak{E}$ the set of bi-univalent functions in $\mathfrak{D}$ satisfying (1.1). In fact, Srivastava et al. [32] refreshed the study of holomorphic and bi-univalent functions in recent years, it was followed by other works as those by Frasin and Aouf [15], Altinkaya and Yalçin

[5], Güney et al. [16] and others (see, for example [1, 3, 8, 10, 11, 18, 21, 22, 23, 26, 27, 28, 29, 30, 31, 33, 34, 35, 38, 39, 41]). The problem to obtain the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n| \qquad (n \in \mathbb{N};\ n \geq 4)$$

for functions $\mathfrak{U} \in \mathfrak{E}$ is still not completely addressed for many of the subfamilies of the bi-univalent function class $\mathfrak{E}$. The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for $\mathfrak{U} \in \mathfrak{S}$ is well known for its rich history in the field of Geometric Function Theory. Its origin was in the disproof by Fekete and Szegö [13] of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity.

A function $\mathfrak{U} \in \mathfrak{A}$ is named starlike of order θ $(0 \leq \theta < 1)$, if

$$\Re\left\{\frac{\xi\mathfrak{U}'(\xi)}{\mathfrak{U}(\xi)}\right\} > \theta, \quad (\xi \in \mathfrak{O}).$$

For $\mathfrak{U} \in \mathfrak{A}$ given by (1.1) and $\mathfrak{J} \in \mathfrak{A}$ defined by

$$\mathfrak{J}(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n,$$

the "Hadamard product" of $\mathfrak{U}$ and $\mathfrak{J}$ is defined by

$$(\mathfrak{U} * \mathfrak{J})(\xi) = \xi + \sum_{n=2}^{\infty} a_n b_n \xi^n, \quad (\xi \in \mathfrak{O}).$$

Ruscheweyh [25] introduced and studied the family of "prestarlike functions" of order θ, that are the function $\mathfrak{U}$ such as $\mathfrak{U} * I_\theta$ is a starlike function of order θ, where

$$I_\theta(\xi) = \frac{\xi}{(1-\xi)^{2(1-\theta)}}, \quad (0 \leq \theta < 1, \xi \in \mathfrak{O}).$$

The function I_θ can be written in the form:

$$I_\theta(\xi) = \xi + \sum_{n=2}^{\infty} \varrho_n(\theta) \xi^n,$$

where

$$\varrho_n(\theta) = \frac{\prod_{i=2}^{n} (i - 2\theta)}{(n-1)!}, \quad n \geq 2.$$

We note that $\varrho_n(\theta)$ is a decreasing function in θ and satisfies

$$\lim_{n \to \infty} \varrho_n(\theta) = \begin{cases} \infty, & if\, \theta < \frac{1}{2} \\ 1, & if\, \theta = \frac{1}{2} \\ 0, & if\, \theta > \frac{1}{2} \end{cases}.$$

With a view to remembering the principle of subordination between holomorphic functions, let the functions $\mathfrak{U}$ and $\mathfrak{J}$ be holomorphic in $\mathfrak{O}$, we name the function $\mathfrak{U}$ is subordinate to $\mathfrak{J}$, if there is a Schwarz function $\hbar$ holomorphic in $\mathfrak{O}$ with

$$\hbar(0) = 0 \quad \text{and} \quad |\hbar(\xi)| < 1 \quad (\xi \in \mathfrak{O})$$

such that

$$\mathfrak{U}(\xi) = \mathfrak{J}\left(\hbar(\xi)\right).$$

This subordination is indicated by

$$\mathfrak{U} \prec \mathfrak{J} \quad \text{or} \quad \mathfrak{U}(\xi) \prec \mathfrak{J}(\xi) \quad (\xi \in \mathfrak{D}).$$

For two polynomials $M(x)$ and $N(x)$ that have real-valued coefficients, the following recurrence relation gives the (M,N)-Lucas Polynomials $L_{M,N,k}(x)$ (see [19]):

$$L_{M,N,k}(x) = M(x)L_{M,N,k-1}(x) + N(x)L_{M,N,k-2}(x) \quad (k \geq 2),$$

with

$$L_{M,N,0}(x) = 2, \quad L_{M,N,1}(x) = M(x) \quad \textit{and} \quad L_{M,N,2}(x) = M^2(x) + 2N(x). \tag{1.3}$$

The function that generates (M,N)-Lucas Polynomial $L_{M,N,k}(x)$ (see [20]) is given by

$$T_{\{L_{M,N,k}(x)\}}(\xi) = \sum_{k=2}^{\infty} L_{M,N,k}(x)\xi^k = \frac{2 - M(x)\xi}{1 - M(x)\xi - N(x)\xi^2}.$$

Remark 1.1. *For particular choices of $M(x)$ and $N(x)$, the (M,N)-Lucas Polynomial $L_{M,N,k}(x)$ leads to various polynomials, among those we list following few here:*
(1) $L_{x,1,k}(x) =: L_k(x)$, the Lucas polynomials,
(2) $L_{2x,1,k}(x) =: P_k(x)$, the Pell-Lucas polynomials,
(3) $L_{1,2x,k}(x) =: J_k(x)$, the Jacobsthal polynomials,
(4) $L_{3x,-2,k}(x) =: F_k(x)$, the Fermat-Lucas polynomials,
(5) $L_{2x,-1,k}(x) =: T_k(x)$, the first kind Chebyshev polynomials.

We also note that the Lucas polynomials and other special polynomials plays an important role in a diversity of disciplines in the mathematical, statistical, physical and engineering sciences. More details associated with these polynomials can be found in [2, 17, 37, 14, 20, 40].

In recent years, the (M,N)-Lucas Polynomial was presented and investigated analogously by the various penmans (see, for example,[2, 4, 6, 7, 9, 24, 36]).

2 Main Results

This section start with defining the families $\mathcal{WN}_{\mathfrak{E}}(\delta, \lambda, \theta; x)$ and $\mathcal{WM}_{\mathfrak{E}}(\tau, \theta; x)$ as follows:

Definition 2.1. *Assume that $\delta \geq 0$, $0 \leq \lambda \leq 1$ and $0 \leq \theta < 1$, a function $\mathfrak{U} \in \mathfrak{E}$ is called in the family $\mathcal{WN}_{\mathfrak{E}}(\delta, \lambda, \theta; x)$ if it fulfills the subordinations:*

$$(1-\delta)\left[(1-\lambda)\frac{\xi\left(\mathfrak{U} * I_\theta\right)'(\xi)}{\left(\mathfrak{U} * I_\theta\right)(\xi)} + \lambda\left(1 + \frac{\xi\left(\mathfrak{U} * I_\theta\right)''(\xi)}{\left(\mathfrak{U} * I_\theta\right)'(\xi)}\right)\right] + \delta\frac{\lambda\xi^2\left(\mathfrak{U} * I_\theta\right)''(\xi) + \xi\left(\mathfrak{U} * I_\theta\right)'(\xi)}{\lambda\xi\left(\mathfrak{U} * I_\theta\right)'(\xi) + (1-\lambda)\left(\mathfrak{U} * I_\theta\right)(\xi)}$$
$$\prec T_{\{L_{M,N,k}(x)\}}(\xi) - 1$$

and

$$(1-\delta)\left[(1-\lambda)\frac{\zeta\left(\mathfrak{J} * I_\theta\right)'(\zeta)}{\left(\mathfrak{J} * I_\theta\right)(\zeta)} + \lambda\left(1 + \frac{\zeta\left(\mathfrak{J} * I_\theta\right)''(\zeta)}{\left(\mathfrak{J} * I_\theta\right)'(\zeta)}\right)\right] + \delta\frac{\lambda\zeta^2\left(\mathfrak{J} * I_\theta\right)''(\zeta) + \zeta\left(\mathfrak{J} * I_\theta\right)'(\zeta)}{\lambda\zeta\left(\mathfrak{J} * I_\theta\right)'(\zeta) + (1-\lambda)\left(\mathfrak{J} * I_\theta\right)(\zeta)}$$
$$\prec T_{\{L_{M,N,k}(x)\}}(\zeta) - 1,$$

where $\mathfrak{J} = \mathfrak{U}^{-1}$ is given by (1.2).

In particular, if we choose $\delta = \lambda = 0$ and $\theta = \frac{1}{2}$ in Definition 2.1, we have $\mathcal{WN}_{\mathfrak{E}}(0,0,\frac{1}{2};x) \equiv S_{\mathfrak{E}}(x)$ for the bi-starlike functions that was given by Altinkaya [4] and satisfying the following subordinations:

$$\frac{\xi\mathfrak{U}'(\xi)}{\mathfrak{U}(\xi)} \prec T_{\{L_{M,N,k}(x)\}}(\xi) - 1$$

and

$$\frac{\zeta\mathfrak{J}'(\zeta)}{\mathfrak{J}(\zeta)} \prec T_{\{L_{M,N,k}(x)\}}(\zeta) - 1.$$

If we choose $\delta = 0$, $\lambda = 1$ and $\theta = \frac{1}{2}$ in Definition 2.1, we have $\mathcal{WN}_{\mathfrak{E}}(0,1,\frac{1}{2};x) \equiv C_{\mathfrak{E}}(x)$ for the bi-convex functions which which was given by Altinkaya [4] and satisfying the following subordinations:

$$1 + \frac{\xi\mathfrak{U}''(\xi)}{\mathfrak{U}'(\xi)} \prec T_{\{L_{M,N,k}(x)\}}(\xi) - 1$$

and

$$1 + \frac{\zeta\mathfrak{J}''(\zeta)}{\mathfrak{J}'(\zeta)} \prec T_{\{L_{M,N,k}(x)\}}(\zeta) - 1.$$

Definition 2.2. *Assume that $0 \le \tau \le 1$ and $0 \le \theta < 1$, a function $f \in \mathfrak{E}$ is called in the family $\mathcal{WM}_{\mathfrak{E}}(\tau,\theta;x)$ if it fulfills the subordinations:*

$$\tau\xi\left(\mathfrak{U} * I_\theta\right)''(\xi) + (2\tau+1)\left(\mathfrak{U} * I_\theta\right)'(\xi) - 2\tau \prec T_{\{L_{M,N,k}(x)\}}(\xi) - 1$$

and

$$\tau\zeta\left(\mathfrak{J} * I_\theta\right)''(\zeta) + (2\tau+1)\left(\mathfrak{J} * I_\theta\right)'(\zeta) - 2\tau \prec T_{\{L_{M,N,k}(x)\}}(\zeta) - 1$$

where $\mathfrak{J} = \mathfrak{U}^{-1}$ is given by (1.2).

In particular, if we choose $\tau = 0$ and $\theta = \frac{1}{2}$ in Definition 2.2, we have $\mathcal{WM}_{\mathfrak{E}}(0,\frac{1}{2};x) \equiv \mathcal{WM}_{\mathfrak{E}}(x)$ which satisfying the following subordinations:

$$\mathfrak{U}'(\xi) \prec T_{\{L_{M,N,k}(x)\}}(\xi)\xi - 1$$

and

$$\mathfrak{J}'(\zeta) \prec T_{\{L_{M,N,k}(x)\}}(\zeta) - 1.$$

Theorem 2.1. *For $\delta \ge 0$, $0 \le \lambda \le 1$ and $0 \le \theta < 1$, let $\mathfrak{U} \in \mathfrak{A}$ belongs to the family $\mathcal{WN}_{\mathfrak{E}}(\delta,\lambda,\theta;x)$. Then*

$$|a_2| \le \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{2\left|\left[(1-\theta)\Omega(\lambda,\delta,\theta) - 2\left(1-\theta\right)^2(\lambda+1)^2\right]M^2(x) - 4\left(1-\theta\right)^2(\lambda+1)^2N(x)\right|}}$$

and

$$|a_3| \le \frac{M^2(x)}{4\left(1-\theta\right)^2(\lambda+1)^2} + \frac{|M(x)|}{2(1-\theta)(3-2\theta)(2\lambda+1)},$$

where

$$\Omega(\lambda,\delta,\theta) = 2\lambda\delta(1-\theta)(1-\lambda) + 2\theta\lambda + 1. \tag{2.1}$$

Proof. Suppose that $\mathfrak{U} \in \mathcal{WN}_{\mathfrak{E}}(\delta,\lambda,\theta;x)$. Then there exists two holomorphic functions $\phi,\psi : \mathfrak{O} \longrightarrow \mathfrak{O}$ given by

$$\phi(\xi) = r_1\xi + r_2\xi^2 + r_3\xi^3 + \cdots \quad (\xi \in \mathfrak{O}) \tag{2.2}$$

and

$$\psi(\zeta) = s_1\zeta + s_2\zeta^2 + s_3\zeta^3 + \cdots \quad (\zeta \in \mathfrak{D}), \tag{2.3}$$

with $\phi(0) = \psi(0) = 0$, $|\phi(\xi)| < 1$, $|\psi(\zeta)| < 1$, $\xi, \zeta \in \mathfrak{D}$ such that

$$\begin{aligned}
&(1-\delta)\left[(1-\lambda)\frac{\xi\left(\mathfrak{U} * I_\theta\right)'(\xi)}{\left(\mathfrak{U} * I_\theta\right)(\xi)} + \lambda\left(1 + \frac{\xi\left(\mathfrak{U} * I_\theta\right)''(\xi)}{\left(\mathfrak{U} * I_\theta\right)'(\xi)}\right)\right] + \delta\frac{\lambda\xi^2\left(\mathfrak{U} * I_\theta\right)''(\xi) + \xi\left(\mathfrak{U} * I_\theta\right)'(\xi)}{\lambda\xi\left(\mathfrak{U} * I_\theta\right)'(\xi) + (1-\lambda)\left(\mathfrak{U} * I_\theta\right)(\xi)}\\
&= -1 + L_{M,N,0}(x) + L_{M,N,1}(x)\phi(\xi) + L_{M,N,2}(x)\phi^2(\xi) + \cdots
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&(1-\delta)\left[(1-\lambda)\frac{\zeta\left(\mathfrak{J} * I_\theta\right)'(\zeta)}{\left(\mathfrak{J} * I_\theta\right)(\zeta)} + \lambda\left(1 + \frac{\zeta\left(\mathfrak{J} * I_\theta\right)''(\zeta)}{\left(\mathfrak{J} * I_\theta\right)'(\zeta)}\right)\right] + \delta\frac{\lambda\zeta^2\left(\mathfrak{J} * I_\theta\right)''(\zeta) + \zeta\left(\mathfrak{J} * I_\theta\right)'(\zeta)}{\lambda\zeta\left(\mathfrak{J} * I_\theta\right)'(\zeta) + (1-\lambda)\left(\mathfrak{J} * I_\theta\right)(\zeta)}\\
&= -1 + L_{M,N,0}(x) + L_{M,N,1}(x)\psi(\zeta) + L_{M,N,2}(x)\psi^2(\zeta) + \cdots .
\end{aligned} \tag{2.5}$$

Combining (2.2), (2.3), (2.4) and (2.5), yield

$$\begin{aligned}
&(1-\delta)\left[(1-\lambda)\frac{\xi\left(\mathfrak{U} * I_\theta\right)'(\xi)}{\left(\mathfrak{U} * I_\theta\right)(\xi)} + \lambda\left(1 + \frac{\xi\left(\mathfrak{U} * I_\theta\right)''(\xi)}{\left(\mathfrak{U} * I_\theta\right)'(\xi)}\right)\right] + \delta\frac{\lambda\xi^2\left(\mathfrak{U} * I_\theta\right)''(\xi) + \xi\left(\mathfrak{U} * I_\theta\right)'(\xi)}{\lambda\xi\left(\mathfrak{U} * I_\theta\right)'(\xi) + (1-\lambda)\left(\mathfrak{U} * I_\theta\right)(\xi)}\\
&= 1 + L_{M,N,1}(x)r_1\xi + \left[L_{M,N,1}(x)r_2 + L_{M,N,2}(x)r_1^2\right]\xi^2 + \cdots
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&(1-\delta)\left[(1-\lambda)\frac{\zeta\left(\mathfrak{J} * I_\theta\right)'(\zeta)}{\left(\mathfrak{J} * I_\theta\right)(\zeta)} + \lambda\left(1 + \frac{\zeta\left(\mathfrak{J} * I_\theta\right)''(\zeta)}{\left(\mathfrak{J} * I_\theta\right)'(\zeta)}\right)\right] + \delta\frac{\lambda\zeta^2\left(\mathfrak{J} * I_\theta\right)''(\zeta) + \zeta\left(\mathfrak{J} * I_\theta\right)'(\zeta)}{\lambda\zeta\left(\mathfrak{J} * I_\theta\right)'(\zeta) + (1-\lambda)\left(\mathfrak{J} * I_\theta\right)(\zeta)}\\
&= 1 + L_{M,N,1}(x)s_1\zeta + \left[L_{M,N,1}(x)s_2 + L_{M,N,2}(x)s_1^2\right]\zeta^2 + \cdots .
\end{aligned} \tag{2.7}$$

It is quite well-known that if $|\phi(\xi)| < 1$ and $|\psi(\zeta)| < 1$, $\xi, \zeta \in \mathfrak{D}$, we get

$$|r_j| \leq 1 \quad and \quad |s_j| \leq 1 \; (j \in \mathbb{N}). \tag{2.8}$$

In the light of (2.6) and (2.7), after simplifying, we find that

$$2(1-\theta)(\lambda+1)a_2 = L_{M,N,1}(x)r_1, \tag{2.9}$$

$$2(1-\theta)(3-2\theta)(2\lambda+1)a_3 - 4(1-\theta)^2(\lambda\delta(\lambda-1)+3\lambda+1)a_2^2 = L_{M,N,1}(x)r_2 + L_{M,N,2}(x)r_1^2, \tag{2.10}$$

$$-2(1-\theta)(\lambda+1)a_2 = L_{M,N,1}(x)s_1 \tag{2.11}$$

and

$$\begin{aligned}
2(1-\theta)(3-2\theta)(2\lambda+1)\left(2a_2^2 - a_3\right) - 4(1-\theta)^2(\lambda\delta(\lambda-1)+3\lambda+1)a_2^2&\\
= L_{M,N,1}(x)s_2 + L_{M,N,2}(x)s_1^2.&
\end{aligned} \tag{2.12}$$

It follows from (2.9) and (2.11) that

$$r_1 = -s_1 \tag{2.13}$$

and

$$8(1-\theta)^2(\lambda+1)^2a_2^2 = L_{M,N,1}^2(x)(r_1^2 + s_1^2). \tag{2.14}$$

If we add (2.10) to (2.12), we obtain

$$4(1-\theta)\left[2\lambda\delta(1-\theta)(1-\lambda) + 2\theta\lambda + 1\right]a_2^2 = L_{M,N,1}(x)(r_2+s_2) + L_{M,N,2}(x)(r_1^2+s_1^2). \tag{2.15}$$

By substitute the value of $r_1^2 + s_1^2$ from (2.14) in the right hand side of (2.15), we conclude that

$$\left[4(1-\theta)\Omega(\lambda,\delta,\theta) - \frac{8L_{M,N,2}(x)}{L_{M,N,1}^2(x)}(1-\theta)^2(\lambda+1)^2\right]a_2^2 = L_{M,N,1}(x)(r_2+s_2), \tag{2.16}$$

where $\Omega(\lambda,\delta,\theta)$ is given by (2.1).

Moreover computations using (1.3), (2.8) and (2.16), we find that

$$|a_2| \le \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{2\left|\left[(1-\theta)\Omega(\lambda,\delta,\theta) - 2(1-\theta)^2(\lambda+1)^2\right]M^2(x) - 4(1-\theta)^2(\lambda+1)^2N(x)\right|}}.$$

Next, if we subtract (2.12) from (2.10), we can easily see that

$$4(1-\theta)(3-2\theta)(2\lambda+1)\left(a_3 - a_2^2\right) = L_{M,N,1}(x)(r_2-s_2) + L_{M,N,2}(x)(r_1^2 - s_1^2). \tag{2.17}$$

In view of (2.13) and (2.14), we get from (2.17)

$$a_3 = \frac{L_{M,N,1}^2(x)}{8(1-\theta)^2(\lambda+1)^2}(r_1^2+s_1^2) + \frac{L_{M,N,1}(x)}{4(1-\theta)(3-2\theta)(2\lambda+1)}(r_2-s_2).$$

Thus applying (1.3), we conclude that

$$|a_3| \le \frac{M^2(x)}{4(1-\theta)^2(\lambda+1)^2} + \frac{|M(x)|}{2(1-\theta)(3-2\theta)(2\lambda+1)}.$$

□

Putting $\delta = \lambda = 0$ and $\theta = \frac{1}{2}$ in Theorem 2.1, we deduce the next outcome:

Corollary 2.1. *[4] If $\mathfrak{U}$ belongs to the family $S_{\mathfrak{E}}(x)$, then*

$$|a_2| \le |M(x)|\sqrt{\left|\frac{M(x)}{2N(x)}\right|}$$

and

$$|a_3| \le M^2(x) + \frac{|M(x)|}{2}.$$

Putting $\delta = 0$, $\lambda = 1$ and $\theta = \frac{1}{2}$ in Theorem 2.1, we deduce the next outcome:

Corollary 2.2. *[4] If $\mathfrak{U}$ belongs to the family $C_{\mathfrak{E}}(x)$, then*

$$|a_2| \le \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{2|M^2(x) + 4N(x)|}}$$

and

$$|a_3| \le \frac{M^2(x)}{4} + \frac{|M(x)|}{6}.$$

Theorem 2.2. *For $0 \le \tau \le 1$ and $0 \le \theta < 1$, let $\mathfrak{U} \in \mathfrak{A}$ belongs to the family $\mathcal{WM}_{\mathfrak{E}}(\tau,\theta;x)$. Then*

$$|a_2| \le \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{\left|\left[3(1-\theta)(3-2\theta)(4\tau+1) - 4(1-\theta)^2(7\tau+3)^2\right]M^2(x) - 8(1-\theta)^2(7\tau+3)^2N(x)\right|}}$$

and

$$|a_3| \le \frac{M^2(x)}{4(1-\theta)^2(7\tau+3)^2} + \frac{|M(x)|}{3(1-\theta)(3-2\theta)(4\tau+1)}.$$

Proof. Suppose that $\mathfrak{U} \in \mathcal{WM}_{\mathfrak{E}}(\tau, \theta; x)$. Then there exists two holomorphic functions $\phi, \psi : \mathfrak{O} \longrightarrow \mathfrak{O}$ such that

$$\begin{aligned} &\tau\xi\left(\mathfrak{U} * I_\theta\right)''(\xi) + (2\tau+1)\left(\mathfrak{U} * I_\theta\right)'(\xi) - 2\tau \\ &= -1 + L_{M,N,0}(x) + L_{M,N,1}(x)\phi(\xi) + L_{M,N,2}(x)\phi^2(\xi) + \cdots \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &\tau\zeta\left(\mathfrak{J} * I_\theta\right)''(\zeta) + (2\tau+1)\left(\mathfrak{J} * I_\theta\right)'(\zeta) - 2\tau \\ &= -1 + L_{M,N,0}(x) + L_{M,N,1}(x)\psi(\zeta) + L_{M,N,2}(x)\psi^2(\zeta) + \cdots, \end{aligned} \tag{2.19}$$

where ϕ and ψ have the forms (2.2) and (2.3). Combining (2.18) and (2.19), yield

$$\begin{aligned} &\tau\xi\left(\mathfrak{U} * I_\theta\right)''(\xi) + (2\tau+1)\left(\mathfrak{U} * I_\theta\right)'(\xi) - 2\tau \\ &= 1 + L_{M,N,1}(x)r_1\xi + \left[L_{M,N,1}(x)r_2 + L_{M,N,2}(x)r_1^2\right]\xi^2 + \cdots \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &\tau\zeta\left(\mathfrak{J} * I_\theta\right)''(\zeta) + (2\tau+1)\left(\mathfrak{J} * I_\theta\right)'(\zeta) - 2\tau \\ &= 1 + L_{M,N,1}(x)s_1\zeta + \left[L_{M,N,1}(x)s_2 + L_{M,N,2}(x)s_1^2\right]\zeta^2 + \cdots. \end{aligned} \tag{2.21}$$

In the light of (2.20) and (2.21), after simplifying, we find that

$$2(1-\theta)(7\tau+3)a_2 = L_{M,N,1}(x)r_1, \tag{2.22}$$

$$3(1-\theta)(3-2\theta)(4\tau+1)a_3 = L_{M,N,1}(x)r_2 + L_{M,N,2}(x)r_1^2, \tag{2.23}$$

$$-2(1-\theta)(7\tau+3)a_2 = L_{M,N,1}(x)s_1 \tag{2.24}$$

and

$$3(1-\theta)(3-2\theta)(4\tau+1)\left(2a_2^2 - a_3\right) = L_{M,N,1}(x)s_2 + L_{M,N,2}(x)s_1^2. \tag{2.25}$$

It follows from (2.22) and (2.24) that

$$r_1 = -s_1 \tag{2.26}$$

and

$$8\left(1-\theta\right)^2(7\tau+3)^2 a_2^2 = L_{M,N,1}^2(x)(r_1^2 + s_1^2). \tag{2.27}$$

If we add (2.23) to (2.25), we obtain

$$6(1-\theta)(3-2\theta)(4\tau+1)a_2^2 = L_{M,N,1}(x)(r_2+s_2) + L_{M,N,2}(x)(r_1^2+s_1^2). \tag{2.28}$$

By substitute the value of $r_1^2 + s_1^2$ from (2.27) in the right hand side of (2.28), we conclude that

$$\left[6(1-\theta)(3-2\theta)(4\tau+1) - \frac{8L_{M,N,2}(x)}{L_{M,N,1}^2(x)}\left(1-\theta\right)^2(7\tau+3)^2\right]a_2^2 = L_{M,N,1}(x)(r_2+s_2), \tag{2.29}$$

Moreover computations using (1.3), (2.8) and (2.29), we find that

$$|a_2| \leq \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{\left|\left[3(1-\theta)(3-2\theta)(4\tau+1) - 4\left(1-\theta\right)^2(7\tau+3)^2\right]M^2(x) - 8\left(1-\theta\right)^2(7\tau+3)^2N(x)\right|}}.$$

Next, if we subtract (2.25) from (2.23), we can easily see that

$$6(1-\theta)(3-2\theta)(4\tau+1)\left(a_3-a_2^2\right)=L_{M,N,1}(x)(r_2-s_2)+L_{M,N,2}(x)(r_1^2-s_1^2). \tag{2.30}$$

In view of (2.26) and (2.27), we get from (2.30)

$$a_3=\frac{L_{M,N,1}^2(x)}{8\left(1-\theta\right)^2(7\tau+3)^2}(r_1^2+s_1^2)+\frac{L_{M,N,1}(x)}{6(1-\theta)(3-2\theta)(4\tau+1)}(r_2-s_2).$$

Thus applying (1.3), we conclude that

$$|a_3|\le\frac{M^2(x)}{4\left(1-\theta\right)^2(7\tau+3)^2}+\frac{|M(x)|}{3(1-\theta)(3-2\theta)(4\tau+1)}.$$

□

Putting $\tau=0$ and $\theta=\frac{1}{2}$ in Theorem 2.2, we deduce the next outcome:

Corollary 2.3. *If $\mathfrak{U}$ belongs to the family $\mathcal{WM}_{\mathfrak{E}}(x)$, then*

$$|a_2|\le\frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{6\left|M^2(x)+3N(x)\right|}}$$

and

$$|a_3|\le\frac{M^2(x)}{9}+\frac{|M(x)|}{3}.$$

In the following theorems, we introduce the Fekete-Szegö Problem of the families $\mathcal{WN}_{\mathfrak{E}}(\delta,\lambda,\theta;x)$ and $\mathcal{WM}_{\mathfrak{E}}(\tau,\theta;x)$.

Theorem 2.3. *For $\delta\ge0$, $0\le\lambda\le1$, $0\le\theta<1$ and $\rho\in\mathbb{R}$, let $\mathfrak{U}\in\mathfrak{A}$ belongs to the family $\mathcal{WN}_{\mathfrak{E}}(\delta,\lambda,\theta;x)$. Then*

$$\left|a_3-\rho a_2^2\right|\le\begin{cases}\frac{|M(x)|}{2(1-\theta)(3-2\theta)(2\lambda+1)}; \\ \qquad for\ \ |\rho-1|\le\frac{\left|(1-\theta)\Omega(\lambda,\delta,\theta)-2(1-\theta)^2(\lambda+1)^2-\frac{4(1-\theta)^2(\lambda+1)^2N(x)}{M^2(x)}\right|}{(1-\theta)(3-2\theta)(2\lambda+1)}, \\ \frac{|M(x)|^3|\rho-1|}{2\left|\left[(1-\theta)\Omega(\lambda,\delta,\theta)-2(1-\theta)^2(\lambda+1)^2\right]M^2(x)-4(1-\theta)^2(\lambda+1)^2N(x)\right|}; \\ \qquad for\ \ |\rho-1|\ge\frac{\left|(1-\theta)\Omega(\lambda,\delta,\theta)-2(1-\theta)^2(\lambda+1)^2-\frac{4(1-\theta)^2(\lambda+1)^2N(x)}{M^2(x)}\right|}{(1-\theta)(3-2\theta)(2\lambda+1)},\end{cases}$$

where $\Omega(\lambda,\delta,\theta)$ is given by (2.1).

Proof. By making use of (2.16) and (2.17), we conclude that

$$\begin{aligned}a_3-\rho a_2^2&=\frac{L_{M,N,1}^3(x)(r_2+s_2)\left(1-\rho\right)}{4\left[L_{M,N,1}^2(x)(1-\theta)\Omega(\lambda,\delta,\theta)-2L_{M,N,2}(x)\left(1-\theta\right)^2\left(\lambda+1\right)^2\right]}\\&\quad+\frac{L_{M,N,1}(x)(r_2-s_2)}{4(1-\theta)(3-2\theta)(2\lambda+1)}\\&=\frac{L_{M,N,1}(x)}{4}\left[\left(\varphi(\rho;x)+\frac{1}{(1-\theta)(3-2\theta)(2\lambda+1)}\right)r_2\right.\\&\quad\left.+\left(\varphi(\rho;x)-\frac{1}{(1-\theta)(3-2\theta)(2\lambda+1)}\right)s_2\right],\end{aligned}$$

where

$$\varphi(\rho;x)=\frac{L_{M,N,1}^{2}(x)(1-\rho)}{L_{M,N,1}^{2}(x)(1-\theta)\Omega(\lambda,\delta,\theta)-2L_{M,N,2}(x)\left(1-\theta\right)^{2}\left(\lambda+1\right)^{2}}.$$

According to (1.3), we find that

$$\left|a_3-\rho a_2^2\right|\le\begin{cases}\frac{|M(x)|}{2(1-\theta)(3-2\theta)(2\lambda+1)}, & 0\le|\varphi(\rho;x)|\le\frac{1}{(1-\theta)(3-2\theta)(2\lambda+1)},\\ \\ \frac{1}{2}\left|M(x)\right|\left|\varphi(\rho;x)\right|, & |\varphi(\rho;x)|\ge\frac{1}{(1-\theta)(3-2\theta)(2\lambda+1)}.\end{cases}$$

After some computations, we obtain

$$\left|a_3-\rho a_2^2\right|\le\begin{cases}\frac{|M(x)|}{2(1-\theta)(3-2\theta)(2\lambda+1)};\\ \qquad for\ |\rho-1|\le\frac{\left|(1-\theta)\Omega(\lambda,\delta,\theta)-2(1-\theta)^2(\lambda+1)^2-\frac{4(1-\theta)^2(\lambda+1)^2N(x)}{M^2(x)}\right|}{(1-\theta)(3-2\theta)(2\lambda+1)},\\ \frac{|M(x)|^3|\rho-1|}{2\left|\left[(1-\theta)\Omega(\lambda,\delta,\theta)-2(1-\theta)^2(\lambda+1)^2\right]M^2(x)-4(1-\theta)^2(\lambda+1)^2N(x)\right|};\\ \qquad for\ |\rho-1|\ge\frac{\left|(1-\theta)\Omega(\lambda,\delta,\theta)-2(1-\theta)^2(\lambda+1)^2-\frac{4(1-\theta)^2(\lambda+1)^2N(x)}{M^2(x)}\right|}{(1-\theta)(3-2\theta)(2\lambda+1)}.\end{cases}$$

□

Putting $\delta=\lambda=0$ and $\theta=\frac{1}{2}$ in Theorem 2.3, we deduce the next outcome:

Corollary 2.4. *[4] If $\mathfrak{U}$ belongs to the family $S_{\mathfrak{E}}(x)$, then*

$$\left|a_3-\rho a_2^2\right|\le\begin{cases}\frac{|M(x)|}{2}; & for\ |\rho-1|\le\frac{|N(x)|}{M^2(x)},\\ \\ \frac{|M(x)|^3|\rho-1|}{2|N(x)|}; & for\ |\rho-1|\ge\frac{|N(x)|}{M^2(x)}.\end{cases}$$

Putting $\delta=\lambda=0$ and $\theta=\frac{1}{2}$ in Theorem 2.3, we deduce the next outcome:

Corollary 2.5. *[4] If $\mathfrak{U}$ belongs to the family $C_{\mathfrak{E}}(x)$, then*

$$\left|a_3-\rho a_2^2\right|\le\begin{cases}\frac{|M(x)|}{6}; & for\ |\rho-1|\le\frac{\left|M^2(x)+4N(x)\right|}{3M^2(x)},\\ \\ \frac{|M(x)|^3|\rho-1|}{2|M^2(x)+4N(x)|}; & for\ |\rho-1|\ge\frac{\left|M^2(x)+4N(x)\right|}{3M^2(x)}.\end{cases}$$

Putting $\rho=1$ in Theorem 2.3, we deduce the next outcome:

Corollary 2.6. *If $\mathfrak{U}$ belongs to the family $\mathcal{WN}_{\mathfrak{E}}(\delta,\lambda,\theta;x)$, then*

$$\left|a_3-a_2^2\right|\le\frac{|M(x)|}{2(1-\theta)(3-2\theta)(2\lambda+1)}.$$

Putting $\rho=1$ in Corollary 2.4, we deduce the next outcome:

Corollary 2.7. *[4] If $\mathfrak{U}$ belongs to the family $S_{\mathfrak{E}}(x)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|M(x)|}{2}.$$

Putting $\rho = 1$ in Corollary 2.5, we deduce the next outcome:

Corollary 2.8. *[4] If $\mathfrak{U}$ belongs to the family $C_{\mathfrak{E}}(x)$, then*

$$\left|a_3 - a_2^2\right| \leq \frac{|M(x)|}{6}.$$

Theorem 2.4. *For $0 \leq \tau \leq 1$, $0 \leq \theta < 1$ and $\rho \in \mathbb{R}$, let $\mathfrak{U} \in \mathfrak{A}$ belongs to the family $\mathcal{WM}_{\mathfrak{E}}(\tau, \theta; x)$. Then*

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \frac{|M(x)|}{3(1-\theta)(3-2\theta)(4\tau+1)}; \\ \qquad for \ \ |\rho - 1| \leq \left|1 - \frac{\frac{4}{3}\left[(1-\theta)^2(7\tau+3)^2 + \frac{2(1-\theta)^2(7\tau+3)^2 N(x)}{M^2(x)}\right]}{(1-\theta)(3-2\theta)(4\tau+1)}\right|, \\ \frac{|M(x)|^3|\rho-1|}{\left|\left[3(1-\theta)(3-2\theta)(4\tau+1)-4(1-\theta)^2(7\tau+3)^2\right]M^2(x)-8(1-\theta)^2(7\tau+3)^2N(x)\right|}; \\ \qquad for \ \ |\rho - 1| \geq \left|1 - \frac{\frac{4}{3}\left[(1-\theta)^2(7\tau+3)^2 + \frac{2(1-\theta)^2(7\tau+3)^2 N(x)}{M^2(x)}\right]}{(1-\theta)(3-2\theta)(4\tau+1)}\right|. \end{cases}$$

Proof. By making use of (2.29) and (2.30), we conclude that

$$\begin{aligned} a_3 - \rho a_2^2 &= \frac{L_{M,N,1}^3(x)(r_2+s_2)(1-\rho)}{2\left[3L_{M,N,1}^2(x)(1-\theta)(3-2\theta)(4\tau+1) - 4L_{M,N,2}(x)(1-\theta)^2(7\tau+3)^2\right]} \\ &+ \frac{L_{M,N,1}(x)(r_2-s_2)}{6(1-\theta)(3-2\theta)(4\tau+1)} \\ &= \frac{L_{M,N,1}(x)}{2}\left[\left(\psi(\rho;x) + \frac{1}{3(1-\theta)(3-2\theta)(4\tau+1)}\right)r_2 \right. \\ &+ \left.\left(\psi(\rho;x) - \frac{1}{3(1-\theta)(3-2\theta)(4\tau+1)}\right)s_2\right], \end{aligned}$$

where

$$\psi(\rho;x) = \frac{L_{M,N,1}^2(x)(1-\rho)}{3L_{M,N,1}^2(x)(1-\theta)(3-2\theta)(4\tau+1) - 4L_{M,N,2}(x)(1-\theta)^2(7\tau+3)^2}.$$

According to (1.3), we find that

$$\left|a_3 - \rho a_2^2\right| \leq \begin{cases} \frac{|M(x)|}{3(1-\theta)(3-2\theta)(4\tau+1)}, & 0 \leq |\psi(\rho;x)| \leq \frac{1}{3(1-\theta)(3-2\theta)(4\tau+1)}, \\ |M(x)|\,|\psi(\rho;x)|, & |\psi(\rho;x)| \geq \frac{1}{3(1-\theta)(3-2\theta)(4\tau+1)}. \end{cases}$$

After some computations, we obtain

$$\left|a_3-\rho a_2^2\right| \leq \begin{cases} \frac{|M(x)|}{3(1-\theta)(3-2\theta)(4\tau+1)}; \\ \qquad for\ \ |\rho-1| \leq \left|1-\frac{\frac{4}{3}\left[(1-\theta)^2(7\tau+3)^2+\frac{2(1-\theta)^2(7\tau+3)^2N(x)}{M^2(x)}\right]}{(1-\theta)(3-2\theta)(4\tau+1)}\right|, \\ \frac{|M(x)|^3|\rho-1|}{\left|\left[3(1-\theta)(3-2\theta)(4\tau+1)-4(1-\theta)^2(7\tau+3)^2\right]M^2(x)-8(1-\theta)^2(7\tau+3)^2N(x)\right|}; \\ \qquad for\ \ |\rho-1| \geq \left|1-\frac{\frac{4}{3}\left[(1-\theta)^2(7\tau+3)^2+\frac{2(1-\theta)^2(7\tau+3)^2N(x)}{M^2(x)}\right]}{(1-\theta)(3-2\theta)(4\tau+1)}\right|. \end{cases}$$

□

Putting $\tau=0$ and $\theta=\frac{1}{2}$ in Theorem 2.4, we deduce the next outcome:

Corollary 2.9. *If $\mathfrak{U}$ belongs to the family $\mathcal{WM}_{\mathfrak{E}}(x)$, then*

$$\left|a_3-\rho a_2^2\right| \leq \begin{cases} \frac{|M(x)|}{3}; & for\ \ |\rho-1| \leq 2\left|1+3\frac{N(x)}{M^2(x)}\right|, \\ \frac{|M(x)|^3|\rho-1|}{6|M^2(x)+3N(x)|}; & for\ \ |\rho-1| \geq 2\left|1+3\frac{N(x)}{M^2(x)}\right|. \end{cases}$$

Putting $\rho=1$ in Theorem 2.4, we deduce the next outcome:

Corollary 2.10. *If $\mathfrak{U}$ belongs to the family $\mathcal{WM}_{\mathfrak{E}}(\tau,\theta;x)$, then*

$$\left|a_3-a_2^2\right| \leq \frac{|M(x)|}{3(1-\theta)(3-2\theta)(4\tau+1)}.$$

Putting $\rho=1$ in Corollary 2.9, we deduce the next outcome:

Corollary 2.11. *If $\mathfrak{U}$ belongs to the family $\mathcal{WM}_{\mathfrak{E}}(x)$, then*

$$\left|a_3-a_2^2\right| \leq \frac{|M(x)|}{3}.$$

References

[1] C. Abirami, N. Magesh and J. Yamini, Initial bounds for certain classes of bi-univalent functions defined by Horadam polynomials, Abstr. Appl. Anal., 2020, Art. ID 7391058, (2020), 1–8.

[2] A. Akgül, (P,Q)-Lucas polynomial coefficient inequalities of the bi-univalent function class, Turkish Journal of Mathematics, 43(2019), 2170–2176.

[3] A. G. Al-Amoush, Certain subclasses of bi-univalent functions involving the Poisson distribution associated with Horadam polynomials, Malaya J. Mat., 7(2019), 618–624.

[4] S. Altinkaya, Inclusion properties of Lucas polynomials for bi-univalent functions introduced through the q-analogue of the Noor integral operator, Turkish J. Math., 43(2019), 620–629.

[5] S. Altinkaya and S. Yalçin, Coefficient estimates for two new subclasses of bi-univalent functions with respect to symmetric points, J. Funct. Spaces, 2015, Art. ID 145242, (2015), 1–5.

[6] S. Altinkaya and S. Yalçin, On the (p,q)-Lucas polynomial coefficient bounds of the bi-univalent function class σ, Boletínde la Sociedad Matemática Mexicana, 25(2019), 567–575.

[7] S. Altinkaya and S. Yalçin, (p,q)-Lucas polynomials and their applications to bi-univalent functions, Proyecciones, 39(5)(2019), 1093–1105.

[8] A. Amourah, Initial bounds for analytic and bi-univalent functions by means of (p,q)-Chebyshev polynomials defined by differential operator, General Letters in Mathematics, 7 (2019), 45–51.

[9] A. Amourah, B. A. Frasin, G. Murugusundaramoorthy and T. Al-Hawary, Bi-Bazilevič functions of order $\vartheta + i\delta$ associated with (p,q)-Lucas polynomials, AIMS Mathematics, 6(5)(2021), 4296–4305.

[10] S. Bulut, Faber polynomial coefficient estimates for a subclass of analytic bi-univalent functions, Filomat, 30(2016), 1567–1575.

[11] M. Caglar, E. Deniz and H. M. Srivastava, Second Hankel determinant for certain subclasses of bi-univalent functions, Turkish J. Math., 41(2017), 694–706.

[12] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[13] M. Fekete and G. Szegö, Eine bemerkung uber ungerade schlichte funktionen, J. London Math. Soc., 2(1933), 85–89.

[14] P. Filipponi and AF. Horadam, Derivative sequences of Fibonacci and Lucas polynomials, Applications of Fibonacci Numbers, 4(1991), 99–108.

[15] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett., 24(2011), 1569–1573.

[16] H. Ö. Güney, G. Murugusundaramoorthy and J. Sokół, Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers, Acta Univ. Sapient. Math., 10(2018), 70–84.

[17] A. F. Horadam and J. M. Mahon, Pell and Pell-Lucas polynomials, The Fibonacci Quarterly, 23(1985), 7–20.

[18] S. Joshi, S. Joshi and H. Pawar, On some subclasses of bi-univalent functions associated with pseudo-starlike functions, J. Egyptian Math. Soc., 24(2016), 522–525.

[19] GY. Lee and M. Aşci, Some properties of the (p,q)-Fibonacci and (p,q)-Lucas polynomials, J. Appl. Math., 2012(2012), 1–18.

[20] A. Lupas, A guide of Fibonacci and Lucas polynomials, Octagon Math. Mag., 7(1999), 2–12.

[21] N. Magesh and S. Bulut, Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, Afr. Mat., 29(2018), 203–209.

[22] E. P. Mazi and T. O. Opoola, On some subclasses of bi-univalent functions associating pseudo-starlike functions with Sakaguchi type functions, General Mathematics, 25(2017), 85–95.

[23] G. Murugusundaramoorthy and S. Yalçin, On the λ-Psedo-bi-starlike functions related to (p,q)-Lucas polynomial, Libertas Mathematica (newseries), 39(2019), 79–88.

[24] H. Orhan and H. Arikan, (P,Q)-Lucas polynomial coefficient inequalities of bi-univalent functions defined by the combination of both operators of Al-Aboudi and Ruscheweyh, Afr. Mat. (2020). https://doi.org/10.1007/s13370-020-00847-5

[25] S. Ruscheweyh, Linear operators between classes of prestarlike functions, *Comment. Math. Helv.*, **52**(4), (1977), 497-509.

[26] H. M. Srivastava and D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egyptian Math. Soc., 23(2015), 242–246.

[27] H. M. Srivastava, S. Bulut, M. Caglar and N. Yagmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat, 27(5)(2013), 831–842.

[28] H. M. Srivastava, S. S. Eker and R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, Filomat, 29(2015), 1839–1845.

[29] H. M. Srivastava, S. S. Eker, S. G. Hamidi and J. M. Jahangiri, Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Bull. Iranian Math. Soc., 44(1)(2018), 149–157.

[30] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some general subclasses of analytic and bi-univalent functions, Afr. Mat., 28(2017), 693–706.

[31] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma-Minda type, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM), 112(2018), 1157–1168.

[32] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23(2010), 1188–1192.

[33] H. M. Srivastava, A. Motamednezhad and E. A. Adegani, Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator, Mathematics, 8(2020), Art. ID 172, 1–12.

[34] H. M. Srivastava, N. Raza, E. S. A. AbuJarad, G. Srivastava and M. H. AbuJarad, Fekete-Szegö inequality for classes of (p, q)-starlike and (p, q)-convex functions, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM), 113(2019), 3563–3584.

[35] H. M. Srivastava and A. K. Wanas, Initial Maclaurin coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination, Kyungpook Math. J., 59(3)(2019), 493–503.

[36] S. R. Swamy, A. K. Wanas and Y. Sailaja, Some special families of holomorphic and Sălăgean type bi-univalent functions associated with (m,n)-Lucas polynomials, Communications in Mathematics and Applications, 11(4)(2020), 563–574.

[37] P. Vellucci and AM. Bersani, The class of Lucas-Lehmer polynomials, Rend Math. Appl., 37(2016), 43–62.

[38] F. Yousef, T. Al-Hawary and G. Murugusundaramoorthy, Fekete-Szegö functional problems for some subclasses of bi-univalent functions defined by Frasin differential operator, Afr. Mat., 30(2019), 495–503.

[39] F. Yousef, B. Frasin and T. Al-Hawary, Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials, Filomat, 32(2018), 3229–3236.

[40] T. Wang and W. Zhang, Some identities involving Fibonacci, Lucas polynomials and their applications, Bull Math. Soc. Sci. Math. Roum., 55(2012), 95–103.

[41] A. Zireh, E. Analouei Adegani and S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions defined by subordination, Bull. Belg. Math. Soc. Simon Stevin, 23(2016), 487–504.