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Coefficient Estimates for Some Subclasses of m-Fold Symmetric Bi-univalent Functions Defined by Linear Operator

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Abstract

The articles introduces and investigates "two new subclasses of the bi-univalent functions $f(z)\&f^{-1}(z)$." These are analytical functions related to the m-fold symmetric function $\mathcal{H}_{\Sigma_m}(\eta,\delta;\alpha)$ and $\mathcal{H}_{\Sigma_m}(\eta,\delta;\beta)$. We calculate the initial coefficients for all the functions that belong to them, as well as the coefficients for the functions that belong to a field where finding these coefficients requires a complicated method. Between the remaining results, the upper bounds for "the initial coefficients $|a_{m+1}|\ \&|a_{2m+1}|$ "are found in our study as well as several examples. We also provide a general formula for the function and its inverse in the m-field. A function $f(z)$ is called analytical if it does not take the same values twice $f(z_1) \neq f(z_2)\ if\ z_1 \neq z_2$. It is called a univalent function if it is analytical at all its points, and the function is called a bi-univalent if it and its inverse are univalent functions together. We also discuss other concepts and important terms. .

Keywords: " Analytic function ,Univalent & Bi-univalent function ,m-fold symmetric function, m-fold symmetric bi-univalent function".

Introduction

"Let $\mathcal{F}$ be the class of analytic functions defined on the open unit disk $U = \{z: z \in \mathbb{C}$ and $|z| < 1\}$ and normalized under the condition $f(0) = 0 = f'(0) - 1$ in U. A function $f \in \mathcal{F}$ has Taylor's series expansion of the form":

$$(1.1) \qquad f(z) = z + \sum_{j=2}^{\infty} a_j z^j \qquad (z \in U).$$

Further, "by S we shall denote the subclass of $\mathcal{F}$ consisting of form (1.1), which is also univalent in U ".

Theory of Koebe One-Quarter [2] is "the image of U under every function f from S contains" one-quarter of the radius.

$$f^{-1}(f(z)) = z,\ \ (z \in U)\ ,\& f^{-1}(f(z)) = w, \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$$

Where

$$(1.2) \qquad f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots$$

and f and f^{-1} are univalent , $f \in \mathcal{F}$ is named "bi-univalent in U."

We symbolize by Σ the class of all "bi-univalent functions" defined in U. $\forall f \in S$

,the function

$$(1.3) \qquad "h(z) = \sqrt[m]{f(z^m)}\ , (z \in U, m \in \mathbb{N})"$$

"is univalent and maps the unit disk U in to a region with m-fold symmetry. A function is said to be m-fold symmetric [4] if it has the following normalized" from:

$$(1.4) \qquad f(z)=z+\sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z\in U. m\in \mathbb{N}).$$

We symbolize" by S_m the class of m-fold symmetric univalent" function "in U, which are normalized by the series expansion "(1.4). "In fact, the functions in class S are one-fold symmetric". Brannan and Taha [3] introduce certain subclasses $S^*(\alpha)$ and$\mathcal{K}(\alpha)$ of" starlike and convex" functions of order $\alpha(0\le\alpha<1)$ respectively [1]. The classes $S^*_\Sigma(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$ "of bi-starlike functions of order β and bi-convex functions of order β corresponding to the function classes $S^*(\alpha)$ and $\mathcal{K}(\alpha)$, were also introduced analogously. For each of the function" class $S^*_\Sigma(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$, "they found non-sharp estimates on the initial coefficients". In [5] Srivastava et al. "specified that m-fold symmetric bi-univalent function analogues to the concept of m-fold symmetric univalent function and these gave some important results, such as each function $f\in\Sigma$ generates an m-fold symmetric bi-univalent function for each $m\in\mathbb{N}$, in their study. Furthermore, for the normalized from of f given by (1.4) is concerned, they obtained the series expansion the expansion for f^{-1} as follows":

$$(1.5) \qquad "g(w)=w-a_{m+1}w^{m+1}+[(m+1)a_{m+1}^2-a_{2m+1}]w^{2m+1}-[\tfrac{1}{2}(m+1)(3m+2)\times a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1}+a_{3m+1}]w^{3m+1}+\cdots"$$

where $f^{-1}(w)=g(w)$.We symbolize by $\sum_m$ "the class of m-fold symmetric bi-univalent functions in U. For $m=1$, formula" (1.5) "coincides with formula" (1.2).

Also many researchers have studied m field such as[7,8,9,10]

Aljarah and Darus [6] defined the following differential operator :

$D^0_{\xi,\sigma,\theta,\tau}f(z)=f(z),$

$D^1_{\xi,\sigma,\theta,\tau}f(z)=[(1-(\theta-\tau)(\sigma-\xi)]f(z)+[(\theta-\tau)(\sigma-\xi)]z\,f'(z),,$

$$(1.6) \qquad D^\delta_{\xi,\sigma,\theta,\tau}f(z)=z+\sum_{k=1}^{\infty}[k(\theta-\tau)(\sigma-\xi)+1]^\delta a_{mk+1}\,z^{mk+1}$$

Where $f(z)\in S_m,\ \xi,\sigma,\theta,\tau\ge 0, \theta>\tau,\sigma>\xi\ ,\delta=1,2,3\ldots$

In the work, we derive "estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$for functions belonging to the general subclasses $\mathcal{H}_{\Sigma_m}(\eta,\delta;\alpha)$ and $\mathcal{H}_{\Sigma_m}(\eta,\delta\,;\beta)$ of $\sum_m$". Also some interesting applications of the results presented here are also discussed . "We now introduce the following general subclasses of" m-fold symmetric "bi-univalent functions".

Definition 1.1: A function $f\in\Sigma_m$ given by (1.4) is said $f\in\mathcal{H}_{\Sigma_m}(\eta,\delta;\alpha)$

$(z,w\in U\ ,\eta\ge 1; 0<\alpha\le 1, m\in N)$ if the following conditions are convinced :

$$(1.7) \qquad f\in\Sigma_m \left|arg\left(z\left(\frac{(1-\eta)D^\delta_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)' + z\left(\eta(D^\delta_{\xi,\sigma,\theta,\tau}f(z))\right)''\right)\right|<\frac{\alpha\pi}{2}$$

and

$$(1.8) \qquad g\in\Sigma_m \left|arg\left(w\left(\frac{(1-\eta)D^\delta_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)' + w\left(\eta(D^\delta_{\xi,\sigma,\theta,\tau}g(w))\right)''\right)\right|<\frac{\alpha\pi}{2}$$

$g=f^{-1}$ is given by (1.5).

Definition 1.2. "A function $f\in\Sigma_m$ given by (1.4) is said " $f\in\mathcal{H}_{\Sigma_m}(\eta,\delta\,;\beta)$

$(z,w\in U\ ,\eta\ge 1; 0\le\beta<1, m\in N)$if the following conditions are convinced :

(1.9) $f \in \Sigma_m \& \ Re\left(z\left(\frac{(1-\eta)D^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)' + z\left(\eta(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))\right)''\right) > \beta$

and

(1.10) $g \in \Sigma_m \& \ Re\left(w\left(\frac{(1-\eta)D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)' + w\left(\eta(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))\right)''\right) > \beta$

$g = f^{-1}$ is given by (1.5) .

for showing our results ,the study need the lemma [**2**].

Lemma 3 [**2**]: "If $h \in \mathcal{P}$, then $|c_k| \le 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h , analytic in U , for which Re($h(z)$)> 0 $(z \in U)$"

Where

$h(z) = 1 + cz + c_2 z^2 + \cdots.\ \ (z \in U).$

2. Main Results

Theorem 2.1. Let $f(z)$ in (1.4) & $f \in \mathcal{H}_{\Sigma\ m}(\eta, \delta; \alpha)$

$(z, w \in U\ , \eta \ge 1; 0 < \alpha \le 1, m \in N)$ Then

(2.1) $|a_{m+1}| \le \frac{2\alpha}{\sqrt{\alpha(2m + 4\eta m^2)(m+1)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta} - (\alpha-1)\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^{\delta}\rangle^2}}$

&

(2.2) $|a_{2m+1}| \le \frac{2\alpha}{(2m + 4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}} + \frac{2\,\alpha^2(m+1)}{\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^{\delta}\rangle^2}$

Proof : If $f \in \mathcal{H}_{\Sigma\ m}(\eta, \delta; \alpha)$ .Then

(2,3) $\left(z\left(\frac{(1-\eta)D^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)' + z\left(\eta(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))\right)''\right) = [p(z)]^{\alpha}$

&

(2,4) $\left(w\left(\frac{(1-\eta)D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)' + w\left(\eta(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))\right)''\right) = [q(w)]^{\alpha}$

where $g(w) = f^{-1}(w), p(z), q(w)$ in $\mathcal{P}$ and have the forms:

(2.5) $p(w) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots$

&

(2.6) $q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots$

"equating the coefficients in (2.3) and (2.4) "

(2.7) $(m + \eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^{\delta} a_{m+1} = \alpha p_m$

(2.8) $(2m + 4\eta m^2)(m+1)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta} a_{2m+1} = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2$

And

(2.9) $-(m + \eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^{\delta} a_{m+1} = \alpha q_m$

(2.10) $(2m + 4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}\left[(m+1)a_{m+1}^2 - a_{2m+1}\right] = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2$

From (2.7) & (2.9) , we get

(2.11) $p_m = -q_m$

And

(2.12) $2\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^\delta\rangle^2 a_{m+1}^2 = \alpha^2(p_m^2+q_m^2)$

Now, by adding (2.8) & (2.10) , we have

(2.13) $(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^\delta(m+1)a_{m+1}^2 = \alpha(p_{2m}+q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2)$

Using (2.12) we get

(2.14) $$(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^\delta(m+1)a_{m+1}^2 = \alpha(p_{2m}+q_{2m}) + \frac{(\alpha-1)\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^\delta\rangle^2 a_{m+1}^2}{\alpha}$$

Therefore ,we obtain

(2.15) $$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{\alpha(2m+4\eta m^2)(m+1)[2(\theta-\tau)(\sigma-\xi)+1]^\delta - (1-\alpha)\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^\delta\rangle^2}$$

Lemma 3 is applied for p_{2m} and q_{2m} ,

(2.16) $$|a_{m+1}| \le \frac{2\alpha}{\sqrt{\alpha(2m+4\eta m^2)(m+1)[2(\theta-\tau)(\sigma-\xi)+1]^\delta - (1-\alpha)\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^\delta\rangle^2}}$$

That provided $|a_{m+1}|$ showed in (2.1).

Next , for finding $|a_{2m+1}|$, by subtracting (2.10) from(2.8)

(2.17) $$(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^\delta\left[2a_{2m+1}-(m+1)a_{m+1}^2\right] = \alpha(p_{2m}-q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2)$$

It follows from (2.11),(2.12) and(2.15), that

$$a_{2m+1} = \frac{\alpha(p_{2m}+q_{2m})}{2(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^\delta} + \frac{\alpha^2(m+1)(2p_m^2)}{4\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^\delta\rangle^2}$$

Lemma 3 applied for " p_m, p_{2m} and q_{2m} "

(2.18) $$|a_{2m+1}| \le \frac{2\alpha}{(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^\delta} + \frac{2\alpha^2(m+1)}{\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^\delta\rangle^2}$$

That provided $|a_{2m+1}|$ as showed(2.2).

Theorem 2.2: Let $f(z)$ given by (1.4) & $f \in \mathcal{H}_{\Sigma_m}(\eta, \delta; \beta)$

$(z, w \in U, \eta \ge 1; 0 \le \beta < 1, m \in N)$.Then

(2.19) $$|a_{m+1}| \le \sqrt{\frac{4(1-\beta)}{(m+1)(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^\delta}}$$

And

(2.20) $$|a_{2m+1}| \le \frac{2(1-\beta)}{(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^\delta} + \frac{4(1-\beta)^2(m+1)}{\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^\delta\rangle^2}$$

Proof : If $f \in \mathcal{H}_{\Sigma_m}(\eta, \delta; \beta)$. Then

(2.21) $\left(z\left(\frac{(1-\eta)D^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)' + z\left(\eta(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))\right)''\right) = \beta + (1-\beta)p(z)$

&

(2.22) $\left(w\left(\frac{(1-\eta)D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)' + w\left(\eta(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))\right)''\right) = \beta + (1-\beta)q(w)$

" $p(z), q(w) \in \mathcal{P}$ ",it has "forms: (2.5) and (2.6)".

Equating the coefficients in (*2.21*) and (*2.22*), we get

(*2.23*) $(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^{\delta} a_{m+1} = (1-\beta)p_m$

(*2.24*) $(2m+4\eta m^2)(m+1)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta} a_{2m+1} = (1-\beta)p_{2m}$

&

(*2.25*) $-\ (m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^{\delta} a_{m+1} = (1-\beta)q_m$

(*2.26*) $(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}\left[(m+1)a^2_{m+1} - a_{2m+1}\right] = (1-\beta)q_{2m}$

Then, by making use of(2.23) & (2.25), we have

(*2.27*) $p_m = -q_m$

&

(*2.28*) $2\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^{\delta}\rangle^2 a^2_{m+1} = (1-\beta)^2(p^2_m + q^2_m)$

Adding(*2.24*)& (*2.26*)we have

(*2.29*) $a^2_{m+1} = \frac{(1-\beta)(p_{2m}+q_{2m})}{(m+1)(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}}$

Lemma 3 applied for p_{2m} and q_{2m}

(*2.30*) $|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(m+1)(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}}}$

That provided $|a_{m+1}|$ as showed in (*2.19*).

Now, for finding $|a_{2m+1}|$, by subtracting (2.26) from (2.24)

(*2.31*) $(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}(2a_{2m+1} - (m+1)a^2_{m+1}) = (1-\beta)(p_{2m} - q_{2m})$

It follows from (*2.27*)&(*2.27*) that

(*2.32*) $a_{2m+1} = \frac{(1-\beta)(p_{2m}-q_{2m})}{2(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}} + \frac{(m+1)(1-\beta)^2(2p^2_m)}{4\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^{\delta}\rangle^2}$

Lemma 3 applied for p_m , p_{2m} and q_m ,

(*2.33*) $|a_{2m+1}| \leq \frac{2(1-\beta)}{(2m+4\eta m^2)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}} + \frac{2(1-\beta)^2(m+1)}{\langle(m+\eta m^2)[(\theta-\tau)(\sigma-\xi)+1]^{\delta}\rangle^2}$

That provided $|a_{2m+1}|$ as showed (2.20).

Corollary 2.3. Let $f(z)$ in (*1.1*) & $f \in \mathcal{H}_{\Sigma_m}(0,\delta;\alpha)$

$(z, w \in U; \eta = 0, 0 < \alpha \leq 1, m \in N)$ Then

(*2.34*) $|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha(2m+4\eta m^2)(m+1) - (\alpha-1)(m+\eta m^2)^2}}$

&

(2.35) $$|a_{2m+1}| \leq \frac{2\alpha}{(2m+4\eta m^2)} + \frac{2\,\alpha^2(m+1)}{(m+\eta m^2)^2}$$

Corollary 2.4. Let $f(z)$ in(1.1) & $f \in \mathcal{H}_{\Sigma_m}(0,\delta;\beta)$

$(z,w \in U, \eta = 0, 0 \leq \beta < 1, m \in N)$.Then

(2.36) $$|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(m+1)(2m+4\eta m^2)}}$$

&

(2.37) $$|a_{2m+1}| \leq \frac{2(1-\beta)}{(2m+4\eta m^2)} + \frac{2(1-\beta)^2(m+1)}{(m+\eta m^2)^2}$$

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