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Coincidence points in θ-metric spaces

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Abstract

In this paper, inspired by the concept of metric space, two fixed point theorems for $\boldsymbol{\alpha}$ −set-valued mapping T:$\mathbb{A}$ → $\boldsymbol{CB}$($\mathbb{A}$), $\mathrm{h}_{\boldsymbol{\theta}}$ $(Tp,Tq) \leq \boldsymbol{\alpha}(d\theta\ (p,q))\ d\theta\ (p,q)$, where $\boldsymbol{\alpha}$: (0,∞) → (0, 1] such that $\boldsymbol{\alpha}(\boldsymbol{r}) < 1$, ∀ t ∈ [0,∞)) are given in complete $\boldsymbol{\theta}$ −metric and then extended for two mappings with $\boldsymbol{R}$-weakly commuting property to obtain a common coincidence point.

Keywords: Generalized metric space, non-commuting mappings, coincidence points.

1. Introduction and preliminaries

Bakhtin [1] defined the b-metric space as a generalization of a usual metric space and proved analogue of Banach's contraction principle. Then several articles have contained fixed points results in this space and its generalizations (e.g. see [1-7] and their **references**). Kamran, Samreen and Ain [8] introduced θ-metric space as an extended to b −metric space and established some fixed points results. Very recent results in this space will appear to the researcher Albundi [9].

Here, the coincidence point results for four mappings. Firstly, start with the following definition [4]:

"Let:$\mathbb{A} \neq \emptyset$ and θ: $\mathbb{A} \times \mathbb{A}$ → [1,∞) and d_θ: :$\mathbb{A}$ ×:$\mathbb{A}$ → [0,∞) be functions. If the following hold ∀ p, q, ∈ $\mathbb{A}$:

(d_θ *1*) $d_\theta(p, q) = 0$ iff $p = q$

(d_θ*2*) $d_\theta(p,q) = d_\theta(q, p)$

(d_θ*3*) $d_\theta(p,r) \leq \theta(p, r)[\, d_\theta(p, q) + d_\theta(q, r)]$.

Then ($\mathbb{A}$, d_θ) is called θ-metric space"

Remark 1.1. If $\theta(p, q) = s$ for $s \geq 1$, then we obtain the definition of a b-metric space.

Example 1.2. If: $\mathbb{A}$ = {1, 2, 3}, and θ: $X \times X$ → [1,∞). A function d_θ: $\mathbb{A} \times \mathbb{A}$ → [0,∞) as:

$$\theta(p, q) = 1 + p + q$$

$d_\theta(1, 1) = d_\theta(2, 2) = d_\theta(3, 3) = 0$

$d_\theta(1, 2) = d_\theta(2, 1) = 80$, $d_\theta(1, 3) = d_\theta(3, 1) = 1000$, $d_\theta(2, 3) = d_\theta(3, 2) = 600$.

Example 1.3." Let $\mathbb{A}$ = ([p, q]) be the space of all continuous real valued functions define on [p, q]. Note that $\mathbb{A}$ is complete extended b -metric space by considering $d_\theta(p,\ q) = \sup_{t\in[p,q]}|p(t) - q(t)|^2$, with $\theta(p,\ q) = |p(t) - q(t)| + 2$, where θ: $\mathbb{A} \times \mathbb{A}$ → [1,∞)" [4].

Definition 1.4 [8]: "Let ($\mathbb{A}$,d_θ) is a θ-metric space and a sequence {p_n} in $\mathbb{A}$ is said to be:

i. Cauchy if and only if $d_\theta(p_n, p_m)$→ 0 as m, n→ ∞.

ii. Converges to a point $p \in \mathbb{A}$ if $d_\theta(p_n, p)$→ 0 as n→ ∞ and we write $\lim_{n\to\infty} p_n = p$.

A θ-metric space is complete if every Cauchy sequence $\mathbb{A}$ is convergent to q in $\mathbb{A}$".

Let $2^{\mathbb{A}} = \{A: \emptyset \neq A \subset \mathbb{A}\}$,

$CB(X)$ = { A: A is a nonempty bounded closed subsets of $\mathbb{A}$}.

"For $p \in ꬰ$ and $A \subseteq X$, $d_\theta(p, A) = \inf\{d_\theta(p, q): q \in A\}$. Let h_θ be the θ-Hausdorff distance [8] with respect to d_θ, that is,

$$h_\theta(A,B) = \max\{d_\theta(p,B), d_\theta(q,A)\}".$$

Immediately, the following is obtained

Lemma 1.5 [8] "If $A, B \in CB(ꬰ)$ and $a \in A$, then $\forall\ \varepsilon > 0$, $\exists\ b \in B$ such that

$$d_\theta(a,b) \le h_\theta(A,B) + \varepsilon".$$

Lemma 1.6 [8] "If $\{A_n\}$ is a sequence in $CB(ꬰ)$ and $h_\theta(A_n, A) = 0$ for $A \in CB(ꬰ)$. If $p_n \in A_n$ and $\lim_{n\to\infty} d_\theta(p_n, p) = 0$, then $p \in A$".

Definition 1.7. "A set valued mapping $T: ꬰ \to 2^{ꬰ}$ is called contraction if $\exists\ k \in (0,1) \ni$

$$h_\theta(T(p), T(q)) \le k\, d_\theta(p, q),\ \forall\ p, q \in ꬰ"$$

Definition 1.8. "A point $p \in ꬰ$ is called fixed point of set-valued mapping $T: ꬰ \to 2^{ꬰ}$ if $p \in Tp$".

Definition 1.9. "The mappings $T: ꬰ \to 2^{ꬰ}$ and $f: ꬰ \to ꬰ$ are coincide at p if $fp \in Tp$."

Definition 1.10. [9], [10] "Let $ꬰ$ be a θ-metric space, $T: ꬰ \to 2^{ꬰ}$ and $f: ꬰ \to ꬰ$ be two mappings then

i. f and T are called commuting if $fTꬰ \subseteq Tfꬰꬰ$.

ii. f and T are called weakly commuting if, $\forall\ p \in ꬰ$, $fTp \in CB(ꬰ)$ and $h_\theta(fTp, Tfp) \le d_\theta(fp, Tp)$.

iii. f and $T$are R-weakly commuting if $\forall\ p \in ꬰ$, $fTp \in CB(ꬰ)$, and $\exists\ R > 0$ such that

$h_\theta(Tf(p), Tf(p)) \le R d_\theta(f(p), T(p))$".

Note the commutativity $\Rightarrow$ weak commutativity $\Rightarrow$ R-weakly commutativity. But the converse is not true. The following example illustrate this when $R > 1$.

Example 1.11. Consider $ꬰ = R$, with $d_\theta = |\ \ |$ (the absolute value) then $(ꬰ, d_\theta)$ is θ-metric space with $\theta(t) = 2$, $\forall\ t$. If $f, g: ꬰ \to ꬰ$, are defined by $T(p) = 2p\text{-}1$, $T(p) = p^2$. Then

$$d_\theta(fgp, gfp) = 2(p\text{-}1)^2, \quad d_\theta(fp, gp) = (p\text{-}1)^2,\ \forall p \in ꬰ.$$

That is, $d_\theta(fgp, gfp) = 2\, d_\theta(fp, gp)$. So, $f$and g are 2-weakly commutating but are not weakly commuting.

In the next section, there are a generalization and an extension of some results in [11] and [12].

2. Main Result

We begin with following theorem.

Theorem 2.1. Let $ꬰ$ be a complete θ- metric space and $T: ꬰ \to CB(ꬰ)$ such that

$$h_\theta(T(p), T(q)) \le k(d_\theta(p, q))\, d_\theta(p, q),\ \ p, q \in ꬰ,$$

where k: $(0,\infty) \to (0, 1]$ is a function $\ni \lim sup_{r\to t^+}\alpha(r) < 1$, for $\forall\ t \in [0,\infty)$. Then, T has a fixed point in $ꬰ$.

Since a function k: $(0,\infty) \to (0, 1]$ such that $\lim sup_{r\to t^+}(r) < 1$, $\forall t \in [0,\infty)$ is special case of the functionα: $(0,\infty) \to (0, 1]$ such that $\alpha(r) < 1$, for $\forall\ t \in [0,\infty)$, so,

A general case which is included in the result below:

Theorem 2.2. Assume $(ꬰ, d_\theta)$ be a complete θ- metric space, and $T: ꬰ \to CB(ꬰ)$.

$$h_\theta(T(p), T(q)) \le \alpha(d_\theta(p,q))\, d_\theta(p,q),\ \forall p, q \in ꬰ,$$

where α: $(0,\infty) \to (0, 1]$ is a function with $\alpha(r) < 1$, $\forall\ t \in [0,\infty)$.

Then T has a fixed point in $ꬰ$.

Proof: Suppose $p_0 \in ꬰ$ and $p_1 \in T(p_0)$. Choose a $n_1 \in N \ni$

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$\alpha^{n_1}(d_\theta(p_0, p_1) \leq \{1-\alpha(d_\theta(p_0, p_1))\}\, d_\theta(p_0, p_1)$.

Choose $p_2 \in T(p_1)$ with definition of the θ-Hausdorff distance,

$d_\theta(p_2, p_1) \leq h_\theta(T(p_1), T(p_0)) + \alpha^{n_1}(d_\theta(p_0, p_1)$.

Therefore,

$d_\theta(p_2, p_1) \leq \alpha(d_\theta(p_1, p_0))\, d_\theta(p_1, p_0) + \alpha^{n_1}(d_\theta(p_0, p_1) < d_\theta(p_1, p_0)$.

Now, choose $n_2 \in N$, $n_2 > n_1 \ni$

$\alpha^{n_2}((d_\theta(p_2, p_1)) < \{1-\alpha(d_\theta(p_2, p_1))\}\, d_\theta(p_2, p_1)$.

Since $T(p_2) \in CB(\not{A})$, choose $p_3 \in T(p_2)$ so

$d_\theta(p_3, p_2) \leq h_\theta(T(p_2), T(p_1)) + \alpha^{n_2}(d_\theta(p_2, p_1))$.

Then

$d_\theta(p_3, p_2) \leq h_\theta(T(p_2), T(p_1)) + \alpha^{n_2}(d_\theta(p_2, p_1))$.

$\leq \alpha(d_\theta(p_2, p_1))\, d_\theta(p_2, p_1) + \alpha^{n_2}(d_\theta(p_2, p_1))$

$< d_\theta(p_2, p_1)$.

Again, for each k with $T(p) \in CB(\not{A})$. Choose $n_k \in N \ni$

$\alpha^{n_k}((d_\theta(p_k, p_{k-1})) < \{1-\alpha(d_\theta(p_k, p_{k-1}))\}\, d_\theta(p_k, p_{k-1})$.

Now choose $p_{k+1} \in T(p_k)$ then

$d_\theta(p_{k+1}, p_k) \leq h_\theta(T(p), T(p_{k-1})) + \alpha^{n_k}(d_\theta(p_k, p_{k-1}))$.

So, $d_\theta(p_{k+1}, p_k) < d_\theta(p_k, p_{k-1})$ then $d_k \equiv d_\theta(p_k, p_{k-1})$ is called a monotone non-increasing sequence of nonnegative number.

Now, the sequence $\{d_k\}$ so generated is Cauchy.

Let $\lim_{k\to\infty} d_{\theta_k} = c \geq 0$. By assumption, $\alpha(t) < 1$.

Hence $\exists\, k_0 \ni k \geq k_0 \Rightarrow \alpha(d_{\theta_k}) < h$, if $\alpha(t) < h < 1$.

Now,

$d_{\theta_{k+1}} = d_\theta(p_{k+1}, p_k)$

$\leq h_\theta(T(p_k), T(p_{k-1})) + \alpha^{n_k}(d_{\theta_k})$

$\leq \alpha(d_{\theta_k})\, d_{\theta_k} + \alpha^{n_k}(d_{\theta_k})$

$\leq \alpha(d_{\theta_k})\ \alpha(d_{\theta_{k-1}})\, d_{\theta_{k-1}} + \alpha(d_{\theta_k})\, \alpha^{n_{k-1}}(d_{\theta_{k-1}})\, \alpha^{n_k}(d_{\theta_k})$

....

$\leq \prod_{i=1}^{k}(d_{\theta_i})\, d_{\theta_1} + \sum_{m=1}^{k-1}\prod_{i=m+1}^{k}\alpha(d_{\theta_i})\, \alpha^{n_m}(d_{\theta_m}) + \alpha^{n_k}(d_{\theta_k})$

$\leq \prod_{i=1}^{k}(d_{\theta_i})\, d_{\theta_1} + \sum_{m=1}^{k-1}\prod_{i=max\{k_0,m+1\}}^{k}\alpha(d_{\theta_i})\, \alpha^{n_m}(d_{\theta_m}) + \alpha^{n_k}(d_{\theta_k}) \equiv A$.

From above inequality, we benefited by the fact that $\alpha < 1$ to delete some α factors from the product.

Now

$\sum_{m=1}^{k-1}\prod_{i=max\{k_0,m+1\}}^{k}\alpha(d_{\theta_i})\, \alpha^{n_m}(d_{\theta_m}) \leq (k_0-1)\, h^{k-k_0+1}\sum_{m=1}^{k_0-1}\alpha^{n_m}(d_{\theta_m})$

$+ \sum_{m=1}^{k_0-1} h^{k-m}\, \alpha^{n_m}(d_{\theta_m})$

$\leq (k_0-1)\, h^{k-k_0+1}\sum_{m=1}^{k_0-1}\alpha^{n_m}(d_{\theta_m}) + + \sum_{m=k_0}^{k-1} h^{k-m+n_m}$

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$\leq Ch^k + \sum_{m=k_0}^{k-1} h^{k-m_{n_m}}$

$\leq Ch^k + h^{k+n_{k_0}-k_0} + h^{k+n_{k_0-1}-(k_0-1)} + \ldots + h^{k+n_{k-1}-(k-1)}$

$\leq Ch^k + \sum_{m=k+n_{k_0}-k_0}^{k+n_{k-1}-(k-1)} h^m$

$= Ch^k + \frac{h^{k+n_{k_0}-k_0+1} - h^{k+n_{k-1}-k+2}}{1-h}$

$= Ch^k + h^k \frac{h^{n_{k_0}-k_0+1}}{1-h}$

$= Ch^k$

where $C > 0$. Now,

$A \leq \prod_{i=1}^{k} \alpha\,(d_{\theta_i})\, d_{\theta_1} + Ch^k + \alpha^{n_k}\,(d_{\theta_k})$

$< h^{k-k_0+1} \prod_{i=1}^{k_0-1} \alpha\,(d_{\theta_i})\, d_{\theta_1} + Ch^k + h^{n_k}$

$< Ch^k + Ch^k + h^k$

$= Ch^k,$

C is a generic constant. If $k \geq k_0$, $m \in N$, so $\{x_k\}$ is Cauchy.

$d_\theta\,(p_k, p_{k+m}) \leq d_\theta\,(p_k, p_{k+1}) + \ldots + d_\theta\,(p_{k+m-1}, p_{k+m})$

$= \sum_{i=k+1}^{k+m} d_{\theta_i}$

$< \sum_{i=k+1}^{k+m} Ch^{i-1}$

$= C\,\frac{h^{k+1} - h^{k+m}}{1-h}$

$\leq h^k,$

which tends to zero as $k \to \infty$. Let $p_k \to \ \in$ Ꞩ, so

$d_\theta\,(p, T(p)) \leq d_\theta\,(p, p_k) + d_\theta\,(p_k, T(p))$

$\leq d_\theta\,(p, p_k) + \alpha\,(d_\theta\,(p_{k-1}, p))\, d_\theta\,(p_{k-1}, p).$

From above expression, both terms tent to zero as $k \to \infty$, then $p \in (p_k)$.

$d_\theta\,(T(p), p) \leq \theta(T(p), p))[\, d_\theta\,(T(p), p_n) + d_\theta\,(p_n, p)].$

≤ 0 as $k \to \infty$

So,

$d_\theta\,(T\,(p), p) \ \leq \theta(T\,(p), p)[k\, d_\theta\,(p, p_{n-1}) + d_\theta\,(p_n, p)]$

$d_\theta\,(T\,(p), p) = 0.$

Hence p is called a fixed point in T.

Theorem 2.3. Let Ꞩ be a complete θ- metric space, if f, g: Ꞩ → Ꞩ, and H, J: Ꞩ → CB(Ꞩ) are continuous mappings ∋ HꞨ ⊆ gꞨ, and JꞨ ⊆ fꞨ such that

$h_\theta\,(Hp, Jq) \leq \alpha(d_\theta\,(gp, fq))\, d_\theta\,(gp, fq)$, $p, q \in$ Ꞩ (1)

where α: $(0,\infty) \to (0, 1]$ ∋ $\lim sup_{r\to t^+}\alpha(r) < 1$, for ∀ $t \in [0,\infty)$. If (g, J) and (f, H) are R-weakly commuting. Then g, H and f, J have a common coincidence point.

Proof: We organize sequences $\{p_n\}$, $\{q_n\}$, and $\{A_n\}$ in X and $CB(X)$. Let $p_0 \in$ Ꞩ, and $q_0 = f\ p_0$.

Since $Hp_0 \subseteq g$Ꞩ, ∃ $p_1 \in$ Ꞩ ∋ $q_1 = g\ p_1 \in H\ p_0 = A_0$. Select $n_1 \in N$ ∋

$\alpha^{n_1}\,((d_\theta\,(q_0, q_1)) < \{1 - \alpha\,(d_\theta\,(q_0, q_1))\}\, d_\theta\,(q_0, q_1).$ (2)

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By Lemma 1.5 and $J\text{Ꞩ} \subseteq f\text{Ꞩ}$, $\exists\ q_2 = f\ p_2 \in J\ p_1 = A_1 \ni$

$$d_\theta (q_2, q_1) \le h_\theta (A_1, A_0) + \alpha^{n_1} ((d_\theta (q_0, q_1)). \quad (3)$$

From (1) and (2) $\Rightarrow d_\theta (q_2, q_1) < d_\theta (q_0, q_1)$. Now select $n_2 \in N \ni n_2 > n_1$ such that

$$\alpha^{n_2} ((d_\theta (q_2, q_1)) < \{1- \alpha (d_\theta (q_2, q_1))\} d_\theta (q_2, q_1). \quad (4)$$

By Lemma 1.5 and $H\text{Ꞩ} \subseteq g\text{Ꞩ}X$, implies that $q_3 = g\ p_3 \in H\ p_2 = A_2 \ni$

$$d_\theta (q_3, q_2) \le h_\theta (A_2, A_1) + \alpha^{n_2} ((d_\theta (q_2, q_1)). \quad (5)$$

So, (1) and (4) $\Rightarrow d_\theta (q_3, q_2) < d_\theta (q_2, q_1)$.

Now, by induction, getting $\{p_n\}$, $\{q_n\}$ in Ꞩ and $\{A_n\}$ in $CB(\text{Ꞩ}) \ni$

$$q_{2k+1} = g\ q_{2k+1} \in H\ p_{2k} = A_{2k}, \qquad q_{2k} = f\ p_{2k} \in J\ p_{2k-1} = A_{2k-1} \quad (6)$$

$$d_\theta (q_{2k+1}, q_{2k}) \le h_\theta (A_{2k}, A_{2k-1}) + \alpha^{n_k} ((d_\theta (q_{2k}, q_{2k-1})). \quad (7)$$

where

$$\alpha^{n_{2k}} ((d_\theta (q_{2k}, q_{2k-1})) < \{1- \alpha (d_\theta (q_{2k}, q_{2k-1}))\} d_\theta (q_{2k}, q_{2k-1}). \quad (8)$$

So, $d_\theta (q_{2k+1}, q_{2k}) < d_\theta (q_{2k}, q_{2k-1})$, $\forall\ k$.

So, the real sequence $\{d_\theta (q_{2k+1}, q_{2k})\}$ is monotone non-increasing.

As proof of Theorem 2.1, $\{q_n\}$ is Cauchy sequence in Ꞩ.

Moreover, (1) implies that $\{A_n\}$ is a Cauchy sequence in $CB(\text{Ꞩ})$. If Ꞩis complete then is $CB(\text{Ꞩ})$. Thus, when $q_n \to r$ and $A_n \to A$, $\exists\ r \in X$ and $A \in CB(\text{Ꞩ})$. So, $g\ p_{2k+1} \to r$ and $f\ p_{2k} \to r$. Since

$$d_\theta (r, A) = d_\theta\ (q_n, A_n) \le \lim_{n\to\infty} h_\theta (A_{n-1}, A_n) = 0 \quad (9)$$

By Lemma 1.6, $r \in A$. Also

$$\lim_{k\to\infty} fp_{2k} = r \in A = \lim_{k\to\infty} H\ p_{2k}, \qquad \lim_{k\to\infty} gp_{2k+1} = r \in A = \lim_{k\to\infty} Jp_{2k-1} \quad (10)$$

By (6) and R-weak commutativity of (g, J) and (f, H), we obtain

$$d_\theta (gfp_{2k+2}, fgp_{2k+1}) \le h_\theta (gJp_{2k+1}, Jgp_{2k+1}) \le R\ d_\theta (gp_{2k+1}, Jp_{2k+1}),$$

$$d_\theta (fgp_{2k+1}, Hfp_{2k}) \le h_\theta (fHp_{2k}, Hfp_{2k}) \le R\ d_\theta (fp_{2k}, Hp_{2k}). \quad (11)$$

Then, the continuity of f, g, J and H give $\in Jr$ and $fr \in Hr$. The proof is complete.

If we set J=H and $f = g$ in Theorem (2.2), the following corollary.

Corollary 2.4. If Ꞩ be a complete θ-metric space and f: Ꞩ → Ꞩ, T: Ꞩ → $CB(\text{Ꞩ})$ are continuous mappings ∋ $T\text{Ꞩ}\subseteq f\text{Ꞩ}$ such that

$$h_\theta(Tp, Tq) \le \alpha(d_\theta(fp,fq))\ d_\theta(fp,fq),\ p,\ q \in \text{Ꞩ},$$

where α: $(0, \infty) \to (0, 1] \ni \lim\ sup_{r\to t^+} \alpha(r) < 1$, $\forall\ t \in [0,\infty)$ and . If f, T are called R-weakly commuting. Then f, T have a coincidence point.

Our results are generalization and an extension of the results in [11] and [12].

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