DOI: https://doi.org/10.24297/jam.v20i.8912

ON POINTWISE PRODUCT VECTOR MEASURE DUALITY

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Abstract:

This article is devoted to the study of pointwise product vector measure duality. The properties of Hilbert function space of integrable functions and pointwise sections of measurable sets are considered through the application of integral representation of product vector measures, inner product functions and products of measurable sets.

Keywords: Measurable sets, Vector measure duality, Integrable functions.

1 Introduction

Over the years, mathematics scholars have studied inner product functions in Banach spaces. In previous results, many theories on integration of vector valued functions with respect to vector measure duality have been proved. This paper explores pointwise projections and sections of measurable sets. The study proves the existence of integral representation of pointwise product vector measure duality with values in a Hilbert space. Throughout this paper, $(\mu_i)_{i\in I}$ and $(\nu_i)_{i\in I}$ denote families of set functions indexed by a finite set I. The functions $(\mu_i)_{i\in I}$ and $(\nu_i)_{i\in I}$ defined on sigma rings ρ and ε with values in Hilbert spaces X and Y respectively i.e $\mu_i : \rho \to X$ and $\nu_i : \varepsilon \to Y$ for each $i \in I$, are called vector measures. The function $\mu_i \times \nu_i : \rho \times \varepsilon \to X \times Y$ denotes the product of vector measures $\mu_i : \rho \to X$ and $\nu_i : \varepsilon \to Y$. The function $(T_{t_{\mu_i \times \nu_i}})_{i\in I}$ denotes a family of non-negative vector measures in $M(\rho \times \varepsilon, X \times Y)$, where T_t is an integrable vector valued function with respect to a vector measure ν_i for each $i \in I$. There exists a vector measure function g_i^t for $t \in \mathbf{R}$ in $M(\rho \times \varepsilon, X \times Y)$, where $M(\rho \times \varepsilon, X \times Y)$ is a set of $X \times Y$ valued vector measures defined on $\rho \times \varepsilon$.

2 Basic concepts

Definition 1 (Okada et al.,2008, Otanga et al., 2015a, Yaogan, 2013)

If $g_i^t \leq T_{t_{\mu_{i_k} \times \nu_{i_k}}}$ for $i < k_i$ is an increasingly directed family of vector

measures, we define the function g_i^t by

$$g_i^t(A \times B) = LUB_{i_k} T_{t_{\mu_{i_k} \times \nu_{i_k}}}(A \times B)$$

where i, $i_k \in I$, $t \in \mathbf{R}$ and $A \times B$ is a fixed measurable set with respect

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to $\rho \times \varepsilon$.

If $\psi^{(A\times B)}$ is a vector valued function, where $A \times B \in \rho \times \varepsilon$ is a fixed set, then the product of $\psi^{(A\times B)}$ and $T_{t\mu_{i_k} \times \nu_{i_k}}$ is given by

$$\psi^{(A\times B)} * T_{t\mu_{i_k} \times \nu_{i_k}} = T_{t\mu_{i_k} \times \nu_{i_k}}(A \times B) \in X \times Y$$

If $b \in B$ is a fixed element, then the set $(A \times B)^b$ is measurable with respect to ρ. Therefore,

$$LUB_{i_k} \psi^{(A\times B)^b} * T_{t\mu_{i_k}} = g_i^t((A \times B)^b) = LUB_{i_k} T_t \mu_{i_k}(A \times B)^b \in X$$

For each $i \in I$ and $G \subset X \times Y$, we define

$\nu_i * T_t\mu_i((A \times B)^b) = \int T^t \mu_i((A \times B)^b)\delta\nu_i \in G.$

Definition 2 (Dorlas, 2010 and Otanga, 2015a)

If $Q \in \rho \times \varepsilon$, then the set $Q^b = (A \in \rho : A \times B \subset Q)$ where $b \in B$ and A is a fixed set of finite measure, is called a *fixed segment* of the set Q.

Definition 3 (Rodriguez, 2006)

If $T_t\mu_i \times \nu_i : \rho \times \varepsilon \to X \times Y$ is a product vector measure, then the integral of the function T_t with respect to the *pointwise product vector measure duality* is given by $< \int T_t\mu_i((A \times B)^b)\delta\nu_i, g^* >$, where $A \times B$ is a fixed measurable set with respect to $\rho \times \varepsilon$, $b \in B$, g^* is an element in G^*, the dual space of the Hilbert space G.

Definition 4 (Otanga et al., 2015b and Sanchez, 2004)

Let $T_t\mu_i \times \nu_i \in M(\rho \times \varepsilon, X \times Y)$. Consider a fixed element $b \in B$ and the integral map $\psi_{g^*}^{(A\times B)}$ where $g^* \in G^*$. The pointwise integral of the function T_t generated by $\psi_{g^*}^{(A\times B)}$ is given by

$$\psi_{g^*}^{(A\times B)^b}(\nu_i * T_t\mu_i) =< \int T_t\mu_i((A\times B)^b)\delta\nu_i, g^* >$$

3 Results

The following propositions are devoted to analyse pointwise integral representation of the product vector measure $\nu_i * T_t\mu_i$ and its relationship with the inner product vector measure duality denoted by $< \int T_t\mu_i((A\times B)^b)\delta\nu_i, g^* >$ for each $i \in I$, where $t \in \mathbf{R}$, $A \times B$ is a measurable set with respect to $\rho \times \varepsilon$, $b \in B$ is a fixed element and g^* is an element in G^*, the dual space of the Hilbert space G.

Proposition 1

Let $(\Phi, \rho \times \varepsilon, \mu_i \times \nu_i)$ for each $i \in I$ be a measure space, where Φ is a non-empty two dimensional set and $\psi^{(\mu_i \times \nu_i)} \in M(\rho \times \varepsilon, X \times Y)$. Then $\int \psi(T_t\mu_i)\delta\nu_i = \psi(\nu_i * T_t\mu_i)$ for each $i \in I$ and $t \in \mathbf{R}$

Proof

Let $\psi_{g^*}^{(A\times B)^b}$ be an integral function, where $A \times B \in \rho \times \varepsilon$, $g^* \in G^*$ and $b \in B$ is a fixed element. As a consequence of integral representation of the product vector measure $\nu_i * T_t\mu_i$ as illustrated in (Otanga et al., 2015a and Yaogan, 2013), we obtain

$$\int \psi_{g^*}^{(A\times B)^b}(T_t\mu_i)\delta\nu_i = < \int (T_t\mu_i)(A\times B)^b \delta\nu_i, g^* > = < (\nu_i * T_t\mu_i(A\times B)^b, g^* >$$

$$= \psi_{g^*}^{(A\times B)^b}(\nu_i * T_t\mu_i)$$

Therefore, $\int \psi(T_t\mu_i)\delta\nu_i = \psi(\nu_i * T_t\mu_i)$ for each $i \in I$ and $t \in \mathbf{R}$

Proposition 2

Let $(\mu_i)_{p_i}^{p_j} = T_s(\mu_i)_{p_i}^{p_j}$ $s \in \mathbf{R}$. Then $\nu_i * (T_t(\mu_i)_{p_i}^{p_j}) = T_t(\nu_i * \mu_i^{p_i})$, where p_i and p_j are measurable sets of finite measure, $p_i \subset p_j$ for each $i, j \in I$, $i \neq j$ and $s, t \in \mathbf{R}$.

Proof

Let $A \times B$ be a measurable set with respect to $\rho \times \varepsilon$ and $g^* \in G^*$. For a fixed element $b \in B$, the set $(A \times B)^b$ is measurable with respect to ρ. It follows that

$$< \nu_i * (T_t(\mu_i)_{p_i}^{p_j})^{(A\times B)^b}, g^* > = < \int T_{t+s}\mu_i^{p_j}(p_i \cap (A\times B)^b)\delta\nu_i, g^* > \text{ for } s, t \in \mathbf{R}$$

Applying the results in (Otanga, 2015b and Otanga et al., 2015b) on contraction of a vector measure μ_i by p_i for each $i \in I$, we obtain

$< \int T_{t+s}\mu_i(p_i \cap p_j \cap (A\times B)^b)\delta\nu_i, g^* > = < \int T_{t+s}\mu_i(p_i \cap (A\times B)^b)\delta\nu_i, g^* >$

$= < \int T_{t+s}(\mu_i)^{p_i}(A\times B)^b)\delta\nu_i, g^* > = \int \psi_{g^*}^{(A\times B)^b}(T_t(T_s\mu_i^{p_i})\delta\nu_i$

$= \int T_t\psi_{g^*}^{(A\times B)^b}(T_s\mu_i^{p_i})\delta\nu_i$. Since $\int T_s\mu_i^{p_i}\delta\nu_i = \nu_i * T_s\mu_i^{p_i}$, it follows that

$\int T_t\psi_{g^*}^{(A\times B)^b}(T_s\mu_i^{p_i})\delta\nu_i = T_t(\psi_{g^*}^{(A\times B)^b})(\nu_i * T_s\mu_i^{p_i}))$. On application of the

relation $T_s\mu_i^{p_i} = \mu_i^{p_i}$ for $i \in I$ and $s \in \mathbf{R}$, we obtain

$T_t(\psi_{g^*}^{(A\times B)^b})(\nu_i * T_s\mu_i^{p_i})) = \psi_{g^*}^{(A\times B)^b}(T_t(\nu_i * \mu_i^{p_i}))$

$= < T_t(\nu_i * \mu_i^{p_i})(A\times B)^b, g^* >.$

In general, $\nu_i * (T_t(\mu_i)_{p_i}^{p_j}) = T_t(\nu_i * \mu_i^{p_i})$.

Proposition 3

Let ψ_{g^*} represent an integral function with respect to vector measure

duality and $\mu_i \times \nu_i \times \alpha_i$ for each $i \in I$, be a vector measure defined on

$\rho \times \varepsilon \times \tau$. If $< \int (T_t\mu_i)\delta\nu_i, g^* > = \psi_{g^*}(\nu_i * T_t\mu_i)$ for each $g^* \in G^*$, then

$(\alpha_i * \nu_i) * T_t\mu_i = \alpha_i * (\nu_i * T_t\mu_i)$ for $t \in \mathbf{R}$

Proof

Let $A \times B \times C \in \rho \times \varepsilon \times \tau$ and $g^* \in G^*$. Application of the Integral function

$(\psi_{g^*})^{(A\times B\times C)}$ as illustrated in (Campo et al., 2010), gives

$(\psi_{g^*})^{(A\times B\times C)}(\alpha_i \times \nu_i) * T_t\mu_i = < \int\int (T_t\mu_i)(A\times B\times C)\delta\alpha_i\delta\nu_i, g^* >$

Suppose $(b, c) \in B \times C$ is a fixed point. The set $(A\times B\times C)^{(b,c)}$ is measurable with respect to $T_t\mu_i$. It follows that

$$(\psi_{g^*})^{(A\times B\times C)^{(b,c)}}(\alpha_i \times \nu_i) * T_t\mu_i =< \int\int (T_t\mu_i)(A\times B\times C)^{(b,c)}\delta\alpha_i\delta\nu_i, g^* >$$

$$=< \int\int (T_t\mu_i)(A)^{(b,c)}\delta\alpha_i\delta\nu_i, g^* >$$

If $b \in B$ is a fixed element, then the set $A \times B \times C$ is projected onto $A \times C$.

Therefore,

$$(\psi_{g^*})^{(A\times B\times C)^{(b,c)}}(\alpha_i \times \nu_i) * T_t\mu_i =< \int (\alpha_i * T_t\mu_i)(A\times C)^{b}\delta\nu_i, g^* >$$

$$= (\psi_{g^*})^{(A\times B\times C)^{(b,c)}}(\nu_i * (\alpha_i * T_t\mu_i))$$

If $c \in C$ is a fixed element, then the set $A \times B \times C$ is projected onto

$A \times B$ (Dorlas, 2010). Therefore,

$$(\psi_{g^*})^{(A\times B\times C)^{(b,c)}}(\alpha_i \times \nu_i) * T_t\mu_i =< \int (\nu_i * T_t\mu_i)(A\times B)^{c}\delta\alpha_i, g^* >$$

$$= (\psi_{g^*})^{(A\times B\times C)^{(b,c)}}(\alpha_i * (\nu_i * T_t\mu_i))$$

Hence, the above relation gives $(\alpha_i \times \nu_i) * T_t\mu_i = \alpha_i * (\nu_i * T_t\mu_i)$

Proposition 4

For a fixed $b \in B$ and for all $t \in \mathbf{R}$, let $g_i^t(A \times B)^b \in X$ such that

$g_i^t \le T_t\mu_{i_k}$ for $i < i_k$ and each i, $i_k \in I$. If $P^{(X\times Y)} : \mathbf{R} \to M(\rho \times \varepsilon, X \times Y)$

is an integrable function with respect to $\mu_i \times \nu_i$, then for a fixed element

$x \in X$, $P_t^{(X\times Y)^x} * g_i^t = LUB_{i_k} P_t^y * T_t\mu_{i_k}$ for $\forall\ y \in Y$

Proof

Let $A \times B \in \rho \times \varepsilon$, fix $b \in B$ and $x \in X$. For every element $y \in Y$, we have

$$\psi_{g^*}^{(A\times B)^b}(\nu_i * P_t^{(X\times Y)^x} * g_i^t) = \psi_{g^*}^{(A\times B)^b}(\nu_i * P_t^y * g_i^t) \ \forall \ y \in Y \qquad (*)$$

The product of vector measures $(\nu_i * P_t^y)$ and g_i^t in $M(\rho \times \varepsilon, X \times Y)$ and the application of the integral function $\psi_{g^*}^{(A\times B)^b}$ (Rodriguez, 2006 and Yaogan, 2013), implies that

$$\psi_{g^*}^{(A\times B)^b}(\nu_i * P_t^y * g_i^t) =< \int g_i^t (A\times B)^b P_y^t \delta\nu_i, g^* > \qquad (**)$$

Since $g_i^t \leq T_t \mu_{i_k}$ for $i < i_k$ and $t \in \mathbf{R}$, then by the property of increasingly directed vector measure duality, we obtain

$$< g_i^t (A\times B)^b P_t^y \delta\nu_i, g^* >= LUB_{i_k} < \int T_t\mu_{i_k}(A\times B)^b P_t^y \delta\nu_i, g^* >$$

$$= LUB_{i_k} \psi_{g^*}^{(A\times B)^b}(\nu_i * (P_t^y * T_t\mu_{i_k})) \qquad (***)$$

Comparing equations (*),(**) and (***), we obtain

$$P_t^{(X\times Y)^x} * g_i^t = LUB_{i_k} P_t^y * T_t\mu_{i_k}$$

Proposition 5

Let t_o and t be real numbers. Let B be a measurable set with respect to ε such that $B \downarrow \emptyset$. Let $P^{\nu_i(B)} * g_i^{t_o} = g_i^{t_o}$ as $\nu_i(B) \downarrow 0$ and $g_i^{t_o} \leq T_{t_o}\mu_{i_k} \times \nu_{i_k}$ for $i < i_k$. If $T_{t-t_O}T_{t_o}\mu_{i_k} \times \nu_{i_k} = T_t\mu_{i_k} \times \nu_{i_k}$, then

$$T_{t-t_O}(g_i^{t_o}) = LUB_{i_k} P^{\nu_i(B)} * T_t\mu_{i_k} \times \nu_{i_k} \text{ as } \nu_i(B) \downarrow 0$$

Proof

Since $P^{\nu_i(B)} * g_i^{t_o} = g_i^{t_o}$ as $\nu_i(B) \downarrow 0$ by hypothesis, it follows that

$T_{t-t_O}(g_i^{t_o}) = T_{t-t_O}(P^{\nu_i(B)} * g_i^{t_o})$ as $\nu_i(B) \downarrow 0)$. Since $g_i^{t_o} \leq T_{t_o}\mu_{i_k} \times \nu_{i_k}$

for $i < i_k$, by the property of increasingly directed vector measure duality,

it follows that $T_{t-t_O}(g_i^{t_o}) = T_{t-t_O}(P^{\nu_i(B)} * g_i^{t_o})$ as $\nu_i(B) \downarrow 0)$

$= LUB_{i_k} T_{t-t_O}(P^{\nu_i(B)} * T_{t_o}\mu_{i_k} \times \nu_{i_k})$ as $\nu_i(B) \downarrow 0$ (*)

On application of the relation $T_{t-t_O}T_{t_o}\mu_{i_k} \times \nu_{i_k} = T_t\mu_{i_k} \times \nu_{i_k}$ (By hypothesis), equation (*) becomes

$LUB_{i_k} P^{\nu_i(B)} * T_{t-t_O}T_{t_o}\mu_{i_k} \times \nu_{i_k} = LUB_{i_k} P^{\nu_i(B)} * T_t\mu_{i_k} \times \nu_{i_k}$ as $\nu_i(B) \downarrow 0$

Hence, $T_{t-t_O}(g_i^{t_o}) = LUB_{i_k} P^{\nu_i(B)} * T_t\mu_{i_k} \times \nu_{i_k}$ as $\nu_i(B) \downarrow 0$

Proposition 6

Let t_o and t be real numbers, $A \times B \in \rho \times \varepsilon$ and $\mu_i \times \nu_i \in M(\rho \times \varepsilon, X \times Y)$. If

$P^{\nu_i(B)} * T_t\mu_i \times \nu_i(A \times B) = T_t\mu_i \times \nu_i(A \times B)$ and $P^{\nu_i(B)} * g_i^{t_o} = g_i^{t_o}$ as

$\nu_i(B) \downarrow 0$, then $P^{\nu_i(B)} * \mu_i \times \nu_i = \mu_i \times \nu_i$ as $\nu_i(B) \downarrow 0$

Proof

By hypothesis, $P^{\nu_i(B)} * T_t\mu_i \times \nu_i(A \times B) = T_t\mu_i \times \nu_i(A \times B)$ as $\nu_i(B) \downarrow 0$ for

$A \times B \in \rho \times \varepsilon$.

Also $T_{t-t_O}(g_i^{t_o}) = LUB_i P^{\nu_i(B)} * T_t\mu_i \times \nu_i$ as $\nu_i(B) \downarrow 0$ (By Theorem 5).

Comparing the two relations, we obtain

$LUB_i T_t \mu_i \times \nu_i(A \times B) = T_{t-t_O}(g_i^{t_o}(A \times B))$

Suppose $t = (\mu_i \times \nu_i)(C \times D)$, where $C \times D \in \rho \times \varepsilon$. Let $C = \emptyset$ or $D = \emptyset$ or both C and D be empty sets. Considering all the three cases, it follows that $t = 0$. Therefore,

$LUB_i \mu_i \times \nu_i(A \times B) = T_{-t_O}(g_i^{t_o}(A \times B))$ (*)

Applying $P^{\nu_i(B)}$ to the above relation, we obtain

$LUB_i P^{\nu_i(B)} * \mu_i \times \nu_i = T_{-t_O}(P^{\nu_i(B)} * g_i^{t_o})$

By hypothesis, $P^{\nu_i(B)} * g_i^{t_o} = g_i^{t_o}$ as $\nu_i(B) \downarrow 0$. Therefore,

$LUB_i P^{\nu_i(B)} * \mu_i \times \nu_i = T_{-t_O}(g_i^{t_o})$. (**)

Comparing equations (*) and (**), we obtain

$P^{\nu_i(B)} * \mu_i \times \nu_i = \mu_i \times \nu_i$ as $\nu_i(B) \downarrow 0$

Corollary

Let (p_i) be a finite measure defined on $\rho \times \varepsilon$ such that

$(p_i)_{(A\times E)} = (\mu_i \times \nu_i)_{(A\times E)} - LUB_{i_k} T_t \mu_{i_k} \geq 0$

for each $i \in I$, $t \in \mathbf{R}$, where $(p_i)_{(A\times E)}$ and $(\mu_i \times \nu_i)_{(A\times E)}$ is the contraction of p_i and $\mu_i \times \nu_i$ by a measurable set $A \times E \in \rho \times \varepsilon$ of finite measure. Given $\epsilon > 0$, if $\epsilon(\mu_i \times \nu_i)_{A\times E} \leq (p_i)_{A\times E}$ and

$Q^{\nu_i(E)} * g_i^t = LUB_{i_k} T_t \mu_{i_k}$ as $\nu_i(E) \downarrow 0$ for a fixed $a \in A$, then $g_i^t \leq (\mu_i)_A$

Proof

By hypothesis, $(p_i)_{(A\times E)} = (\mu_i \times \nu_i)_{(A\times E)} - LUB_{i_k} T_t \mu_{i_k}$. For a fixed

element $e \in E$, let $(p_i)_A = (\mu_i)_A - LUB_{i_k} T_t \mu_{i_k}$. Since

$Q^{\nu_i(E)} * g_i^t = g_i^t$ as $\nu_i(E) \downarrow 0$ (by proposition 5) and

$Q^{\nu_i(E)} * g_i^t = LUB_{i_k} T_t \mu_{i_k}$ as $\nu_i(E) \downarrow 0$ (by hypothesis), it follows that

$g_i^t = LUB_{i_k} T_t \mu_{i_k}$. For each $\epsilon > 0$, let $g_i^t = \nu_i * LUB_{i_k} T_t \mu_{i_k} + \epsilon(\mu_i)_A$, where

A is a measurable set of finite measure. Consider the function $\psi_{g^*}^M$ defined

by $\psi_{g^*}^M(g_i^t) =< g_i^t(M), g^* >$ where $M \in \rho$ and $g^* \in G^*$, the dual space of

G. Therefore, $\psi_{g^*}^M(g_i^t) =< g_i^t(M), g^* >$

$= LUB_{i_k} < \int T_t \mu_{i_k}(M)\delta_{\nu_i}, g^* > + < \epsilon(\mu_i)_A(M), g^* >$

$\leq LUB_{i_k} < \int T_t \mu_{i_k}(M)\delta_{\nu_i}, g^* > + < (p_i)_A(M), g^* >$

$= LUB_{i_k}[< \int T_t \mu_{i_k}(M)\delta_{\nu_i}, g^* > +$

$$(< (\mu_i)_A(M), g^* > - < \int T_t \mu_{i_k}(M)\delta_{\nu_i}, g^* >)]$$

$=< (\mu_i)_A(M), g^* >$

In general $g_i^t \leq (\mu_i)_A$ for each $i \in I$, $t \in \mathbf{R}$ and $A \in \rho$

4 Conclusion

The results obtained in this paper is a clear demonstration of the existence of integral representation of pointwise product vector measure duality co-domained in a Hilbert space. To this end, the relationship between pointwise integral representation of the product vector measure with the inner product vector measure duality has been revealed.

Acknowledgement: This research was supported by University of Kabianga

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