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Modified New Iterative Method for Solving Nonlinear Partial Differential Equations

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Abstract

In this paper, the iterative method, proposed by Gejji and Jafari in 2006, has been modified for solving nonlinear initial value problems. The Laplace transform was used in this modification to eliminate the linear differential operator in the differential equation. The convergence of the solution was discussed according to the modification proposed. To illustrate this modification, some examples were presented.

Keywords: Iterative Method, partial differential equations, initial value problem, numerical solution, Laplace transform.

Introduction

Differential equations appear in many scientific and life fields such as engineering, physics, chemistry, economics, and biology. Therefore, it became necessary to focus on finding the best methods to solve it. There are many numerical and analytical methods used to solve partial differential equations, such as Admoain decomposition method [1 – 6], homotopy perturbation method [7 – 11], Laplace decomposition method [12, 13], variational iteration method [14 – 16], collocation method [17-19] and artificial neural network (Ann) [20 – 24].

In 2006, Gejji and Jafari proposed a new iterative method (NIM) for solving functional equations [25]. They described the general functional equation as following:

$$u = N(u) + f \qquad (1)$$

where N is a nonlinear operator and f is a known function. The solution u expressed as the form:

$$u = \sum_{m=0}^{\infty} u_m \qquad (2)$$

They decomposed the nonlinear operator N as:

$$N\left(\sum_{m=0}^{\infty} u_m\right) = N(u_0) + \sum_{m=1}^{\infty}\left\{N\left(\sum_{j=0}^{m} u_j\right) - N\left(\sum_{j=0}^{m-1} u_j\right)\right\} \qquad (3)$$

From (2) and (3), equation (1) is equivalent to

$$\sum_{m=0}^{\infty} u_m = f + N(u_0) + \sum_{m=1}^{\infty}\left\{N\left(\sum_{j=0}^{m} u_j\right) - N\left(\sum_{j=0}^{m-1} u_j\right)\right\} \qquad (4)$$

Finally, the recurrence relation was defined as:

$$\begin{cases} u_0 = f \\ u_1 = N(u_0) \\ u_{m+1} = N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}), \quad m = 1,2,\ldots \end{cases} \qquad (5)$$

Later, NIM was used and modified to solve nonlinear equations such as partial differential equations of integer and fractional order, integral equations, a system of equations, and algebraic equations [26-30].

In this paper, the NIM was modified to solve partial differential equations. The Laplace transform was used to convert the partial differential equation into a formula similar to the formula (1) as it will be illustrated in the next section.

Modified New Iterative Method

To illustrate the modified method, rewrite a general IVP as:

$$L\big(u(X,t)\big)+R\big(u(X,t)\big)+N\big(u(X,t)\big)=g(X,t) \qquad (6)$$

With the initial condition

$$\frac{\partial^k u(X,t)}{\partial t^k}|_{t=0}=f_k(X)\ ,k=0,1,\dots,n-1 \qquad (7)$$

where $L(.)=\frac{\partial^n(.)}{\partial t^n}$,$n=1,2,3,\dots$ is a linear operator of the partial derivative with respect to t, $R(.)$ the remained of the linear operator, $N(.)$ the nonlinear operator, $g(X,t)$ is the inhomogeneous part which is known function and X is a variable with one or more dimensions.

Taking the Laplace transform (with respect to the variable *t*) for the equation (6) to get:

$$\mathcal{L}\{L(u)\}+\mathcal{L}\{R(u)+N(u)\}=\mathcal{L}\{g\} \qquad (8)$$

By the properties of the Laplace transform and put $\mathcal{F}=R+N$, the equation (6) becomes:

$$s^n\mathcal{L}\{u\}-\sum_{i=0}^{n-1}s^{n-i-1}\frac{\partial^i u}{\partial t^i}|_{t=0}+\mathcal{L}\{\mathcal{F}(u)\}=\mathcal{L}\{g\} \qquad (9)$$

From (7) we have:

$$s^n\mathcal{L}\{u\}-\sum_{i=0}^{n-1}s^{n-i-1}f_i+\mathcal{L}\{\mathcal{F}(u)\}=\mathcal{L}\{g\} \qquad (10)$$

And hence:

$$\mathcal{L}\{u\}=\sum_{i=0}^{n-1}s^{-i-1}f_i+\frac{1}{s^n}\mathcal{L}\{g\}-\frac{1}{s^n}\mathcal{L}\{\mathcal{F}(u)\} \qquad (11)$$

Taking the inverse of the Laplace transform on both sides of equation (11), to get:

$$u=\sum_{i=0}^{n-1}f_i\frac{t^i}{i!}+\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{g\}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{\mathcal{F}(u)\}\right\} \qquad (12)$$

By the linearity property of the Laplace transform and it's inverse:

$$u=\sum_{i=0}^{n-1}f_i\frac{t^i}{i!}+\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{g\}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{\mathcal{F}(u)\}\right\} \qquad (13)$$

Then (13) is equivalent to (6), i.e. the two equations have the same solutions.

Substituting (2) in (13) we have:

$$\sum_{m=0}^{\infty}u_m(X,t)=f-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\mathcal{F}\left(\sum_{m=0}^{\infty}u_m(X,t)\right)\right\}\right\} \qquad (14)$$

Where

$$f=\sum_{i=0}^{n-1}f_i\frac{t^i}{i!}+\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{g\}\right\} \qquad (15)$$

The operator $\mathcal{F}$ in (14) can be decomposed as:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\mathcal{F}\left(\sum_{m=0}^{\infty}u_m\right)\right\}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\mathcal{F}(u_0) + \sum_{m=1}^{\infty}\left[\mathcal{F}\left(\sum_{j=0}^{m}u_j\right) - \mathcal{F}\left(\sum_{j=0}^{m-1}u_j\right)\right]\right\}\right\} \quad (16)$$

Substituting (16) in (14), we have:

$$\sum_{m=0}^{\infty}u_m = f - \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\mathcal{F}(u_0) + \sum_{i=1}^{\infty}\left[\mathcal{F}\left(\sum_{j=0}^{m}u_j\right) - \mathcal{F}\left(\sum_{j=0}^{m-1}u_j\right)\right]\right\}\right\} \quad (17)$$

Then, compared to the NIM (1-5), the recurrence relation can be defined as follows:

$$\begin{cases} u_0 = f \\ u_1 = -\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{\mathcal{F}(u_0)\}\right\} \\ u_{m+1} = -\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{\mathcal{F}(u_0+\cdots+u_m) - \mathcal{F}(u_0+\cdots+u_{m-1})\}\right\}, \quad m = 1,2,\dots \end{cases} \quad (18)$$

Convergence analysis

In this section, the convergence of the series (2), calculated by (18), will be discussed, and it will also be proved that it satisfies the initial value problem (6). But first, the following three lemmas will be presented.

Lemma 1: If $L^{-1}(F)$ and $L^{-1}(\|F\|)$ are exist, where L is the Laplace transform, for any function F in a Banach space B, then

$$\|L^{-1}(F)\| = L^{-1}(\|F\|)$$

Proof:

The Post-Widder inversion formula for the Laplace transform is: [31]

$$L^{-1}(F) = \lim_{k\to\infty}\left[\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)\right], \quad t > 0 \text{ and } k = 0,1,2,\dots$$

Since $L^{-1}(F)$ is exist then there is $f(t)$ such that

$$L^{-1}(F) = \lim_{k\to\infty}\left[\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)\right] = f(t)$$

Then

$$\|f(t)\| = \|L^{-1}(f)\| = \left\|\lim_{k\to\infty}\left[\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)\right]\right\| = \lim_{k\to\infty}\left[\left\|\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)\right\|\right]$$
$$= \lim_{k\to\infty}\left[\left\|\frac{(-1)^k}{k!}\right\|\left\|\left(\frac{k}{t}\right)^{k+1}\right\|\left\|F^{(k)}\left(\frac{k}{t}\right)\right\|\right] = \lim_{k\to\infty}\left[\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1}\left\|F^{(k)}\left(\frac{k}{t}\right)\right\|\right]$$

Then the sequence $\left\{\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1}\left\|F^{(k)}\left(\frac{k}{t}\right)\right\|\right\}$ converges to $\|f(t)\|$.

Now since $L^{-1}(\|F\|)$ is exist then there is β such that $L^{-1}(\|F\|) = \beta$ i.e. by the Post-Widder inversion formula for the Laplace transform we have

$$L^{-1}(\|F\|) = \lim_{k\to\infty}\left[\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1}\left\|F^{(k)}\left(\frac{k}{t}\right)\right\|\right] = \beta$$

Then the sequence $\left\{\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1}\left\|F^{(k)}\left(\frac{k}{t}\right)\right\|\right\}$ converges to β. But this sequence has a subsequence $\left\{\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1}\left\|F^{(k)}\left(\frac{k}{t}\right)\right\|\right\}$ which is converging to $\|f(t)\|$, and every convergent subsequence of a convergent sequence has the same convergence point, that implies $\beta = \|f(t)\|$ and hence $L^{-1}(\|F\|) = \|f(t)\|$, then

$$\left\|L^{-1}(F)\right\| = L^{-1}(\|F\|)$$

Lemma 2: If $L(f)$ and $L(\|f\|)$ are exist, where L is the Laplace transform, for any function f in a Banach space B, then

$$\|L(f)\| \leq L(\|f\|)$$

Proof:

The Laplace transform is defined as:

$$L(f) = \int_0^\infty e^{-s\,t} f(t)\, dt$$

Therefore,

$$\|L(f)\| = \left\|\int_0^\infty e^{-s\,t} f(t)\, dt\right\| \leq \int_0^\infty \|e^{-s\,t} f(t)\|\, dt = \int_0^\infty e^{-s\,t} \|f(t)\|\, dt = L(\|f\|)$$

Then

$$\|L(f)\| \leq L(\|f\|)$$

Lemma 3: For any function f in a Banach space B

$$\left\|L^{-1}\left[\tfrac{1}{s^n} L(f)\right]\right\| \leq L^{-1}\left[\tfrac{1}{s^n} L(\|f\|)\right] \quad , \ s > 0, \ \ n = 1,2, \dots$$

Proof:

By Lemma 1 and Lemma 2 and since $s > 0$, we have

$$\left\|L^{-1}\left[\frac{1}{s^n} L(f)\right]\right\| = L^{-1}\left[\left\|\frac{1}{s^n} L(f)\right\|\right] = L^{-1}\left[\frac{1}{s^n} \|L(f)\|\right] \leq L^{-1}\left[\frac{1}{s^n} L(\|f\|)\right]$$

Theorem 1: If $\mathcal{F}$ is a contraction operator, i.e., there is $\alpha \in (0,1)$ such that $\|\mathcal{F}(u(t,X)) - \mathcal{F}(v(t,X))\| \leq \alpha\, \|u(t,X) - v(t,X)\|$, $t \geq 0$. Then, the infinite series (2), which is computed by (18), is absolutely convergent if $t < \sqrt[n]{n!/\alpha}$.

Proof:

For any $m = 1,2,3, \dots$, by (18):

$$\|u_{m+1}\| = \left\|\mathcal{L}^{-1}\left\{\frac{1}{s^n} \mathcal{L}\{\mathcal{F}(u_0 + \cdots + u_m) - \mathcal{F}(u_0 + \cdots + u_{m-1})\}\right\}\right\|$$

By Lemma 3 we have:

$$\begin{aligned}
\|u_{m+1}\| &= \left\|\mathcal{L}^{-1}\left\{\frac{1}{s^n} \mathcal{L}\{\mathcal{F}(u_0 + \cdots + u_m) - \mathcal{F}(u_0 + \cdots + u_{m-1})\}\right\}\right\| \\
&\leq \mathcal{L}^{-1}\left\{\frac{1}{s^n} \mathcal{L}\{\|\mathcal{F}(u_0 + \cdots + u_m) - \mathcal{F}(u_0 + \cdots + u_{m-1})\|\}\right\} \\
&\leq \mathcal{L}^{-1}\left\{\frac{1}{s^n} \mathcal{L}\{\alpha\, \|(u_0 + \cdots + u_m) - (u_0 + \cdots + u_{m-1})\|\}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^n} \mathcal{L}\{\alpha\, \|u_m\|\}\right\} \\
&= \alpha\, \|u_m\|\, \mathcal{L}^{-1}\left\{\frac{1}{s^n} \mathcal{L}\{1\}\right\} = \alpha\, \frac{t^n}{n!} \|u_m\|
\end{aligned}$$

Then

$$\|u_{m+1}\| \leq \alpha\, \frac{t^n}{n!} \|u_m\| \quad \to \quad \frac{\|u_{m+1}\|}{\|u_m\|} \leq \alpha\, \frac{t^n}{n!} \qquad m = 1,2,3, \dots$$

By the ratio test, the series (2) is convergent if $\frac{\|u_{m+1}\|}{\|u_m\|} < 1$, this implies that

$$\alpha\frac{t^n}{n!} < 1 \quad \rightarrow \quad t^n < \frac{n!}{\alpha} \quad \rightarrow \quad t < \sqrt[n]{n!/\alpha}$$

Theorem 2: The infinite series (2), which is computed by (18), is a solution to (6).

Proof:

Since R is a linear operator then:

$$\begin{aligned}\mathcal{F}(u_0+\cdots+u_m)-\mathcal{F}(u_0+\cdots+u_{m-1}) &= R(u_0+\cdots+u_m)+N(u_0+\cdots+u_m)-R(u_0+\cdots+u_{m-1})-N(u_0+\cdots+u_{m-1})\\ &= R(u_0)+\cdots+R(u_m)-R(u_0)-\cdots-R(u_{m-1})+N(u_0+\cdots+u_m)-N(u_0+\cdots+u_{m-1})\\ &= R(u_m)+N(u_0+\cdots+u_m)-N(u_0+\cdots+u_{m-1})\end{aligned}$$

Now, Let

$$\begin{aligned}S_m &= u_1+\cdots+u_{m+1}\\ &= -\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{\mathcal{F}(u_0)\}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{\mathcal{F}(u_0+u_1)-\mathcal{F}(u_0)\}\right\}\\ &-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{\mathcal{F}(u_0+u_1+u_2)-\mathcal{F}(u_0+u_1)\}\right\}-\cdots\\ &-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{\mathcal{F}(u_0+\cdots+u_m)-\mathcal{F}(u_0+\cdots+u_{m-1})\}\right\}\\ &= -\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{R(u_0)+N(u_0)\}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{R(u_1)+N(u_0+u_1)-N(u_0)\}\right\}\\ &-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{R(u_2)+N(u_0+u_1+u_2)-N(u_0+u_1)\}\right\}-\cdots\\ &-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{R(u_m)+N(u_0+\cdots+u_m)-N(u_0+\cdots+u_{m-1})\}\right\}\\ &= -\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{R(u_0+\cdots+u_m)+N(u_0+\cdots+u_m)\}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{R\left(\sum_{j=0}^{m}u_j\right)+N\left(\sum_{j=0}^{m}u_j\right)\right\}\right\}\end{aligned}$$

Then

$$\begin{aligned}u=\sum_{i=0}^{\infty}u_i=u_0+\sum_{i=1}^{\infty}u_i=f+\lim_{m\to\infty}(S_m)&=f+\lim_{m\to\infty}\left(-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{R\left(\sum_{j=0}^{m}u_j\right)+N\left(\sum_{j=0}^{m}u_j\right)\right\}\right\}\right)\\ &=f-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{R\left(\sum_{j=0}^{\infty}u_j\right)+N\left(\sum_{j=0}^{\infty}u_j\right)\right\}\right\}=f-\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{R(u)+N(u)\}\right\}\end{aligned}$$

i.e $u=\sum_{i=0}^{\infty}u_i$ calculated according to (18) satisfies equation (13) and hence it is a solution to (6).

4. Applications

In this section, some examples will be introduced to illustrate the modified of NIM.

Example 4.1: Consider the following 3rd order nonlinear homogeneous PDE:

$$u_t+6\,u^2u_x+u_{xxx}=0\,, \qquad u(x,0)=k\ \text{sech}\,(kx)$$

Then $L(u)=\frac{\partial u}{\partial t}$ i.e. $n=1,\ R(u)=u_{xxx},\ N(u)=6\,u^2u_x$ and since $g=0$ then:

$$u_0 = f = f_0 + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{g\}\right\} = k \text{ sech }(kx) + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{0\}\right\} = k \text{ sech }(kx)$$

$$u_1 = -\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{R(u_0) + N(u_0)\}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{R(k\,\text{sech}(kx)) + N(k\,\text{sech}(kx))\}\right\}$$
$$= -\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{k^4 \tanh(kx)(-\text{sech}(kx) + 6\text{ sech}^3(kx)) - 6\,k^4 \tanh(kx)\text{ sech}^3(kx)\}\right\}$$
$$= k^4 \tanh(kx)\,\text{sech}(kx)\;t$$

$$u_2 = -\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{R(u_1) + N(u_0 + u_1) - N(u_0)\}\right\}$$
$$= -\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{R(k^4\tanh(kx)\,\text{sech}(kx)\;t) + N(k\,\text{sech}(kx) + k^4 \tanh(kx)\,\text{sech}(kx)\;t)\right.$$
$$\left. - N(k\,\text{sech}(kx))\}\right\}$$
$$= \frac{1}{2}k^7\tanh(kx)\;(1 - 2\text{ sech}^2(kx))\,t^2 + k^{10}\tanh(kx)\;(6\,\text{sech}^3(kx) - 10\,\cosh^5(kx))\,t^3$$
$$+ k^{13}(3\,\text{sech}^3(kx) - 9\,\text{sech}^5(kx) + 6\text{ sech}^7(kx)\,)\,t^4$$

Then from (2), we have:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$
$$= k\,\text{sech}(kx) + k^4\tanh(kx)\,\text{sech}(kx)\;t + \frac{1}{2}k^7\tanh(kx)\;(1 - 2\text{ sech}^2(kx))\,t^2$$
$$+ k^{10}\tanh(kx)\;(6\,\text{sech}^3(kx) - 10\,\cosh^5(kx))\,t^3$$
$$+ k^{13}(3\,\text{sech}^3(kx) - 9\,\text{sech}^5(kx) + 6\text{ sech}^7(kx)\,)\,t^4 + \cdots$$
$$= k\,\text{sech}(kx) + k^4\tanh(kx)\,\text{sech}(kx)\;t + \frac{1}{2}k^7\tanh(kx)\;(1 - 2\text{ sech}^2(kx))\,t^2$$
$$+ \frac{1}{6}k^{10}\tanh(kx)\,(\text{sech}(kx) - 6\text{ sech}^3(kx))\,t^3 + \cdots$$

This is closed to the exact solution:

$$u(x,t) = k\text{ sech }(k(x - k^2t))$$

Example 4.2: Consider the following 2nd order nonlinear homogeneous PDE:

$$u_{tt} - u + \frac{1}{4}u_x^2 = 0\,, \qquad u(x,0) = 1 + x^2,\;\; u_t(x,0) = 1$$

Then $L(u) = \frac{\partial^2 u}{\partial t^2}$ i.e. $n = 2,\;\; R(u) = -u,\;\; N(u) = \frac{1}{4}u_x^2$ and since $g = 0$ then:

$$u_0 = f = f_0 + f_1 t + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{g\}\right\} = 1 + x^2 + t + 0 = x^2 + 1 + t$$

$$u_1 = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{R(u_0) + N(u_0)\}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{R(x^2 + 1 + t) + N(x^2 + 1 + t)\}\right\}$$
$$= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{-x^2 - 1 - t + \frac{1}{4}(2\,x)^2\right\}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{1 + t\}\right\} = \frac{t^2}{2} + \frac{t^3}{6} = \frac{t^2}{2} + \frac{t^3}{3!}$$

$$u_2 = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{R(u_1) + N(u_0 + u_1) - N(u_0)\}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{R\left(\frac{1}{2}t^2 + \frac{1}{6}t^3\right) + N\left(x^2 + 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3\right) - \frac{1}{4}(2\,x)^2\right\}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{-\frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{4}(2\,x)^2 - \frac{1}{4}(2\,x)^2\right\}\right\} = \frac{t^4}{24} + \frac{t^5}{120} = \frac{t^4}{4!} + \frac{t^5}{5!}$$

$$u_3 = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{R(u_2) + N(u_0 + u_1 + u_2) - N(u_0 + u_1)\}\right\} = \frac{t^6}{720} + \frac{t^7}{5040} = \frac{t^6}{6!} + \frac{t^7}{7!}$$

Then from (2), we have:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots = x^2 + 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \cdots$$

This is closed to the exact solution:

$$u(x,t) = x^2 + e^t$$

Example 4.3: Consider the following 3rd order nonlinear inhomogeneous PDE:

$$u_{ttt} + u^2 - uu_x - u = 3e^{x+t}, \qquad u(x,0) = 0,\ u_t(x,0) = e^x, \qquad u_{tt}(x,0) = 2\,e^x$$

Then $L(u) = \frac{\partial^3 u}{\partial t^3}$ i.e. $n = 3$, $R(u) = -u$, $N(u) = u^2 - uu_x$ and since $g = 3e^{x+t}$ then:

$$u_0 = f = f_0 + f_1 t + f_2\frac{t^2}{2!} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{g\}\right\} = 0 + e^x t + e^x t^2 + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{g3e^{x+t}\}\right\}$$

$$= e^x t + e^x t^2 + \frac{3}{2}e^x\left(2\,e^t - t^2 - 2\,t - 2\right) = e^x\left(3\,e^t - 3 - 2\,t - \frac{t^2}{2}\right)$$

$$u_1 = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{R(u_0) + N(u_0)\}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{R\left(e^x\left(3\,e^t - 3 - 2\,t - \frac{t^2}{2}\right)\right) + N\left(e^x\left(3\,e^t - 3 - 2\,t - \frac{t^2}{2}\right)\right)\right\}\right\}$$

$$= e^x\left(3\,e^t - 3 - 3\,t - \frac{3\,t^2}{2} - \frac{t^3}{2} - \frac{t^4}{12} - \frac{t^5}{120}\right)$$

$$u_2 = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{R(u_1) + N(u_0 + u_1) - N(u_0)\}\right\}$$

$$= e^x\left(3\,e^t - 3 - 3\,t - \frac{3\,t^2}{2} - \frac{t^3}{2} - \frac{t^4}{8} - \frac{t^5}{40} - \frac{t^6}{240} - \frac{t^7}{2520} - \frac{t^8}{40320}\right)$$

Then from (2), we have:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots =$$
$$= e^x\left(9\,e^t - 9 - 8\,t - \frac{7\,t^2}{2!} - \frac{6\,t^3}{3!} - \frac{5\,t^4}{4!} - \frac{4\,t^5}{5!} - \frac{3\,t^6}{6!} - \frac{2\,t^7}{7!} - \frac{t^8}{8!} - \cdots\right)$$
$$= e^x\left(9\,e^t - 9 - 8\,t - \frac{7\,t^2}{2!} - \frac{6\,t^3}{3!} - \frac{5\,t^4}{4!} - \frac{4\,t^5}{5!} - \frac{3\,t^6}{6!} - \frac{2\,t^7}{7!} - \frac{t^8}{8!} - \cdots \right.$$
$$+ \left[-t + t - \frac{2\,t^2}{2!} + \frac{2\,t^2}{2!} - \frac{3\,t^3}{3!} + \frac{3\,t^3}{3!} - \frac{4\,t^4}{4!} + \frac{4\,t^4}{4!} - \frac{5\,t^5}{5!} + \frac{5\,t^5}{5!} - \frac{6\,t^6}{6!} + \frac{6\,t^6}{6!} - \frac{7\,t^7}{7!} + \frac{7\,t^7}{7!} - \frac{8\,t^8}{8!}\right.$$
$$\left.\left. + \frac{8\,t^8}{8!} - \cdots\right]\right)$$
$$= e^x\left(9\,e^t - 9 - 9\,t - \frac{9\,t^2}{2!} - \frac{9\,t^3}{3!} - \frac{9\,t^4}{4!} - \frac{9\,t^5}{5!} - \frac{9\,t^6}{6!} - \frac{9\,t^7}{7!} - \frac{9\,t^8}{8!} - \cdots \right.$$
$$\left. + \left[t + t^2 + \frac{t^3}{2!} + \frac{t^4}{3!} + \frac{t^5}{4!} + \frac{t^6}{5!} + \frac{t^7}{6!} + \frac{t^8}{7!} + \cdots\right]\right)$$
$$= e^x\left(9\,e^t - 9\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!} + \cdots\right]\right.$$
$$\left. + t\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \cdots\right]\right)$$

This is closed to the exact solution:

$$u(x,t) = te^{x+t}$$

5. Conclusion

In this research, a new modification of the iterative method to solve PDEs is proposed. The experimental results show that the suggested modification is computationally efficient for solving non-linear, non-homogenous PDEs and can easily be implemented since it is free of using Adomian polynomials when dealing with the nonlinear terms like in the ADM and being free of using the Lagrange multiplier as in the VIM. The method proves to be simple in its principles and convenient for computer algorithms. It has also been witnessed that a few approximations can be used to achieve a high degree of accuracy.

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