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Some Inclusion Properties for Meromorphic Functions Defined by New Generalization of Mittag-Leffler Function

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Abstract

In this paper, we introduce $\mathbb{I}_{\alpha,\beta,\eta}^{\gamma,k,m}$, which is a new operator by using generalized Mittag-Leffler function. Also, we defined meromorphic subclasses associated $\mathbb{I}_{\alpha,\beta,\eta}^{\gamma,k,m}$. Finally, we calculated inclusion relations by using $\mathbb{I}_{\alpha,\beta,\eta}^{\gamma,k,m}$ and integral operator F_μ

Keywords: Meromorphic functions, Hadamard product, Mittag-Leffler function, inclusion.

1. Introduction

rst, we prepared a definition of $\sum$ as follows:

$$f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

which is analytic in the punctured unit disk $\mathbb{U}^* = \mathbb{U}\backslash\{0\}$ where $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For f given by (1.1) and g given by

$$g(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n, \tag{1.2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^{-1} + \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.3}$$

A function $f(z) \in \Sigma$ is said to meromorphically starlike function of order δ in $\mathbb{U}^*$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} < -\delta \quad (0 \le \delta < 1; z \in \mathbb{U}^*). \tag{1.4}$$

We denote by $\Sigma S^*(\delta)$ the class of all meromorphically starlike functions of order δ. A function $f(z) \in \Sigma$ is said to be meromorphically convex function of order δ in $\mathbb{U}^*$, if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < -\delta \quad (0 \le \delta < 1; z \in \mathbb{U}^*). \tag{1.5}$$

We denote by $\Sigma C(\delta)$ the class of all meromorphically convex functions of order δ. It is easy to observe from (1.4) and (1.5) that

$$f(z) \in \Sigma C(\delta) \Longleftrightarrow -zf'(z) \in \Sigma S^*(\delta). \tag{1.6}$$

A function $f(z) \in \Sigma$ is said to be meromorphically close-to-convex function of order σ and type δ in $\mathbb{U}^*$, if there exists a function $g \in \Sigma S^*(\delta)$ such that

$$\Re\left\{\frac{zf'(z)}{g(z)}\right\} < -\sigma \quad (0 \leq \delta, \sigma < 1; z \in \mathbb{U}^*). \tag{1.7}$$

We denote by $\Sigma K(\sigma, \delta)$ the class of all meromorphically close-to-convex function of order σ and type δ. A function $f(z) \in \Sigma$ is said to be meromorphically quasi-convex functions of order σ and type δ in $\mathbb{U}^*$, if there exists a function $g \in \Sigma C(\delta)$ such that

$$\Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} < -\sigma \quad (0 \leq \delta, \sigma < 1; z \in \mathbb{U}^*). \tag{1.8}$$

We denote by $\Sigma K^*(\sigma, \delta)$ the class of all meromorphically quasi-convex functions of order σ and type δ. It follows from (1.7) and (1.8) that

$$f(z) \in \Sigma K^*(\sigma, \delta) \Longleftrightarrow -zf'(z) \in \Sigma K(\sigma, \delta). \tag{1.9}$$

Let the Mittag-Leffler function $E_\alpha(z)$ (see [14], see also [2], [8], [9], [12], [13] and [18]) defined as follows:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\alpha \in \mathbb{C}, Re(\alpha) > 0). \tag{1.10}$$

A more general function $E_\alpha(z)$ is $E_{\alpha,\beta}(z)$ was introduced by Wiman (see [15, 16]) and given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (z \in \mathbb{C}). \tag{1.11}$$

For $\alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha) > \max\{0, Re(k) - 1\}$ and $Re(k) > 0$ Srivastava and Tomovski [17] introduced the function $E_{\alpha,\beta}^{\gamma,k}(z)$ in the form

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}\, z^n}{\Gamma(\alpha n + \beta) n!} \quad (z \in \mathbb{C}). \tag{1.12}$$

Now, by using (1.12) we define the function $\mathfrak{K}_{\alpha,\beta}^{\gamma,k}(z)$ as follows:

$$\mathfrak{K}_{\alpha,\beta}^{\gamma,k}(z) = \Gamma(\beta) z^{-1} E_{\alpha,\beta}^{\gamma,k}(z).$$

It follows that, for $\alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha) > \max\{0, Re(k) - 1\}$ and $Re(k) > 0$ that

$$\mathfrak{K}_{\alpha,\beta}^{\gamma,k}(z) = z^{-1} + \sum_{n=0}^{\infty} \frac{(\gamma)_{(n+1)k}\;\Gamma(\beta) z^n}{\Gamma[\alpha(n+1) + \beta](n+1)!} (z \in \mathbb{C}). \tag{1.13}$$

By using the convolution, we can define the function $\mathbb{K}_{\alpha,\beta}^{\gamma,k}(f)(z)$ as follows:

$$\begin{aligned} \mathbb{K}_{\alpha,\beta}^{\gamma,k}(f)(z) &= \mathfrak{K}_{\alpha,\beta}^{\gamma,k}(z) * f(z), \\ &= z^{-1} + \sum_{n=0}^{\infty} \frac{\Gamma[\gamma + k(n+1)]\;\Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+1) + \beta](n+1)!} a_n z^n .1.14 \end{aligned} \tag{1}$$

For $\alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha) > \max\{0, Re(k) - 1\}, Re(k) > 0, \eta \geq 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N} = \{1, 2, ...\}$, we define a new linear operator $\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z) : \sum \to \sum$ as follows:

$$\mathbb{I}^{\gamma,k,0}_{\alpha,\beta,\eta}(f)(z) = \mathbb{K}^{\gamma,k}_{\alpha,\beta}(f)(z),$$

$$\begin{aligned}\mathbb{I}^{\gamma,k,1}_{\alpha,\beta,\eta}(f)(z) &= (1-\eta)\mathbb{K}^{\gamma,k}_{\alpha,\beta}(f)(z) + \eta z^{-1}[z^2\mathbb{K}^{\gamma,k}_{\alpha,\beta}(f)(z)]' \\ &= z^{-1} + \sum_{n=0}^{\infty} \frac{\Gamma[\gamma + k(n+1)]\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+1)+\beta](n+1)!}\ [1+\eta(n+1)]a_n z^n,\end{aligned}$$

$$\begin{aligned}\mathbb{I}^{\gamma,k,2}_{\alpha,\beta,\eta}(f)(z) &= \mathbb{I}^{\gamma,k,1}_{\alpha,\beta,\eta}[\mathbb{I}^{\gamma,k,1}_{\alpha,\beta,\eta}(f)(z)] = (1-\eta)\mathbb{I}^{\gamma,k,1}_{\alpha,\beta,\eta}(f)(z) + \eta z^{-1}[z^2\mathbb{I}^{\gamma,k,2}_{\alpha,\beta,\eta}(f)(z)]' \\ &= z^{-1} + \sum_{n=0}^{\infty} \frac{\Gamma[\gamma + k(n+1)]\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+1)+\beta](n+1)!}\ [1+\eta(n+1)]^2 a_n z^n.\end{aligned}$$

By induction we prove that

$$\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z) = z^{-1} + \sum_{n=0}^{\infty} \frac{\Gamma[\gamma + k(n+1)]\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma[\alpha(n+1)+\beta](n+1)!}\ [1+\eta(n+1)]^m a_n z^n. \tag{1.15}$$

Note that by taking $m = 0$ in (1.15), we get (1.14)

Remark 1:

(i) $\mathbb{I}^{1,1,0}_{0,\beta,\eta,}(f)(z) = f(z)$;

(ii) $\mathbb{I}^{2,1,0}_{0,\beta,\eta}(f)(z) = 2f(z) + zf'(z)$;

(iii) $\mathbb{I}^{1,1,0}_{1,1,\eta}(\frac{1}{z(1-z)}) = z^{-1}e^z$

(iv) $\mathbb{I}^{1,1,0}_{2,1,\eta}(\frac{1}{z(1-z)}) = z^{-1}\cosh(\sqrt{z})$

(v) $\mathbb{I}^{1,1,0}_{2,2,\eta}(\frac{1}{z(1-z)}) = \frac{\sinh(\sqrt{z})}{\sqrt{z^3}}$

Observe that:

(a) $\mathbb{I}^{1,1,m}_{0,\beta,\eta}(f)(z) = D^m_\eta f(z)$ (see [1], ([4-6]) with $l = p = 1$ and [3] with $p = 1$);

(b) $\mathbb{I}^{1,1,m}_{0,\beta,1}(f)(z) = D^m_{1,1}f(z)$ (see [1]);

(c) $\mathbb{II}^{\gamma,k,0}_{\alpha,\beta,\eta}(f)(z) = M^{\gamma,k}_{1,\beta,\eta}f(z)$ (see [11] with p=1)

2. Materials and Methods

Lemma 1. *Let $f \in \sum$, then the operator $I^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z)$ achieve the follwing relations*

$$(i) \quad z[\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z)]' = \frac{\gamma}{k}\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}(f)(z) - \left(\frac{\gamma+k}{k}\right)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z), \tag{2.1}$$

$$(ii) \quad z\alpha[\mathbb{I}^{\gamma,k,m}_{\alpha,\beta+1,\eta}(f)(z)]' = \beta\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z) - (\alpha+\beta)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta+1,\eta}(f)(z), \tag{2.2}$$

$$(iii) \quad z[\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z)]' = \frac{1}{\eta}\mathbb{I}^{\gamma,k,m+1}_{\alpha,\beta,\eta}(f)(z) - \left(1+\frac{1}{\eta}\right)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(f)(z), \tag{2.3}$$

$$(\alpha,\beta,\gamma \in \mathbb{C},\ Re(\alpha) > \max\{0, Re(k)-1\}, Re(k) > 0, \eta \geq 0, m \in \mathbb{N}_0).$$

Next, by using the operator $\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}$, the classes $\Sigma S^*(\delta)$, $\Sigma C(\delta)$, $\Sigma K(\delta,\sigma)$ and $\Sigma K^*(\delta,\sigma)$ which defined, respectively, by relations (1.4), (1.5), (1.7) and (1.8), we introduce the following new classes of meromorphic functions for $0 \leq \delta, \sigma < 1$:

$$\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta} = \left\{f \in \Sigma : \mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f \in \Sigma S^*(\delta)\right\},$$

$$\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} = \left\{f \in \Sigma : \mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f \in \Sigma C(\delta)\right\},$$

$$\Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta} = \left\{f \in \Sigma : \mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f \in \Sigma K(\delta,\sigma)\right\}$$

and

$$\Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta} = \left\{f \in \Sigma : \mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f \in \Sigma K^*(\delta,\sigma)\right\}.$$

We can see that:

$$f(z) \in \Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} \iff -zf'(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta} \tag{2.4}$$

and

$$f(z) \in \Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta} \iff -zf'(z) \in \Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}. \tag{2.5}$$

Lemma 2 [7]. *Let $\varphi(u,v)$ be complex-valued function such that,*

$$\varphi : D \longrightarrow \mathbb{C},\ (D \subset \mathbb{C} \times \mathbb{C})$$

$\mathbb{C}$ being (as usual) the complex plane and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that $\varphi(u,v)$ satisfies the following conditions:

i) $\varphi(u,v)$ is continuous in D;

ii) $(1,0) \in D$ and $\Re\{\varphi(1,0)\} > 0$;

iii) $\Re\{\varphi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -\frac{(1+u_2^2)}{2}$.

Let

$$h(z) = 1 + h_1 z + h_2 z^2 + ..., \tag{2.6}$$

be regular in $\mathbb{U}$ such that $(h(z),\ zh'(z)) \in D$ for all $z \in \mathbb{U}$. If

$$\Re\{\varphi(h(z),\ zh'(z))\} > 0 \ (z \in \mathbb{U}),$$

then

$$\Re\{h(z)\} > 0 \ (z \in \mathbb{U}).$$

Lemma 3 [9]. *Let the (nonconstant) function $w(z)$ be analytic in $\mathbb{U}=\{z\colon |z|<1\}$,with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in\mathbb{U}$, then*

$$z_0 w^{'}(z_0)=\xi w(z_0)\,,$$

where ξ is a real number and $\xi\geq 1$.

In the following section, we will get inclusion properties which associate the operator $\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}$ with the classes $\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, $\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}$, $\Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$ and $\Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta}$.

3. Results and Discussion

Unless otherwise mentioned, we assume throughout this paper that $\alpha,\beta,\gamma\in\mathbb{C},\ Re(\alpha)>\max\{0,Re(k)-1\},Re(k)>0,\eta>0$ and $m\in\mathbb{N}_0$.

Theorem 1. *If $f(z)\in\Sigma,\ Re(\beta)>1,\ Re(\frac{\gamma}{k})>0$, then*

$$\Sigma S^{*,\gamma+1,k,m}_{\alpha,\beta,\eta}\subset\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}\subset\Sigma S^{*,\gamma,k,m}_{\alpha,\beta+1,\eta}, \tag{3.1}$$

and

$$\Sigma S^{*,\gamma,k,m+1}_{\alpha,\beta,\eta}\subset\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}. \tag{3.2}$$

Proof. To prove the first part of (3.1), let $f\in\Sigma S^{*,\gamma+1,k,m}_{\alpha,\beta,\eta}$ and

$$\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}f(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}f(z)}=-\delta-(1-\delta)h(z)\ , \tag{3.3}$$

where h is given by (2.6). Applying (2.1) in (3.3), we obtain

$$\frac{\gamma}{k}\frac{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}f(z)}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}f(z)}=-\delta-(1-\delta)h(z)+\frac{k+\gamma}{k}. \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to z, we have

$$\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}f(z)}=\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}f(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}f(z)}+\frac{(1-\delta)zh'(z)}{(1-\delta)h(z)+\delta-\left(\frac{k+\gamma}{k}\right)}\quad(z\in\mathbb{U}),$$

which, by (3.3), we get

$$\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}f(z)}=-\delta-(1-\delta)h(z)+\frac{(1-\delta)zh'(z)}{(1-\delta)h(z)+\delta-\left(\frac{k+\gamma}{k}\right)}. \tag{3.5}$$

Let

$$\varphi(u,v)=(1-\delta)u-\frac{(1-\delta)v}{(1-\delta)u+\delta-\left(\frac{k+\gamma}{k}\right)}, \tag{3.6}$$

with $h(z)=u=u_1+iu_2,\ zh'(z)=v=v_1+iv_2$. Then *i)* $\varphi(u,v)$ *is continuous in* $D = \mathbb{C}\backslash\left\{1+\frac{(\gamma/k)}{1-\delta}\right\}\times\mathbb{C}$,

ii) $(1,0)\in D$ *and* $\Re\{\varphi(1,0)\} = 1-\delta$,

iii) $\Re\{\varphi(iu_2,v_1)\}\leq 0$ *for all* $(iu_2,v_1)\in D$ *and such that* $v_1 \leq -\frac{(1+u_2^2)}{2}$,

$$\begin{aligned}\Re\{\varphi(iu_2,v_1)\} &= \Re\left\{\frac{-(1-\delta)v_1}{(1-\delta)iu_2+\delta-\left(\frac{k+\gamma}{k}\right)}\right\}\\ &= \frac{(1-\delta)\left[\left(\frac{k+\gamma}{k}\right)-\delta\right]v_1}{\left[\delta-\left(\frac{k+\gamma}{k}\right)\right]^2+(1-\delta)^2u_2^2}\\ &\leq -\frac{(1-\delta)(1+u_2^2)\left[\left(\frac{k+\gamma}{k}\right)-\delta\right]}{2\left[\left[\delta-\left(\frac{k+\gamma}{k}\right)\right]^2+(1-\delta)^2u_2^2\right]}\\ &< 0.\end{aligned}$$

Therefore, the function $\varphi(u,v)$ satisfies the conditions in Lemma 2. Thus, we have $Re\{h(z)\}>0$, that is, $f\in\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$. By using the similar arguments to those details above with (2.2) instead of (2.1), we can see that the conditions of Lemma 2 are satisfied for the second part in (3.1) with $\mathcal{D} = \mathbb{C}\backslash\left\{1+\frac{\alpha+\beta-1}{1-\delta}\right\}\times\mathbb{C}$.

We can prove (3.2) by using the similar arguments to those detailed above with (2.3) instead of (2.1) with $\mathbf{D} = \mathbb{C}\backslash\left\{1+\frac{1/\eta}{1-\delta}\right\}\times\mathbb{C}$, so we omitted the proof of (3.2). Therefore, the proof of Theorem 1 is completed.

Theorem 2. *If* $f(z)\in\Sigma$, *then*

$$\Sigma C^{\gamma+1,k,m}_{\alpha,\beta,\eta}\subset\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}\subset\Sigma C^{\gamma,k,m}_{\alpha,\beta+1,\eta}, \tag{3.7}$$

and

$$\Sigma C^{\gamma,k,m+1}_{\alpha,\beta,\eta}\subset\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}. \tag{3.8}$$

Proof. To prove (3.7) applying (2.4) and using Theorem 1, we observe that

$$f(z)\in\Sigma C^{\gamma+1,k,m}_{\alpha,\beta,\eta} \iff -zf'(z)\in\Sigma S^{*,\gamma+1,k,m}_{\alpha,\beta,\eta}$$

$$\implies -zf'(z)\in\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta} \iff f(z)\in\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}.$$

Also

$$= f(z)\in\Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} \iff -zf'(z)\in\Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$$

$$\implies -zf'(z)\in\Sigma S^{*,\gamma,k,m}_{\alpha,\beta+1,\eta} \iff f(z)\in\Sigma C^{\gamma,k,m}_{\alpha,\beta+1,\eta}.$$

By the same manner we can prove (3.8) which evidently completes Theorem 2.

Theorem 3. *If* $f(z)\in\Sigma$, *then*

$$\Sigma K^{\gamma+1,k,m}_{\alpha,\beta,\eta}\subset\Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}\subset\Sigma K^{\gamma,k,m}_{\alpha,\beta+1,\eta}, \tag{3.9}$$

and

$$\Sigma K^{\gamma,k,m+1}_{\alpha,\beta,\eta} \subset \Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}. \tag{3.10}$$

Proof. To prove the first inclusion, let $f(z) \in \Sigma K^{\gamma+1,k,m}_{\alpha,\beta,\eta}$. Then, there exists a function $g(z) \in \Sigma S^{*,\gamma+1,k,m}_{\alpha,\beta,\eta}$ such that

$$\Re\left(\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} g(z)}\right) < -\sigma.$$

Let

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} = -\sigma - (1-\sigma)h(z), \tag{3.11}$$

where $h(z)$ is given by (2.6). Using (2.1), we have

$$\begin{aligned}
\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} g(z)} &= \frac{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta}(zf'(z))}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} g(z)} \\
&= \frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))\right)' + \left(\frac{k+\gamma}{k}\right)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))}{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)\right)' + \left(\frac{k+\gamma}{k}\right)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} \\
&= \frac{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} + \left(\frac{k+\gamma}{k}\right)\frac{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)}}{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} + \left(\frac{k+\gamma}{k}\right)}.
\end{aligned}$$

Since $g(z) \in \Sigma S^{*,\gamma+1,k,m}_{\alpha,\beta,\eta} \subset \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, from Theorem 1, we have

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} = -\delta - (1-\delta)\,\chi(z), \tag{3.12}$$

where $\chi(z) = g_1(x,y) + ig_2(x,y)$ and $\Re\{\chi(z)\} = g_1(x,y) > 0$ in $\mathbb{U}$. Then, by using (3.11), we have

$$\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} g(z)} = \frac{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(zf'(z))\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} - \left(\frac{k+\gamma}{k}\right)[\sigma + (1-\sigma)h(z)]}{-\delta - (1-\delta)\chi(z) + \left(\frac{k+\gamma}{k}\right)}. \tag{3.13}$$

Differentiating (3.11) with respect to z, we have

$$\frac{z\left(z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} = -(1-\sigma)zh'(z) + [\delta + (1-\delta)\,\chi(z)]\,[\sigma + (1-\sigma)h(z)]\,. \tag{3.14}$$

By substituting (3.14) into (3.13), we have

$$\frac{z(\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma+1,k,m}_{\alpha,\beta,\eta} g(z)} + \sigma = -\left\{(1-\sigma)h(z) - \frac{(1-\sigma)zh'(z)}{(1-\delta)\,\chi(z) + \delta - \left(\frac{k+\gamma}{k}\right)}\right\}.$$

Let

$$\varphi(u,v) = (1-\sigma)u - \frac{(1-\sigma)v}{(1-\delta)\chi(z) + \delta - \left(\frac{k+\gamma}{k}\right)},$$

with $h(z) = u = u_1 + iu_2,\ zh'(z) = v = v_1 + iv_2$. Then

i) $\varphi(u,v)$ is continuous in $\mathring{D} = \mathbb{C}\backslash D^* \times \mathbb{C}$, where

$$D^* = \{z : z \in \mathbb{C} \text{ and } \Re\{\chi(z)\} = g_1(x,y) > 1 + \frac{\gamma}{k(1-\delta)}\},$$

ii) $(1,0) \in D$ *and* $\Re\{\varphi(1,0)\} = (1-\sigma)$,

iii) $\Re\{\varphi(iu_2, v_1)\} \le 0$ *for all* $(iu_2, v_1) \in D$ *and such that* $v_1 \le -\frac{(1+u_2^2)}{2}$,

$$\begin{aligned}
\Re\{\varphi(iu_2, v_1)\} &= \Re\left\{\frac{-(1-\sigma)v_1}{(1-\delta)\chi(z)+\delta-\left(\frac{k+\gamma}{k}\right)}\right\} \\
&= \frac{(1-\sigma)\left[\left(\frac{k+\gamma}{k}\right)-\delta-(1-\delta)\,g_1(x,y)\right]v_1}{\left[(1-\delta)\,g_1(x,y)+\delta-\left(\frac{k+\gamma}{k}\right)\right]^2+\left[(1-\delta)g_2(x,y)\right]^2} \\
&\le -\frac{(1-\sigma)(1+u_2^2)\left[\left(\frac{k+\gamma}{k}\right)-\delta-(1-\delta)\,g_1(x,y)\right]}{2\left\{\left[(1-\delta)\,g_1(x,y)+\delta-\left(\frac{k+\gamma}{k}\right)\right]^2+\left[(1-\delta)g_2(x,y)\right]^2\right\}} \\
&< 0.
\end{aligned}$$

Therefore, the function $\varphi(u,v)$ satisfies the conditions of Lemma 2. Thus we have $Re\{h(z)\} > 0$, that is, $f \in \Sigma K_{\alpha,\beta,\eta}^{\gamma,k,m}$. By using the similar arguments to those details above with (2.2) instead of (2.1), we can see that the conditions of Lemma 2 are satisfied for the second part of (3.9) with $Q = \mathbb{C}\backslash Q^* \times \mathbb{C}$, where $Q^* = \{z : z \in \mathbb{C} \text{ and } \Re\{\chi(z)\} = g_1(x,y) > 1 + \frac{\alpha+\beta-1}{1-\delta}\}$.

We can prove (3.10) by using the similar arguments to these precedent details with (2.3) instead of (2.1) with $B = \mathbb{C}\backslash B^* \times \mathbb{C}$, where $B^* = \{z : z \in \mathbb{C} \text{ and } \Re\{\chi(z)\} = g_1(x,y) > 1 + \frac{1/\eta}{1-\delta}\}$, so we omitted the proof of (3.10). Therefore, the proof of Theorem 3 is completed.

Theorem 4. *If* $f(z) \in \Sigma$, *then*

$$\Sigma K_{\alpha,\beta,\eta}^{*,\gamma+1,k,m} \subset \Sigma K_{\alpha,\beta,\eta}^{*,\gamma,k,m} \subset \Sigma K_{\alpha,\beta+1,\eta}^{*,\gamma,k,m},$$

and

$$\Sigma K_{\alpha,\beta,\eta}^{*,m+1,k,m} \subset \Sigma K_{\alpha,\beta,\eta}^{*,\gamma,k,m}$$

Proof. Just as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (2.4), we can also prove Theorem 4 by using Theorem 3 in conjunction with the equivalence (2.5).

Let F_μ be the integral operator

$$F_\mu(f)(z) = \frac{\mu}{z^\mu}\int_0^z t^\mu f(t)dt = (z^{-1} + \sum_{k=0}^{\infty}\frac{\mu}{\mu+k+1}z^k) * f(z)$$

$$(f \in \Sigma; \mu > 0; z \in \mathbb{U}^*).$$

$$z(\mathbb{I}_{\alpha,\beta,\eta}^{\gamma,k,m}F_\mu(f)(z))' = \mu\mathbb{I}_{\alpha,\beta,\eta}^{\gamma,k,m}(f)(z) - (\mu+1)\mathbb{I}_{\alpha,\beta,\eta}^{\gamma,k,m}F_\mu(f)(z) \;\; (\mu > 0)\,. \tag{3.15}$$

In the following theorems we will get inclusion properties which associate the operator F_μ with the classes $\Sigma S_{\alpha,\beta,\eta}^{*,\gamma,k,m}$, $\Sigma C_{\alpha,\beta,\eta}^{\gamma,k,m}$, $\Sigma K_{\alpha}^{\gamma}$ $\Sigma K_{\alpha,\beta,\eta}^{*,\gamma,k,m}$.

Theorem 5. *If* $f(z) \in \Sigma,\ \mu > 0$ *and* $f(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, *then* $F_\mu(f)(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$.

Proof. Let $f \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$ and set

$$\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(f)(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(f)(z)} = -\frac{1+(1-2\delta)w(z)}{1-w(z)}\ , \tag{3.16}$$

where $w(0) = 0$. Using (3.15) in (3.16), we obtain

$$\frac{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(f)(z)} = \frac{\mu-(\mu+2-2\delta)w(z)}{\mu\left[1-w(z)\right]}. \tag{3.17}$$

Differentiating (3.17) logarithmically with respect to z, we have

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)} = -\frac{1+(1-2\delta)w(z)}{1-w(z)} + \frac{zw'(z)}{1-w(z)} - \frac{(\mu+2-2\delta)zw'(z)}{\mu-(\mu+2-2\delta)w(z)}\ , \tag{3.18}$$

so that

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z)} + \delta = \frac{(1-\delta)\,(1+w(z))}{1-w(z)} + \frac{zw'(z)}{1-w(z)} - \frac{(\mu+2-2\delta)zw'(z)}{\mu-(\mu+2-2\delta)w(z)}. \tag{3.19}$$

Let $max_{|z|\leq|z_0|}|w(z)| = |w(z_0))| = 1,\ z_0 \in \mathbb{U}$ and applying Lemma 3, we have

$$z_0 w'(z_0) = \zeta w(z_0)\ \zeta \geq 1.$$

If we set $w(z_0) = e^{i\theta},\ \theta \in \mathbb{R}$ in (3.19) and observe that

$$\Re\left\{\frac{(1-\delta)\,(1+w(z_0))}{1-w(z_0)}\right\} = 0,$$

then, we have

$$\begin{aligned}
\Re\left\{\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z_0))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z_0)} + \delta\right\} &= \Re\left\{\frac{z_0 w'(z_0)}{1-w(z_0)} - \frac{(\mu+2-2\delta)z_0 w'(z_0)}{\mu-(\mu+2-2\delta)w(z_0)}\right\} \\
&= \Re\left\{-\frac{2(1-\delta)\zeta e^{i\theta}}{(1-e^{i\theta})\,(\mu-(\mu+2-2\delta)e^{i\theta})}\right\} \\
&= \frac{2\zeta(1-\delta)(\mu+1-\delta)}{\mu^2-2\mu(\mu+2-2\delta)\cos\theta+(\mu+2-2\delta)^2} \\
&\geq 0,
\end{aligned}$$

which obviously contradicts the hypothesis $f(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$. Consequently, we can deduce that $|w(z)| < 1$ for any $z \in \mathbb{U}$, which, in view of (3.16), proves the integral-preserving property asserted by Theorem 5.

Theorem 6. *If* $f(z) \in \Sigma,\ \mu > 0$ *and* $f(z) \in \Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}$, *then* $F_\mu(f)(z) \in \Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}$.

Proof. Applying (2.4) and using Theorem 5, we observe that

$$\begin{aligned}
f(z) &\in \Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} \iff -zf'(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta} \implies F_\mu\left(-zf'(z)\right) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta} \\
&\iff -z\left(F_\mu f(z)\right)' \in \Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta} \implies F_\mu(f)(z) \in \Sigma C^{\gamma,k,m}_{\alpha,\beta,\eta}
\end{aligned}$$

which evidently proves Theorem 6.

Theorem 7. *If* $f(z) \in \Sigma,\ \mu > 0$ *and* $f(z) \in \Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$, *then* $F_\mu(f)(z) \in \Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$.

Proof. Let $f(z) \in \Sigma K^{\gamma,k,m}_{\alpha,\beta,\eta}$. Then, there exists a function $g(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$ such that

$$\Re\left(\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)}\right) < -\sigma.$$

Let

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(f)(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(g)(z)} = -\sigma - (1-\sigma)h(z), \tag{3.20}$$

where $h(z)$ is given by (2.6). Using (3.15), we have

$$\begin{aligned}
\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} &= -\frac{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta}(-zf'(z))}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} \\
&= -\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu\left(-zf'(z)\right)\right)' + (\mu+1)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu\left(-zf'(z)\right)}{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(g)(z)\right)' + (\mu+1)\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(g)(z)} \\
&= -\frac{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(-zf'(z))\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(g)(z)} + (\mu+1)\frac{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(-zf'(z))}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(g)(z)}}{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(g)(z)\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(g)(z)} + \mu + 1}.
\end{aligned}$$

Since $g(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, then from Theorem 5, we have $F_\mu(f)(z) \in \Sigma S^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, we set

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu g(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu g(z)} = -\delta - (1-\delta)\,\chi(z), \tag{3.21}$$

where $\chi(z) = g_1(x,y) + ig_2(x,y)$. Then

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} = \frac{\frac{z\left(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(-zf'(z))\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu(g)(z)} + (\mu+1)[\sigma + (1-\sigma)h(z)]}{\delta + (1-\delta)\chi(z) - \mu - 1}. \tag{3.22}$$

Differentiating (3.20) with respect to z, we have

$$\frac{z\left(z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu f(z))'\right)'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} F_\mu g(z)} = -(1-\sigma)zh'(z) + [\delta + (1-\delta)\,\chi(z)]\,[\sigma + (1-\sigma)h(z)]\,. \tag{3.23}$$

By substituting (3.23) into (3.22), we have

$$\frac{z(\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} f(z))'}{\mathbb{I}^{\gamma,k,m}_{\alpha,\beta,\eta} g(z)} + \sigma = -\left\{(1-\sigma)h(z) - \frac{(1-\sigma)zh'(z)}{(1-\delta)\,\chi(z) + \delta - \mu - 1}\right\}.$$

Let

$$\varphi(u,v) = (1-\sigma)u - \frac{(1-\sigma)v}{(1-\delta)\chi(z) + \delta - \mu - 1},$$

with $h(z) = u = u_1 + iu_2,\ zh'(z) = v = v_1 + iv_2$. We can see that the conditions of Lemma 2 are satisfied with $\mathcal{Q} = \mathbb{C}\backslash\mathcal{Q}^* \times \mathbb{C}$, where $\mathcal{Q}^* = \{z : z \in \mathbb{C} \ and\ \Re\{\chi(z)\} = g_1(x,y) > 1 + \frac{\mu}{1-\delta}\}$. The remainder of our proof of Theorem 7 is similar to that of Theorem 3, so we choose to omit the analogous details involved. This completes the proof of Theorem 7.

Theorem 8. *If* $f(z) \in \Sigma,\ \mu > 0$ *and* $f(z) \in \Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta}$, *then* $F_\mu(f)(z) \in \Sigma K^{*,\gamma,k,m}_{\alpha,\beta,\eta}$.

Conclusions

This article have many application in the future we can calculate applications in differential subordination.

References

[1] F. M. Al-Oboudi, H. A. Al-Zkeri, Applications of Briot Bouquet differential subordination to certain classes of meromorphic functions, Arab J. Math. Sci.,12 (2005) 1-14.

[2] A.H. Ansari, X. Liu, V.N. Mishra, On Mittag-Leffler function and beyond, Nonlinear Science Letters A, Vol. 8, No. 2, pp. 187-199, June 2017.

[3] M.K. Aouf and H.M. Hossen, New certeria for meromorphic p-valent starlike functions, Tsukuba J. Math., 17 (1993), no. 2, 481-486.

[4] M. K. Aouf, A. Shamandy, A.O. Mostafa and S. M. Madian, Properties of some families of meromorphic p-valent functions involving certain dierential operator, Acta Univ. Apulensis, 20 (2009), 7-16.

[5] R. M. El-Ashwah, A note on certain meromorphic p-valent functions, Appl. Math. Letters, 22 (2009), 1756-1759.

[6] R. M. El-Ashwah, M. K. Aouf, Some properties of certain subclasses of meromorphically p-valent functions involving extended multiplier transformations, Comput. Math. Appl., 59 (2010) 2111-2120.

[7] R. M. El-Ashwah, M. K. Aouf, Differential subordination and superordination on p-valent meromorphic function defined by extended multiplier transformations, Europ. J. Pure Appl. Math. 3 (2010) 1070-1085.

[8] G. Farid, A.U. Rehman, V.N. Mishra, S. Mehmood, Fractional Integral Inequalities of Gruss Type via Generalized Mittag-Leffler Function, Int. J. Anal. Appl., Vol. 17, Issue 4, (2019), pp. 548-558.

[9] G. Farid, V.N. Mishra, S. Mehmood, Hadamard And Fej\'{e}$r-HadamardTypeinequalitiesforconvexandrelativeconvexfun$ $Lefflerfunction, InternationalJournalofAnalysisandApplications, Vol.17, No.$5, (2019), $pp.$892 − 903.

[9] I.S. Jack, Functions starlike and convex of order α, J. London Math. Soc., 3 (1971), no. 2, 469-474.

[10] S. S. Miller and P. T. Mocanu, Second differential inequalities in the complex plane, J. Math. Anal. Appl., 65 (1978), 289-305.

[11] G. M. Mittag-Leffler, Sur la nouvelle fonction $E(x)$, C. R. Acad. Sci. Paris, 137(1903), 554-558.

[12] E. Mittal, S. Joshi, R.M. Pandey, and V.N. Mishra, Fractional integral and integral transform formulae using generalized Appell hypergeometric function, Nonlinear Science Letters A, Vol. 8, No. 2, pp. 221-227, June 2017.

[13] V.N. Mishra, D.L. Suthar, S.D. Purohit, Marichev-Saigo-Maeda Fractional Calculus Operators, Srivastava Polynomials and Generalized Mittag-Leffler Function, Cogent Mathematics, (2017), 4: 1320830. DOI: 10.1080/23311835.2017.1320830, pp. 1-11.

[14] A. O. Mostafa and M.K. Aouf, Some inequalities for meromorphic multivalent functions associated with Mittage-Leffler function, Int. J. Open Problems Complex Analysis, Vol. 10, 1 (2018), 1-8.

[15] A. Wiman, Über den Fundamental satz in der Theorie der Funcktionen $E(x)$, Acta Math., 29(1905), 191-201.

[16] A. Wiman, Über die Nullstellun der Funcktionen $E(x)$, Acta Math., 29 (1905), 217-134.

[17] H.M. Srivastava and Z. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, Appl. Math. Comp., 211(2009), 198-210.

[18] G. Tomar, V.N. Mishra, Maximum term of the transcendental entire function and Spider's web, Mathematica Slovaca, Vol. 70, No. 1, (2020), pp. 81-86.

Conflicts of Interest

The authors don't have competing for any interests

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