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New Approach for Solving Partial Differential Equations Based on Collocation Method

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Abstract

In this paper, a new approach for solving partial differential equations was introduced. The collocation method based on LA-transform and proposed the solution as a power series that conforming Taylor series. The method attacks the problem in a direct way and in a straightforward fashion without using linearization, or any other restrictive assumption that may change the behavior of the equation under discussion. Five illustrated examples are introduced to clarifying the accuracy, ease implementation and efficiency of suggested method. The LA-transform was used to eliminate the linear differential operator in the differential equation.

Keywords: Partial Differential Equations, Integral Transform, LA-Transform, Collocation Method.

Introduction

Differential equations can be used to describe physical, engineering, biological and chemical phenomena as a mathematical manner, as well as their use in economic, sciences and engineering. Differential equations have developed and become increasingly important in all fields of science and their applications. Therefore, getting the solution of the differential equation is very important in mathematics and these fields. Only the simplest differential equations can be solved to obtain the exact solution. Many methods have been proposed to obtain approximate or analytic solutions to solve it, such as, homotopy analysis method (HAM) [1 – 5], homotopy perturbation method (HPM) [6 – 11], Admoain decomposition method (ADM) [12 – 17], variational iteration method (VIM) [18 – 20], artificial neural network (Ann) [21 – 25], Laplace decomposition method [26, 27], Sumudu decomposition method [28-30] and Collocation Method [31-33]. All decomposition methods are proposed the solution as a series form and then the solution obtained iteratively.

In this paper, we present a new approach for solving PDEs based on suggested the solution as a series form that actually matches with Taylor series.

In the next section, we will introduce the definition of the LA-transform and main of its properties that are used in suggested approch to eliminate the linear differential operator in the differential equation.

1. LA-Transform and its Inverse

LA-transform is integral transform suggested by Luma and Alaa in [35] which is defined as follows:

$$\bar{f}(v) = \mathbb{T}\{f(t)\} = \int_0^\infty e^{-t} f\left(\frac{t}{v}\right) dt, \qquad (1)$$

Where v is a real number, which is improper integral converges. Table (1) gives the main properties of this transform.

Table 1: Main Properties of LA- transform

$f(t)$	$\bar{f}(v) = \mathbb{T}\{f(t)\}$	$D_{\bar{f}}$
t^n , n=0,1,...	$\frac{n!}{v^n}$	$v \neq 0$
e^{at}	$\frac{v}{v-a}$	$v \in R\backslash[0,a] \quad a \geq 0$ $v \in R\backslash[a,0] \quad a < 0$
$\sin(at)$	$\frac{av}{a^2+v^2}$	$v \neq 0$
$\cos(at)$	$\frac{v^2}{a^2+v^2}$	$v \neq 0$
$\sinh(at)$	$\frac{-av}{a^2-v^2}$	$\lvert v\rvert > \lvert a\rvert$
$\cosh(at)$	$\frac{-v^2}{a^2-v^2}$	$\lvert v\rvert > \lvert a\rvert$
Linear combination	$\mathbb{T}\{af(t)+bg(t)\} = a\mathbb{T}\{f(t)\} + b\mathbb{T}\{g(t)\}$	
The Transform of Derivative	$\mathbb{T}\{f^{(n)}\} = v^n\mathbb{T}\{f\} - \sum_{k=0}^{n-1} v^{n-k} f^{(k)}(0)\,, n = 1,2,3,\dots$	
Derivatives of other variables	$\mathbb{T}\left\{\frac{\partial^n}{\partial x^n} f(t,x)\right\} = \frac{\partial^n}{\partial x^n}\mathbb{T}\{f(t,x)\}\,, \qquad n = 1,2,3,\dots$	
where *f* and *g* are functions, *a* and *b* are constant.		

Let the functions $\bar{f}(v) = \mathbb{T}\{f\}$ is the LA-transform of the function *f*(*t*), then *f*(*t*) called the inverse transform of the function $\bar{f}(v)$ and denoted by: $f(t) = \mathbb{T}^{-1}\{\bar{f}(v)\}$

We noted that The inverse transform has a linear combination property, i.e.,

$$\mathbb{T}^{-1}\left\{\sum_{k=1}^{n} a_k \bar{f}_k(v)\right\} = \sum_{k=1}^{n} a_k \mathbb{T}^{-1}\{\bar{f}_k(v)\}$$

For more details and the advantages of this transform see [35].

2. Suggested Method

To illustrate suggested method, rewrite a general IVP as:

$$L\big(u(X,t)\big) + R\big(u(X,t)\big) + N\big(u(X,t)\big) = g(X,t) \qquad (2)$$

With the initial conditions (ICs):

$$\frac{\partial^k u(X,t)}{\partial t^k}\Big|_{t=0} = f_k(X)\ , k = 0, 1, \dots, n-1 \qquad (3)$$

where $L(.) = \frac{\partial^n (.)}{\partial t^n}$, $n = 1, 2, 3, \dots$ is a linear operator of the partial derivative with respect to t, $R(.)$ is the remained of the linear term, $N(.)$ is a nonlinear term, $g(X, t)$ is the inhomogeneous part and X is space independent variable. $R(.)$ and $N(.)$ are free of partial derivatives with respect to t.

In this method the unknown function $u(X, t)$ can be expressed as infinite series of the form:

$$u(X,t) = u_0(X) + u_1(X)\, t + u_2(X)\, t^2 + \cdots = \sum_{k=0}^{\infty} u_k(X) t^k \tag{4}$$

Where

$$u_k(X) = \frac{1}{k!} \frac{\partial^k u(X,t)}{\partial t^k} |_{t=0} \tag{5}$$

The next step is to determine the terms u_n (n= 0, 1, 2 ...).

Taking the LA-transform (with respect to the variable *t*) for the equation (2) to get:

$$\mathbb{T}\{L(u)\} + \mathbb{T}\{R(u)\} + \mathbb{T}\{N(u)\} = \mathbb{T}\{g(X,t)\} \tag{6}$$

From the properties in the Table (1), equation (6) becomes:

$$v^n \mathbb{T}\{u\} - \sum_{k=0}^{n-1} v^{n-k} \frac{\partial^k u(X,t)}{\partial t^k} |_{t=0} + \mathbb{T}\{R(u)\} + \mathbb{T}\{N(u)\} = \mathbb{T}\{g(X,t)\} \tag{7}$$

From equation (3) we have:

$$v^n \mathbb{T}\{u\} - \sum_{k=0}^{n-1} v^{n-k} f_k(X) + \mathbb{T}\{R(u)\} + \mathbb{T}\{N(u)\} = \mathbb{T}\{g(X,t)\} \tag{8}$$

So:

$$\mathbb{T}\{u\} = \sum_{k=0}^{n-1} v^{-k} f_k(X) - \frac{1}{v^n} \mathbb{T}\{R(u)\} - \frac{1}{v^n} \mathbb{T}\{N(u)\} + \frac{1}{v^n} \mathbb{T}\{g(X,t)\} \tag{9}$$

Taking the inverse of the LA-transform for both sides of equation (9), to get:

$$u(X,t) = \sum_{k=0}^{n-1} f_k(X) \frac{t^k}{k!} - \mathbb{T}^{-1}\left\{\frac{1}{v^n} \mathbb{T}\{R(u)\}\right\} - \mathbb{T}^{-1}\left\{\frac{1}{v^n} \mathbb{T}\{N(u)\}\right\} + \mathbb{T}^{-1}\left\{\frac{1}{v^n} \mathbb{T}\{g(X,t)\}\right\} \tag{10}$$

Now in equation (10) we can get (depending on equation (4) and since the operator *R* is independent of

t):$\mathbb{T}^{-1}\left\{\frac{1}{v^n} \mathbb{T}\{R(u)\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v^n} \mathbb{T}\{R(\sum_{k=0}^{\infty} u_k(X) t^k)\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v^n} \sum_{k=0}^{\infty} R\big(u_k(X)\big)\, \mathbb{T}\{t^k\}\right\} =$
$\mathbb{T}^{-1}\left\{\frac{1}{v^n} \sum_{k=0}^{\infty} R\big(u_k(X)\big) \frac{k!}{v^k}\right\} = \sum_{k=0}^{\infty} R\big(u_k(X)\big)\, k!\ \mathbb{T}^{-1}\left\{\frac{1}{v^{n+k}}\right\} = \sum_{k=0}^{\infty} R\big(u_k(X)\big) \frac{k!}{(n+k)!} t^{n+k}$ (11)

Also, the nonlinear part $N(u)$ of equation (10), can be written as follows:

$$N(u) = \sum_{k=0}^{\infty} N_k t^k \tag{12}$$

Where

$$N_k = \frac{1}{k!}\frac{\partial^k N\big(u(X,t)\big)}{\partial t^k}|_{t=0} \qquad (13)$$

Then the nonlinear part of equation (10) can be written as:

$$\mathbb{T}^{-1}\left\{\frac{1}{v^n}\mathbb{T}\{N(u)\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v^n}\mathbb{T}\left\{\sum_{k=0}^{\infty} N_k t^k\right\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v^n}\sum_{k=0}^{\infty} N_k \mathbb{T}\{t^k\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v^n}\sum_{k=0}^{\infty} N_k \frac{k!}{v^k}\right\} = \sum_{k=0}^{\infty} N_k\, k!\, \mathbb{T}^{-1}\left\{\frac{1}{v^{n+k}}\right\}$$
$$= \sum_{k=0}^{\infty} N_k \frac{k!}{(n+k)!} t^{n+k} \qquad (14)$$

Finally, we can write the inhomogeneous term as follows:

$$G(X,t) = \mathbb{T}^{-1}\left\{\frac{1}{v^n}\mathbb{T}\{g(X,t)\}\right\} = \sum_{k=0}^{\infty} g_k t^k \qquad (15)$$

Where

$$g_k = \frac{1}{k!}\frac{\partial^k G(X,t)}{\partial t^k}|_{t=0} \qquad (16)$$

Substituting equations (11), (14) and (15) in equation (10) we have:

$$u(X,t) = \sum_{k=0}^{n-1} f_k(X)\frac{t^k}{k!} - \sum_{k=0}^{\infty} R\big(u_k(X)\big)\frac{k!}{(n+k)!}t^{n+k} - \sum_{k=0}^{\infty} N_k \frac{k!}{(n+k)!}t^{n+k} + \sum_{k=0}^{\infty} g_k t^k \qquad (17)$$

For all $j \geq n$ substituting equation (17) in (5) to get:

$$u_j(X) = \frac{1}{j!}\frac{\partial^j u(X,t)}{\partial t^j}|_{t=0} = \frac{1}{j!}\frac{\partial^j}{\partial t^j}\left[\sum_{k=0}^{n-1} f_k(X)\frac{t^k}{k!} - \sum_{k=0}^{\infty}\frac{k!}{(n+k)!}\big(R\big(u_k(X)\big) + N_k\big)t^{n+k} + \sum_{k=0}^{\infty} g_k t^k\right]_{t=0} \qquad (18)$$

Since $j \geq n$ then $\frac{\partial^j}{\partial t^j}\left[\sum_{k=0}^{n-1} f_k(X)\frac{t^k}{k!}\right] = 0$ and

$$\frac{\partial^j}{\partial t^j}t^k = \begin{cases} 0 & j > k \\ \dfrac{k!}{(k-j)!}t^{k-j} & k \geq j \end{cases}$$

So equation (18) becomes:

$$u_j(X) = \frac{1}{j!}\left[-\sum_{k=j-n}^{\infty}\frac{k!}{(n+k)!}\big(R\big(u_k(X)\big) + N_k\big)\frac{(n+k)!}{(n+k-j)!}t^{n+k-j} + \sum_{k=j}^{\infty} g_k \frac{k!}{(k-j)!}t^{k-j}\right]_{t=0}$$
$$= \frac{1}{j!}\left[-\frac{(j-n)!}{j!}\left(R\left(u_{j-n}(X)\right) + N_{j-n}\right)\frac{j!}{0!} + g_j\frac{j!}{0!}\right]$$

$$hence \qquad u_j(X) = g_j - \frac{(j-n)!}{j!}\left(R\left(u_{j-n}(X)\right) + N_{j-n}\right), \quad j \geq n \qquad (19)$$

Then substituting equation (19) in (4) to get $u(X,t)$.

3. Applications

In this section, we will introduce some examples to illustrate reliability of suggested method.

Example 3.1: Consider the following 1st order nonlinear inhomogeneous PDE:

$$u_t - u_x^2 = t^2\,, \qquad u(x,0) = x^2$$

It is clear that $L(u) = \frac{\partial u}{\partial t}$ i.e. $n = 1$, $R(u) = 0$, $N(u) = -u_x^2$ and since $g(x,t) = t^2$ then:

$$\mathbb{T}\{g(X,t)\} = \mathbb{T}\{t^2\} = \frac{2}{v^2}$$

so,

$$G = \mathbb{T}^{-1}\left\{\frac{1}{v}\mathbb{T}\{g(X,t)\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v}\frac{2}{v^2}\right\} = 2\,\mathbb{T}^{-1}\left\{\frac{1}{v^3}\right\} = \frac{2}{6}\,t^3 = \frac{1}{3}\,t^3$$

Then by equation (16) we have:

$$g_3 = \frac{1}{3}\ \ and\ \ g_k = 0\ \ \forall\ k \neq 3$$

From the ICs $u_0 = x^2$. Also, by (13) we get:

$$N_0 = \tfrac{1}{0!}N\big(u(X,t)\big)|_{t=0} = -\big(u_{0_x}\big)^2 = -4x^2$$

And by equation (19) we have:

$$u_1(X) = g_1 - \tfrac{(1-1)!}{1!}\big(R\big(u_{1-1}(X)\big) + N_{1-1}\big) = 0 + \ (0 + 4x^2) = 4x^2$$

$$N_1 = \tfrac{1}{1!}\tfrac{\partial\, N(u(X,t))}{\partial t}|_{t=0} = \tfrac{\partial\, N(-u_x^2)}{\partial t}|_{t=0} = [-2u_x u_{xt}]_{|t=0} = -2u_{0_x}u_{1_x} = -32x^2$$

$$u_2(X) = g_2 - \tfrac{(2-1)!}{2!}\big(R\big(u_{2-1}(X)\big) + N_{2-1}\big) = \tfrac{1}{2}(32x^2) = 16\,x^2$$

$$N_2 = \frac{1}{2!}\frac{\partial^2\, N\big(u(X,t)\big)}{\partial t^2}|_{t=0} = \frac{1}{2!}\frac{\partial^2\, N(-u_x^2)}{\partial t^2}|_{t=0} = -\frac{1}{2!}[2\,(u_{xt})^2 + 2\,u_x u_{xtt}]_{|t=0} = -\left[\big(u_{1_x}\big)^2 + 2u_{0_x}u_{2_x}\right]$$
$$= -[64\,x^2 + 128\,x^2] = -192\,x^2$$

$$u_3(X) = g_3 - \tfrac{(3-1)!}{3!}\big(R\big(u_{3-1}(X)\big) + N_{3-1}\big) = \tfrac{1}{3} + \tfrac{1}{3}(192\,x^2) = \tfrac{1}{3} + 64\,x^2$$

$$N_3 = \frac{1}{3!}\frac{\partial^3\, N\big(u(X,t)\big)}{\partial t^3}|_{t=0} = -\frac{1}{3!}[6\,u_{xt}u_{xtt} + 2\,u_x u_{xttt}]_{|t=0} = -\frac{1}{3!}\left[12\,u_{1_x}u_{2_x} + 12\,u_{0_x}u_{3_x}\right] = -(256\,x^2 + 256\,x^2)$$
$$= -1024\,x^2$$

$$u_4(X) = g_4 - \tfrac{(4-1)!}{4!}\big(R\big(u_{4-1}(X)\big) + N_{4-1}\big) = \tfrac{1}{4}(1024\,x^2) = 256\,x^2$$

Similarly

$$u_5(X) = 1024\,x^2\,,\ u_6(X) = 4096\,x^2\,, \dots, u_k(X) = 4^k x^2\,, k = 7, 8, 9, \dots$$

Then from equation (4), we have:

$$u(x,t) = u_0(x) + u_1(x)\,t + u_2(x)\,t^2 + \cdots = x^2 + 4x^2t + x^2(4t)^2 + \left(\frac{1}{3} + x^2 4^3\right)t^3 + x^2(4t)^4 + \cdots$$
$$= \frac{1}{3}\,t^3 + x^2 \sum_{k=1}^{\infty} (4t)^k$$

This is closed to the exact solution:

$$u(x,t) = \frac{t^3}{3} + \frac{x^2}{1-4t}$$

Example 3.2: Consider the following 2nd order nonlinear PDE:

$$u_{tt}(x,t) - u(x,t) + \frac{1}{4}u_x^2 = 0\,, \qquad u(x,0) = 1 + x^2\,, \quad u_t(x,0) = 1$$

It is clear that $L(u) = \frac{\partial^2 u}{\partial t^2}$ i.e. $n = 2,\ R(u) = -u,\ N(u) = \frac{1}{4}u_x^2$ and since $g(x,t) = 0$ then $g_k = 0\ \forall k = 0, 1, 2, \ldots$

From ICs $u_0 = 1 + x^2$ and $u_1 = 1$. From equation (13) we get:

$N_0 = \frac{1}{0!}N\big(u(X,t)\big)|_{t=0} = \frac{1}{4}u_{0_x}^2 = \frac{1}{4}(2x)^2 = x^2$

$N_1 = \frac{1}{1!}\frac{\partial\, N(u(X,t))}{\partial t}|_{t=0} = \frac{1}{4}\frac{\partial\,(u_x^2)}{\partial t}|_{t=0} = \frac{1}{4}\,2\,u_{0_x}\,u_{1_x} = 0$

By equation (19)

$u_2(X) = g_2 - \frac{(2-2)!}{2!}\big(R\big(u_{2-2}(X)\big) + N_{2-2}\big) = -\frac{1}{2}\,(-1 - x^2 + x^2) = \frac{1}{2}$

$N_2 = \frac{1}{2!}\frac{\partial^2\, N(u(X,t))}{\partial t^2}|_{t=0} = \frac{1}{4}\,\frac{1}{2!}\frac{\partial^2\, u_x^2}{\partial t^2}|_{t=0} = \frac{1}{4}\,\big(u_{1_x}^2 + 2\,u_{0_x}\,u_{2_x}\big) = \frac{1}{4}\,(0+0) = 0$

$u_3(X) = g_3 - \frac{(3-2)!}{3!}\big(R\big(u_{3-2}(X)\big) + N_{3-2}\big) = -\frac{1}{6}\,(-1+0) = \frac{1}{3!}$

Similarly

$u_4(X) = \frac{1}{4!}\,,\ u_5(X) = \frac{1}{5!}\,, \ldots, u_k(X) = \frac{1}{k!}\,, \ldots$

Then by equation (4), we have:

$$u(x,t) = u_0(x) + u_1(x)\,t + u_2(x)\,t^2 + \cdots = x^2 + 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots = x^2 + \sum_{k=0}^{\infty}\frac{1}{k!}t^k$$

This is closed to the exact solution:

$$u(x,t) = x^2 + e^t$$

Example 3.3: Consider the following 3-dimentions 2nd order nonlinear homogeneous PDE

$$u_t + u\,u_z - u_{xx} - u_{yy} - u_{zz} = 0\ ,$$

$$subject\ to\ IC: u(x,y,z,0) = \frac{2\ e^{\mu}}{e^{\mu}+1}\ , where\ \mu = \frac{-1}{3}(x+y+z)$$

It is clear that $L(u) = \frac{\partial u(x,y,z,t)}{\partial t}$ i.e. $n = 1,\ R(u) = -u_{xx} - u_{yy} - u_{zz},\ N(u) = u\,u_z$ and since $g(t) = 0$ then:

$$g_k = 0 \quad , \qquad \forall\, k = 0, 1, 2, \ldots$$

By IC: $u_0 = \frac{2\,e^{\mu}}{e^{\mu}+1}$; and from equation (13) we get:

$N_0 = \frac{1}{0!} N\big(u(t)\big)|_{t=0} = u_0 u_{0_z} = \frac{-4\,e^{2\mu}}{3\,(e^{\mu}+1)^3}$

By equation (19)

$u_1 = g_1 - \frac{(1-1)!}{1!}\big(R(u_{1-1}) + N_{1-1}\big) = \frac{2\ e^{2\mu}}{3\,(e^{\mu}+1)^2}$

$N_1 = \frac{1}{1!}\frac{\partial\, N(u(t))}{\partial t}|_{t=0} = \frac{4\,e^{2\mu}\,(e^{\mu}-2)}{9\,(e^{\mu}+1)^4}$

$u_2 = g_2 - \frac{(2-1)!}{2!}\big(R(u_{2-1}) + N_{2-1}\big) = \frac{e^{\mu}\,(1-e^{\mu})}{9\,(e^{\mu}+1)^3}$

$N_2 = \frac{1}{2!}\frac{\partial^2\, N(u(t))}{\partial t^2}|_{t=0} = \frac{2\,e^{2\mu}\,(7\,e^{\mu}-e^{2\mu}-4)}{27\,(e^{\mu}+1)^5}$

$u_3 = g_3 - \frac{(3-1)!}{3!}\big(R(u_{3-1}) + N_{3-1}\big) = \frac{e^{\mu}\,(1-4\,e^{\mu}+e^{2\mu})}{81\,(e^{\mu}+1)^4}$

$N_3 = \frac{1}{3!}\frac{\partial^3\, N(u(t))}{\partial t^3}|_{t=0} = \frac{2\,e^{2\mu}\,(e^{3\mu}+33\,e^{\mu}-18\,e^{2\mu}-8)}{243\,(e^{\mu}+1)^6}$

$u_4 = g_4 - \frac{(4-1)!}{4!}\big(R(u_{4-1}) + N_{4-1}\big) = \frac{e^{\mu}\,(1-11\,e^{\mu}+11\,e^{2\mu}-e^{3\mu})}{972\,(e^{\mu}+1)^5}$

Then from equation (4), we have:

$$\begin{aligned} u(t) &= u_0 + u_1\,t + u_2\,t^2 + \cdots \\ &= \frac{2\,e^{\mu}}{e^{\mu}+1} + \frac{2\ e^{2\mu}}{3\,(e^{\mu}+1)^2}\,t + \frac{e^{\mu}\,(1-e^{\mu})}{9\,(e^{\mu}+1)^3}\,t^2 + \frac{e^{\mu}\,(1-4\,e^{\mu}+e^{2\mu})}{81\,(e^{\mu}+1)^4}\,t^3 \\ &+ \frac{e^{\mu}\,(1-11\,e^{\mu}+11\,e^{2\mu}-e^{3\mu})}{972\,(e^{\mu}+1)^5}\,t^4 + \cdots \end{aligned}$$

This is closed to the exact solution:

$$u(x,y,z,t) = \frac{2\ \mathrm{e}^{\mu+t/3}}{e^{\mu+t/3}+1}$$

Example 3.4: Consider the following 3rd order nonlinear inhomogeneous PDE:

$$u_{ttt} - u\,u_x + u^2 - u = 3\,e^{x+t}\,,$$

$$\textit{subject to ICs}:\ u(x,0) = 0\,,\quad u_t(x,0) = e^x,\ \ u_{tt}(x,0) = 2\,e^x$$

It is clear that $L(u) = \frac{\partial^3 u}{\partial t^3}$ i.e. $n = 3,\ R(u) = -u,\ N(u) = -u\,u_x + u^2$ and since $g(x,t) = 3\,e^{x+t}$ then:

$$\mathbb{T}\{g(X,t)\} = \mathbb{T}\{3\,e^{x+t}\} = 3\,e^x\,\frac{v}{v-1}$$

$$so, G = \mathbb{T}^{-1}\left\{\frac{1}{v^3}\mathbb{T}\{g(X,t)\}\right\} = \mathbb{T}^{-1}\left\{\frac{1}{v^3}3\,e^x\,\frac{v}{v-1}\right\} = 3\,e^x\,\mathbb{T}^{-1}\left\{\frac{1}{v^2\,(v-1)}\right\} = \;3\,e^x\,\mathbb{T}^{-1}\left\{-\frac{v+1}{v^2}+\frac{1}{v-1}\right\}$$
$$= 3\,e^x\,(e^t - t - \frac{t^2}{2} - 1)$$

Then by equation (16):

$$g_0 = 0$$

$$g_1 = \frac{1}{1!}\frac{\partial G(X,t)}{\partial t}|_{t=0} = 0$$

$$g_2 = \frac{1}{2!}\frac{\partial^2 G(X,t)}{\partial t^2}|_{t=0} = 0$$

$$g_3 = \frac{1}{3!}\frac{\partial^3 G(X,t)}{\partial t^3}|_{t=0} = \frac{3\,e^x}{3!}$$

$$g_k = \frac{3\,e^x}{k!} \qquad k = 3, 4, 5, \ldots$$

From the ICs we have: $u_0 = 0$, $u_1 = e^x$, and $u_2 = \frac{2\,e^x}{2!} = e^x$.

From equation (13) we get:

$$N_0 = \frac{1}{0!}N\big(u(X,t)\big)|_{t=0} = -u_0 u_{0_x} + u_0^2 = 0$$

$$N_1 = \frac{1}{1!}\frac{\partial\, N\big(u(X,t)\big)}{\partial t}|_{t=0} = -\frac{\partial\,(u\,u_x - u^2)}{\partial t}|_{t=0} = [-uu_{xt} - u_x u_t + 2uu_t]_{|t=0} = -u_0 u_{1_x} - u_1 u_{0_x} + 2\,u_0 u_1 = 0$$

$$N_2 = \frac{1}{2!}\frac{\partial^2\, N\big(u(X,t)\big)}{\partial t^2}|_{t=0} = -\frac{1}{2!}\frac{\partial^2\,(u\,u_x - u^2)}{\partial t^2}|_{t=0} = \frac{1}{2!}[-u_t u_{xt} - uu_{xtt} - u_{xt}u_t - u_x u_{tt} + 2u_t^2 + 2uu_{tt}]_{|t=0}$$
$$= -u_1 u_{1_x} - u_0 u_{2_x} - u_{0_x} u_2 + u_1^2 + 2\,u_0 u_2 = -e^{2x} + e^{2x} = 0$$

By equation (19) we have:

$$u_3(X) = g_3 - \frac{(3-3)!}{3!}\big(R\big(u_{3-3}(X)\big) + N_{3-3}\big) = \frac{3\,e^x}{3!} - \frac{1}{6}\,(0+0) = \frac{e^x}{2}$$

$$N_3 = \frac{1}{3!}\frac{\partial^3\, N\big(u(X,t)\big)}{\partial t^3}|_{t=0} = 0$$

$$u_4(X) = g_4 - \frac{(4-3)!}{4!}\big(R\big(u_{4-3}(X)\big) + N_{4-3}\big) = \frac{3\,e^x}{4!} + \frac{1}{4!}e^x = \frac{e^x}{3!}$$

Similarly

$$u_5(X) = \frac{e^x}{4!}\ ,\ u_6(X) = \frac{e^x}{5!}\ , \ldots, u_k(X) = \frac{e^x}{(k-1)!}\ , k = 3, 4, 5, \ldots$$

Then from equation (4), we get:

$$u(x,t) = u_0(x) + u_1(x)\,t + \;u_2(x)\,t^2 + \cdots = 0 + t\,e^x + t^2\,e^x + \frac{1}{2!}t^3 e^x + \frac{1}{3!}t^4\,e^x + \cdots = 0 + \sum_{k=1}^{\infty}\frac{e^x}{(k-1)!}t^k$$
$$= te^x\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!} = \;te^x\sum_{k=0}^{\infty}\frac{t^k}{k!}$$

This is closed to the exact solution:

$$u(x,t) = te^{x+t}$$

4. Conclusions

In this research, new approach to solve PDEs is proposed. Suggested method basis on combination of LA-transform with power series is proposed to get the exact solution of non-linear, non-homogenous PDEs. The experimental results show that the suggested method is computationally efficient for solving those types of problems and can easily be implemented. The obtained results show that our proposed methods have several advantages such like being free of using Adomian polynomials when dealing with the nonlinear terms like in the ADM and being free of using the Lagrange multiplier as in the VIM.

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