DOI: https://doi.org/10.24297/jam.v18i.8549

Approximating Fixed Points of The General Asymptotic Set Valued Mappings

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Abstract:

The purpose of this paper is to introduce a new generalization of asymptotically non-expansive set-valued mapping G and to discuss its demi-closeness principle. Then, under certain conditions, we prove that the sequence defined by $y_{n+1} = t_n z + (1 - t_n)u_n$, $u_n \in G^n(y_n)$ converges strongly to some fixed point in reflexive Banach spaces. As an application, existence theorem for an iterative differential equation as well as convergence theorems for a fixed point iterative method designed to approximate this solution is proved

Keywords: Asymptotically Non- Expansive Set-Valued Mappings, Demi-Closeness, Fixed Points, Iterative Schemes.

1. Introduction

It is well known that the concept of asymptotically nonexpansive by Goebel and Kirk [1] was introduced. Additionally, every asymptotically nonexpansive mapping of a Banach space has a fixed point is proved. Since then, iteration processes for asymptotically nonexpansive mappings in Banach spaces have studied extensively by many authors (see [2–8]). Alber, Chideme, and Zegeye [10] introduced a new class of asymptotically nonexpansive. As well as approximating methods for finding their fixed points are studied. Saluja [11] strong and weak convergence for approximating common fixed point for generalized asymptotically quasi-nonexpansive mappings in a Banach space are established. One of the first results of iterative procedures was obtained by Browder [12]. He studied the iterative scheme for non-expansive mapping $G: \mathcal{A} \to \mathcal{A}$

$$x_\lambda = \lambda x_0 + (1 - \lambda)Gx_\lambda \quad \ldots \quad (1.1)$$

where $\mathcal{A}$is a closed convex subset of a Hilbert space, $x_0 \in \mathcal{A}$ $G: \mathcal{A} \to \mathcal{A}$, with $\mathcal{F}(G) \neq \emptyset$. Halpern [13] introduced the following:

$$y_1 \in \mathcal{A}, y_{n+1} = \lambda_n x_0 + (1 - \lambda_n)Gy_n \quad \ldots. \quad (1.2)$$

Later on, this result was extended by many authors, as stated in [14] when $\mathcal{X}$ is a uniformly smooth Banach space, and then it has been investigated in many references with different additional condition on the sequence $\{\lambda_n\}$. By similar way, Lim and Xu [15] studied the following iterative sequence for asymptotically non-expansive:

$$y_n = \lambda_n y_0 + (1 - \lambda_n)G^n y_n \quad \ldots. \quad (1.3)$$

and proved that the sequence (1.3) converges strongly to a fixed point of G in a uniformly smooth Banach space under suitable conditions. Recently, Alber and el at [10] introduced the class of total asymptotic non-expansive mappings which are unified various definitions of classes of nonlinear mappings. Also, Na and Tang [16] proved the set-valued version of total asymptotic non-expansive mapping and some theorems about weak and strong convergence in uniformly convex Banach spaces for two steps iterative sequence which depending on the projection mapping. Throughout this article, $\to$ and $\rightharpoonup$are denoted by strong convergence and weak

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convergence, respectively. An element $z \in \mathcal{A}$ is called a fixed point of a single-valued mapping G, if $p = Gp$ and of a set-valued mapping G if $p \in Gp$. The set of fixed points of G is denoted by $\mathcal{F}(G)$ and

$2^{\mathcal{A}}$ is the class of all non-empty convex closed bounded subsets of $\mathcal{A}$,

$CB(\mathcal{A})$ is the class of all non-empty closed bounded subsets of $\mathcal{A}$,

$CCB(\mathcal{A})$ is the class of all non-empty convex closed bounded subsets of $\mathcal{A}$.

Let $\mathfrak{D}$ be a Hausdorff distance induced by a norm, that is

$\mathfrak{D}(\mathcal{A},\mathcal{B}) = \max\{sup_{x\in\mathcal{A}} d(x,\mathcal{B}), sup_{y\in\mathcal{B}} d(y,\mathcal{A})\}$, for $\mathcal{A}, N \in CB(\mathcal{X})$

and $d(x,\mathcal{B}) = inf_{y\in\mathcal{B}}\|x-y\|$.

The purpose of this paper is to consider a class set-valued asymptotically mappings $G\colon\mathcal{A}\to 2^{\mathcal{A}}$ and obtain strong convergence theorems in Banach spaces for the following sequence of iterates $\{y_{n+1}\}$ with respect to $z\in\mathcal{A}$:

$y_0 \in A,\ y_{n+1} = t_n z + (1-t_n)u_n,\quad n = 1,2,\ldots$ ... (1.4)

where $u_n \in G^n(y_n)$, $0 < t_n \le 1, \lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} t_n$. When G is single-valued mapping (1.4) will be $y_0 \in \mathcal{A}$, $y_{n+1} = t_n z + (1-t_n)G^n y_n$, $n = 1,2,\ldots$. The results presented in this paper mainly improve the known corresponding results in the literature sources listed within this work. Let $\boldsymbol{\mu}$ be the class of all continuous and strictly increasing function $\mu\colon R^+\to R^+$ with $\mu(0) = 0$.

Definition (1.1) A set-valued mapping $G\colon\mathcal{A}\to 2^{\mathcal{A}}$ is called the general asymptotic set-valued mapping if for each $x\in\mathcal{A}$ there exists null non- negative real sequences $\{a_n\}$ and $\{b_n\}$ such that

$$\mathfrak{D}(G^n x, G^n y) \le \|x-w\| + a_n\,\mu(\|x-w\|) + b_n \qquad \ldots\ (1.5)$$

for any $y\in A$, $w\in G^n y$ and $\mu\in\boldsymbol{\mu}$

Remark 1.2 As special cases of (1.5) are total asymptotically nonexpensive set (or, single)valued mappings [6]or [5], semi asymptotically nonexpansive set (single) valued mappings, asymptotically non-expansive set-valued [2] and the set (single) valued non-expansive mapping [7]. In order to show the need to study a type of set-valued mapping that is wider than the total asymptotically nonexpansive set-valued mappings, let us use the following example [8].

Example 1.3 Consider $\mathcal{X} = C[0,1]$ with $\|x\| = sup_{t\in[0,1]}|x(t)|$, $\mathcal{A} = \{x\in\mathcal{X}\colon x(t)\ge 0, \forall t\in[0,1]\}$ and a, b any elements in $(0,1)$ with $a<b$. Let $G\colon\mathcal{A}\to 2^{\mathcal{A}}$ is defined by

$$G(x) = \begin{cases}\{y\in\mathcal{A}\colon a\le x(t)-y(t)\le b, \forall t\in[0,1],\ \text{if} x(t) > 1, \forall t\in[0,1]\} & \\ 0, & \text{otherwise}\end{cases}$$

Then for any $x, y\in\mathcal{A}$ and any sequences $a_n\to 0, b_n\to 0$ and $\mu(t) = t$, G is total asymptotically nonexpansive set-valued mappings only if exactly one $x = 0$ or $y = 0$ holds. Such mapping is called quasi $\mu-$ asymptotically nonexpansive set-valued mapping.

2. Preliminaries

Let $\mathcal{X}$ be a real Banach space, and $\mathcal{X}^*$ be its dual space. As usual the duality pairing of $\mathcal{X}$ and $\mathcal{X}^*$ is denoted by $\langle g,x\rangle$, where x in $\mathcal{X}$ and g in $\mathcal{X}^*$ (i.e., is the value of g at x). $\mathcal{X}$ is called smooth if for each x in $S_{\mathcal{X}} = \{x\in\mathcal{X}\colon\|x\| = 1\}$, there exists a unique functional $g_x\in\mathcal{X}^*$ such that $\langle g_x, x\rangle = \|x\|$ and $\|g_x\|$ Also, $\mathcal{X}$ is uniformly

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convex if for any $0 < \epsilon \leq 2$, the inequalities $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \geq \epsilon$ implies that there exists a $\delta = \delta(\epsilon) > 0$ such that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. For example, every Hilbert space is uniformly convex space, and every uniformly convex space is reflexive. Let μ: R+ →R be continuous strictly increasing function such that $\mu(t) \to \infty$ as $t \to \infty$ and $\mu(0) = 0$. A mapping $J_\mu : \mathcal{X} \to 2^{\mathcal{X}^*}$ is called a generalized duality associated to gauge function μ if $J_\mu = \{x^* \in \mathcal{X}^* : \langle x, x^* \rangle = \mu(\|x\|)\|x\|, \|x^*\| = \mu(\|x\|)\}, x \in \mathcal{X}$. When $\mu(t) = t$ then $J_\mu = J$ which is called normalized duality mapping. If $\mathcal{X}$ is reflexive, then J is a mapping of $\mathcal{X}$ into $\mathcal{X}^*$. A Banach space $\mathcal{X}$ has a weakly continuous duality mapping if there exists a gauge μ for which the generalized mapping J_μ is single-valued and weak-to weak* sequentially continuous. In the sense of convex analysis, J_μ is sub-differential of the convex function

$$\phi(t) = \int_0^t \mu(s)ds, \quad s \geq 0 \qquad \text{... (2.1)}$$

and J_μ is single-valued if and only if $\mathcal{X}$ is smooth. The following sub-differential inequality which is held in smooth spaces:

$$\phi(\|x + y\|) \leq \phi(\|x\|) + \langle y, J_\mu(x + y) \rangle, \text{for all } x, y \text{ in } \mathcal{X}. \qquad \text{... (2.2)}$$

For more details about the above preliminaries, see [17]. In the following, we recall some definition and auxiliary results which are needed in the sequel.

Definition 2.1 [6] Let $\mathcal{A}$ be a closed subset of $\mathcal{X}$, a set-valued mapping $G: \mathcal{A} \to 2^{\mathcal{A}}$ is called demi-closed at x_0 if and only if for any sequence $\{x_n\}$ in $\mathcal{A}$ converges weakly to x_0 and $\{y_n\}$ converges strongly to $\{y_0\}$ with $y_n \in G(y_n)$ for each $n \in N$, implies that $y_0 \in G(x_0)$.

Definition 2.2 [10] A Banach space $\mathcal{X}$ is said to be satisfied Opial's condition if for each sequence $\{x_n\}$ in $\mathcal{X}$, converges to x implies that

$$\lim_{n\to\infty} sup\|x_n - x\| < \lim_{n\to\infty} sup\|x_n - y\|, \text{ for all } y \in \mathcal{X},\ x \neq y.$$

The demi-closeness principle for set-valued mappings are more devoted than that single-valued mapping. As we know, the demi-closeness principle of non-expansive mapping is satisfied only when this Banach space satisfies Opial's condition [11]. Here, the demi-closeness for set-valued mappings with convex values will be given, so, recall that a subset $\mathcal{A}$ of $\mathcal{X}$ is called proximal if, for each x ∈ $\mathcal{X}$, there exists $a \in \mathcal{A}$ such that $d(x, \mathcal{A}) = inf\{\|x - y\| : y \in \mathcal{A}\} = d(x, a)$

Remark 2.2 [9] It is known that weakly compact convex subset of a Banach space and closed convex subset of a uniformly convex Banach space are proximinal.

Now, we prove the demi-closeness principle of a general asymptotic set-valued mapping.

Proposition (2.3) Let $\mathcal{X}$ be uniformly convex Banach space, $\emptyset \neq \mathcal{A} \subseteq \mathcal{X}$, $\mathcal{A}$ be closed convex satisfying Opial's condition and $G: \mathcal{A} \to CCB(\mathcal{A})$ be general asymptotic set-valued mapping. If the sequence $\{x_n\}$ in $\mathcal{A}$ converges to $p \in \mathcal{A}$ and $\lim_{n\to\infty} d(x_n, Gx_n)$, then $p \in Gp$, i.e., $I - G$ is a demi-closed at 0.

Proof: by hypothesis and the remark (2.2), Gp is proximinal set. Therefore, for each x_n in $\mathcal{A}$ there is $u_n \in Gp$ such that $\|x_n - u_n\| = d(x_n, Gp) \leq d(x_n, Gx_n) + \mathfrak{D}(Gx_n, Gp) \leq d(x_n, Gx_n) + \|x_n - p\| + a_n\, \mu(\|x_n - p\|) + b_n, n \geq 1$ ... (2.3)

Taking superior limit on both sides of (2.3), we obtain $\lim_{n\to\infty} sup\|x_n - u_n\| \leq \lim_{n\to\infty}\|x_n - p\|$ Opial's condition implies that $u_n = p, n \geq 1$ and then, $p \in Gp$, this completes the proof.

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The concept of a sunny nonexpansive retraction [17, p.121] is needed, especially, its description by means of duality mapping in a smooth Banach space

Definition (2.4): Let $\mathcal{X}$ be a Banach space, $\emptyset \neq \mathcal{A} \subseteq \mathcal{X}$ and $\mathcal{B} \subseteq \mathcal{A}$. A mapping $r: \mathcal{A} \to \mathcal{B}$ is said to be a retraction onto $\mathcal{B}$ if $r^2 = r$. If it also satisfies the condition

1. $\|rx - ry\| \leq \|x - y\|$, for all x, y in $\mathcal{A}$. Then r is called a non-expansive retraction.
2. And, if for $x \in \mathcal{A}$ and for $t \geq 0, r(rx + t(x - rx)) = rx$ whenever $rx + t(x - rx) \in \mathcal{A}$, then it is called sunny retraction.

Remark 2.5 [10]

1. Every closed convex subset $\mathcal{A}$ of a uniformly convex Banach space is retract of $\mathcal{X}$.
2. By definition (2.4) a non-expansive mapping $r: \mathcal{A} \to \mathcal{B}$ is sunny retraction if and only if for all y in $\mathcal{B}$, $\langle x - rx, J(y - rx)\rangle \leq 0$.

The following lemma will be applied in proof of next results

Lemma 2.6 [12] Let $\{\delta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ be non-negative real sequences with $\alpha_n > 0, \forall n \geq 1$, such that

$$\delta_{n+1} \leq \delta_n - \delta_n \alpha_n , \forall n \geq 1 \quad \text{and} \quad \frac{\gamma_n}{\alpha_n} \leq c_1, \;\; a_n \alpha_n, \;\; n \geq 1 \qquad \ldots \;(2.4)$$

Moreover, if $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\gamma_n}{\alpha_n}$, then $\lim_{n\to\infty} \delta_n = 0$.

3. Convergence Results

Consider the iterative sequence with respect to $z \in \mathcal{A}$ in (1.4) with two addition conditions:

$$\frac{a_{n+b_n}}{t_n} < 1 \; for \; n \geq 1 \qquad \ldots \;(3.1)$$

and

$$\lim_{n\to\infty} \|y_n - \omega_n\| = 0, \quad \text{for some } \omega_n \in \; G^n y_{n+1} \qquad \ldots \;(3.2)$$

Theorem 3.1 Let be $\emptyset \neq \mathcal{A}$ be a closed convex subset of a smooth reflexive J- Banach space $\mathcal{X}$ and $G: \mathcal{A} \to 2^{\mathcal{A}}$ be a uniformly continuous general asymptotic set-valued mapping with $\mathcal{F}(G) \neq \phi$. Suppose that $r: \mathcal{A} \to \mathcal{F}(G)$ is sunny non-expansive retraction mapping and $\exists c_0, \; c_1 > 0$ such that $\mu(\delta) \leq c_0 \delta \; for \; c_1 \leq \delta$ then the sequence$\{y_{n+1}\}$, in (1.4), (3.1) and (3.2), converges strongly to the fixed point p= $r(z)$ of G

Proof: the proof is broken into three steps, we prove that:

i- The sequence $\{y_n - p\}$ is bounded,

ii- $\lim_n d(y_n, Gy_n) = 0$

iii- $\lim_n \|y_{n+1} - p\| = 0$

Firstly, $\|y_{n+1} - p\| = \|t_n z + (1 - t_n)u_n - p\|, \quad u_n \in G^n y_n$

$$\leq \quad t_n \|(z - p\| + (1 - t_n)\|u_n - p\|$$

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$$\leq \quad t_n\|(z-p\| + (1-t_n)\mathfrak{D}(G^n y_n, G^n p)$$

$$\leq \quad t_n + (1-t_n)(\|y_n - p\| + a_n\mu(\|y_n - p\|) + b_n$$

Putting $\delta_n = \|y_n - p\|$ implies that

$$\delta_{n+1} \leq (1-t_n)\delta_n + t_n\|(z-p\| + (1-t_n)(a_n\mu(\delta_n) + b_n) \qquad \ldots \qquad (3.3)$$

By continuity of μ on the closed interval [0,c1], μ attains its maximum C. So, for all $\delta \in [0,\infty)$ we get$\mu(\delta) \leq C + c_0\delta$. Apply this in inequality (3. 3) to have $\delta_{n+1} \leq \delta_n - (t_n - (1-t_n)a_n c_0)\delta_n + f_n$, where $f_n = (1-t_n)(a_n C + b_n) + t_n\|z - p\|$

For (i) it is enough to assume that there exist $k \in (0,1) and\ T > 0 \ni \forall n, n \geq 1$ $\frac{a_n(1-t_n)}{t_n} \leq c_0(1-k)$ and $\frac{f_n}{t_n} \leq kT$.By using Lemma (2.6), we get that $\delta_n \leq \max\{\delta_1(1+k)T\}$, which implies that $\{\|y_n - p\|\}$ is bounded. Therefore, $\{y_n\}$is a bounded sequence. For (ii) suppose that c2 bounds of the sequence$\{\|y_n - p\|\}$, so

$\|y_n\| \leq \|y_n - p\| + \|p\| \leq c_2 + \|p\| = c_3$. It is clear that if $c_2 \leq c_1$then $\mu(\|y_n - p\|) \leq C$. Also, if $c_2 \geq c_1$, then $\mu(\|y_n - p\|) \leq c_0\|y_n - p\| \leq c_0 c_2$then $\mu(\|y_n - p\|) \leq T^*\max\{C, c_o c_2\}$ and, $diam(G^n y_n) \leq \mathfrak{D}(G^n y_n, G^n p) + \|p\|$

$$\leq \|y_n - p\| + a_n\mu(\|y_n - p\|) + b_n + \|p\| \qquad \ldots (3.4)$$

All this yields that the sequence $\{G^n y_n\}$ is also bounded. Then, by (1.4), we get

$$\lim_{n\to\infty} d(y_{n+1}, G^n y_n) \leq \lim_{n\to\infty}\|y_{n+1} - u_n\|, \quad u_n \in G^n y_n$$

$$= \lim_{n\to\infty}\big(t_n(z - u_n)\big) = 0$$

So, $\lim_{n\to\infty} d(y_{n+1}, G^n y_n) = 0 \qquad \ldots (3.5)$

Since

$$d(y_{n+1}, G^n y_{n+1}) \leq d(y_{n+1}, G^n y_n) + \mathfrak{D}(G^n y_n, G^n y_{n+1})$$

$$\leq d(y_{n+1}, G^n y_n) + \|y_n - \omega_n\| + a_n\mu(\|y_n - \omega_n\|) + b_n, \quad \omega_n \in G^n y_{n+1}$$

From (3.5) and hypothesis (ii) we obtain that

$d(y_{n+1}, G^n y_{n+1}) \to 0\ as\ n \to \infty$Therefore, $\lim_{n\to\infty\infty}(y_n, G^{n-1}y_n) = 0$ ... (3.6) (3.6)

Now, $d(y_n, Gy_n) \leq d(y_n, G^n y_n) + \mathfrak{D}(G^n y_n, Gy_n) \qquad \ldots (3.7)$

As in (3.5), $d(G^n y_n, y_n) \leq d(G^n y_n, y_{n+1}) + \|y_{n+1} - y_n\| \to 0\ as\ n \to \infty$ ...(3.8)

The uniform continuity of G implies that there is an increasing continuous real function f with $f(0) = 0$(in view of modulus of continuity definition) such that $\mathfrak{D}(G^n y_n, Gy_n) = \mathfrak{D}(G(G^{n-1}y_n), Gy_n) \leq f(d(G^{n-1}y_n, y_n))$. By (3.6) $\mathfrak{D}(G^n y_n, Gy_n) \to 0\ as\ n \to \infty$, thus $lim\, d(y_n, Gy_n) = 0$,this complete part (ii). Finally, for (iii) we apply the sub-differential inequality to $(y_{n+1} - r(z))$. But $y_{n+1} - r(z) = (1-t_n)(u_n - r(z)) + t_n(z - r(z))$, $u_n \in G^n y_n$, then

$$\phi(\| y_{n+1} - r(z)\| \leq \phi(1-t_n)\|u_n - r(z)\| + t_n\langle z - r(z), J(y_{n+1} - r(z))\rangle$$

$$\leq \phi(1-t_n)\mathfrak{D}(G^n y_n . r(z)) + t_n\langle z - r(z), J(y_{n+1} - r(z))\rangle$$

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Since G is PSNT and $r(z) \in (G)$, then we get $d(G^n y_n.r(z)) \leq \|y_n - r(z)\| + Ca_n + c_0 a_n \|y_n - r(z)\| + b_n$,where $c_n = Ca_n + c_0 a_n \|y_n - r(z)\| + b_n$vanishes $as\ n \to \infty$. Also, since ϕ is convex and non-decreasing, we have $\phi\left(d(G^n y_n.r(z))\right) \leq (1 - c_n)\phi(\|y_n - r(z)\|) + c_n\phi(\|y_n - r(z)\|) + 1) \leq \phi(\|y_n - r(z)\|) + c_n T_1$, where T_1 is suitable constant. Consequently

$$\phi(\|y_{n+1} - r(z)\|) \leq (1 - t_n)\phi(\|y_n - r(z)\|) + (1 - t_n) \quad c_n T_1 + t_n\langle z - r(z), J(y_{n+1} - r(z))\rangle \quad \ldots (3.9)$$

In the following we must prove that $\lim_{n\to\infty} \sup\langle z - r(z), J(y_n - r(z))\rangle \leq 0$

Since $\langle y_n \rangle$ is bounded and X is reflexive space then there is a weakly convergent subsequence $\{y_{n_i}\}$ to y in $\mathcal{A}$. So $\lim_{n\to\infty} \sup\left\langle z - r(z), J(y_n - r(z))\right\rangle = \lim_{n\to\infty}\left\langle z - r(z), J\left(y_{n_i} - r(z)\right)\right\rangle$,But $\lim_n dist(y_n, Gy_n) \to 0\ as\ n \to \infty$, therefore, from the demiclosedness principle, we get $y \in Gy$. The weak continuity of J and Remark (2.5-ii) implies that$\lim_{n\to\infty} \sup\left\langle z - r(z), J(y_n - r(z))\right\rangle = \lim_{n\to\infty}\langle z - r(z), J(y - r(z))\rangle \leq 0$. Rewrite (3.5) as follows $\delta_{n+1} \leq \delta_n - t_n\delta_n + f_n'$, where $\delta_n = \phi(\|y_n - r(z)\|)$ and $f_n' = (1 - t_n)\, c_n T_1 + t_n\langle z - r(z), J(y_{n+1} - r(z))\rangle$. Apply Lemma (2.6) to deduce that $\delta_n \to 0\ as\ n \to \infty$. Therefore, $\{y_n\}$ converges strongly to $r(z)$, and this completes the proof.

Example1. Consider $\mathcal{X} = R$ with its usual distance, $\mathcal{A} = [-1, 1]$ Define the mapping : $\mathcal{A} \to 2^{\mathcal{A}}\ by\ G(x) =$ the line segment between $\frac{x}{2}$ and 0 . Let $\mu: R^+ \to R^+$ with $\mu(0) = 0$ be a strictly increasing function and $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n^3}$,$\forall n \geq 1$. Since $a_n, b_n \to 0$, as $n \to \infty$, then for all $x, y \in \mathcal{A}$, we get (1.4) converges to 0.

4- An application

Although many works on functional-differential equation exist, there are a few of iterative functional differential equations. Here we give such results. In the Banach space $C([a, b], \|\ \ \|)$, where $\|\ \ \|$ is the usu

supremum norm, consider the following initial value problem

$$y'(x) = h\left(x, y(y(x))\right), x \in [a, b],\ y(x_0) = y_0 \quad \ldots (4.1)$$

where $x_0, y_0 \in [a, b]\ and\ h \in \mathcal{C} = C([a, b] \times [a, b])$ are given. Let

$$M_x = \max\{|x - a|, |b - x|\}, x \in [a, b]$$

And $\mathcal{C}_L = \{y \in \mathcal{C}: |y(t_1) - y(t_2)| \leq L|t_1 - t_2|, \forall t_1, t_2 \in [a, b]\}$ ... (4.2)

Where $L > 0$. By [14], $\mathcal{C}_L$ is nonempty convex compact subset of $C[a, b]$. We formulate the following application:

Theorem 4.1 Suppose that the following conditions are satisfied for the problem (4.1):

i- For $h \in \mathcal{C}\ \exists L_1 > 0: |h(s, u) - h(s, v)| \leq K_1|u - v|, for\ all\ u, v, s \in [a, b]$;

ii- If L in (+) is the Lipschitz constant, then $m = max\{h(s, u): (s, u) \in [a, b] \times [a, b]\} \leq L$ and $K_1 M_{x_0}(L + 1) = 1$

iii- One of the following conditions holds:

a- $mM_{x_0} \leq M_{y_0}$,

b- $x_0 = a, m(b - a) \leq b - y_0, h(s, u) \geq 0$, for all $s, u \in [a, b]$,

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c- $x_0 = b, m(b-a) \leq y_0 - a, , h(s,u) \geq 0$, for all $s, u \in [a,b]$.

Then (4.1) has at least one solution $in\ \boldsymbol{C}_L$ which is: $y_1 \in \boldsymbol{C}_L,\ y_{n+1}(t) = \lambda y_n(t) + \lambda y_0 + (1-\lambda)\int_{x_0}^{t} h(s, y_n(y_n(s)))ds, t \in [a,b]$, where $n \geq 1$ *and* $\lambda \in (0,1)$.

Proof: Consider the following integral equation

$$Qy(t) = y_0 + \int_{x_0}^{t} h(s, y(y(s)))ds, \quad t \in [a,b], y \in \boldsymbol{C}_L \qquad \dots (4.3)$$

It is clear that $y \in \boldsymbol{C}_L$ is a solution of + if and only if y is a fixed point of Q. Firstly, we prove that$Q(\boldsymbol{C}_L) \subset \boldsymbol{C}_L$. By condition (iii), we have for any $y \in \boldsymbol{C}_L$ and $t \in [a,b]$ that $|(Qy)(t)| \leq |y_0| + \left|\int_{x_0}^{t} h(s, y(y(s)))ds\right| \leq |y_0| + m|t - x_0| \leq b$ and $|(Qy)(t)| \geq |y_0| - \left|\int_{x_0}^{t} h(s, y(y(s)))ds\right| \geq |y_0| - m|t - x_0| \geq |y_0| + mM_{x_0} \geq y_0 - M_{y_0} \geq a$, which implies that for any $y \in \boldsymbol{C}_L$, $Qy(t) \in [a,b], t \in [a,b]$. Now, for any $t_1, t_1 \in [a,b]$ we get $|(Qy)(t_1) - (Qy)(t_2)| = \left|\int_{t_1}^{t_2} h(s, y(y(s)))ds\right| \leq m|t_2 - t_1| \leq L|t_2 - t_1|$. thus, $Qy \in \boldsymbol{C}_L$ for $y \in \boldsymbol{C}_L$ In a similar way, we treat the cases (iii b) and (iii -c). Therefore, $Q$is self mapping on $\boldsymbol{C}_L, i.e., Q: \boldsymbol{C}_L \to \boldsymbol{C}_L$. Let $y, z \in \boldsymbol{C}_L, t \in [a,b]$then

$$|(Qy)(t) - (Qz)(t)| \leq \left|\int_{x_0}^{t} |h(s, y(y(s))) - h(s, z(z(s)))|\, ds\right| \leq \left|\int_{x_0}^{t} K_1 |y(y(s)) - z(z(s))|\, ds\right| \leq$$

$$K_1 \left|\int_{x_0}^{t} (|y(y(s)) - y(z(s))| + |y(z(s)) - z(z(s))|)\, ds\right| \leq K_1 \left|\int_{x_0}^{t} (L|y(s) - z(s)| + \max_{r \in [a,b]} |y(r) - z(r)|)\, ds\right| \quad \dots$$

$$(4.4) \qquad = K_1 \left|\int_{x_0}^{t} (L|y(s) - z(s)| + \|y - z\|)\, ds\right| \leq K_1 \left|\int_{x_0}^{t} (L+1)\|y - z\|\, ds\right| = K_1(L+1)\|y - z\||t - x_0| \leq$$

$K_1 M_{x_0}(L+1)\|y - z\|$. Now, by taking the maximum in (4.4), we obtain that $\|Q(y) - Q(z)\| \leq K_1 M_{x_0}(L+1)\|y - z\|$.This proves that Q is single-valued quasi total non-expansive mapping, hence, continuous. Appling Schauder fixed point theorem [17] to get the first part of the conclusion and Theorem (3.1) to get the second part.

Open Problem

Recently, Abed and Mohamed Hasan [8] introduced an iteration algorithm for two finite families of total asymptotically quasi-nonexpansive mappings in Banach spaces and studied the weak and strong convergence of this algorithm for approximation common fixed points. We suggest studying this algorithm for two finite families of the general asymptotic set-valued mappings. In [18], there is a study for Fibonacci- Mann random scheme of a monotone asymptotically nonexpansive random operators. We suggest an analogous study for the mappings in (1.5). Also, some procedures can be followed as in [19] or [20] to obtain other results in the convex modular spaces or $\boldsymbol{b}-$ Menger Probabilistic spaces.

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