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The Dynamics in the Soft Numbers Coordinate System

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Abstract

"Soft Logic" extends the number 0 from a single point to a continuous line, which we term "The zero axis". One of the challenges of modern science is finding a bridge between the real world outside the observer and the observer's inner world. In "Soft Logic" we suggested a constructive model of bridging the two worlds by defining, on the basis of the zero axis, a new kind of number, which we called 'Soft Numbers'.

Inspired by the investigation and visualization of fractals by Mandelbrot, we investigate in this paper the dynamics of soft functions on the plane strip with a special coordinate system. The recursive process that creates this soft dynamics allows us to discover new dynamics sets in a plane.

Keywords: Soft Logic, Soft Number, coordinate system, plane strip, soft function, dynamics, recursive process, fractal, Mandelbrot, dynamics set.

1. Introduction

According to traditional mathematics, the expression 0/0 is undefined, although in fact the whole set of real numbers could represent this expression, since $a * 0 = 0$ for all real numbers a. This observation opens a new area for investigation, which is a part of what we call "Soft Logic".

According to Pradip K. Datta [2], the special vision of the two zeros quotient was among Ramanujan's unique capabilities: "... *Ramanujan has been quoted to have said: Zero divided by zero may be anything. The zero of the numerator may be several times the zero of the denominator and vice versa*."

Inspired by Ramanujan's perceiving the multiples a0, we assume the existence of a continuum of multiples $a\bar{0}$ where a is any real number, and by $\bar{0}$ we symbolize some special object that may be called (along with any of its multiples) a 'soft zero'. On the basis of soft zeros, the theory of soft numbers was developed [6].

In Soft Logic, we present a new language that is more flexible than the traditional true-false dichotomy. According to Marcelo Dascal [1], Leibniz envisioned the development of a ''soft rationality" and "soft language".

Subsequently, we invented a constructive model for a continuum set of some special entities that are close to infinitesimals and are called by us 'soft zeros'. This was done in a way that is similar but different from Abraham Robinson's [8] "Nonstandard Analysis" and without using the Zorn lemma. This model and the constructions arising from it are described in Section 2.

2. Methods

2.1 Soft numbers

In our previous papers [3,4,5,6], we axiomatically defined three new kinds of numbers: soft zeros, bridge numbers, and soft numbers. The algebraic operations over numbers of every kind were defined by suitable rules, preserving the usual properties of these operations. To identify new numbers with points, as is done for real and complex numbers, we invented a new coordinate system on a plane strip, shown in Figure 1.

This strip serves for the presentation of soft numbers by points, and therefore is called by us the 'Soft Numbers Strip', or briefly, the SNS. It contains three parallel vertical axes with the same unit of measure. The central axis presents numbers of the form $a\bar{0}$, where *a* is any real number and $\bar{0}$ is a special object of an infinitesimal kind with a unique property: $\bar{0}^2 = 0$. These numbers are called by us soft zeros, and the axis presenting them is called a zero axis. The soft zero $0\bar{0}$ is called an absolute zero. The zero axis extends the notion of a zero from a single point to a straight line. Two other axes bound the strip on either side at a distance of 1 from the zero axis. They are called real axes, right and left, as every one of them presents the set of real numbers in such a manner that their 0-points and the absolute zero point on the zero axis lie on one horizontal segment I_0, orthogonal to all three axes.

The zero axis divides the whole strip into two congruent parts: a right part and a left part. The segment I_0 also divides the whole strip into two congruent parts: a positive part, above I_0, and a negative part, below I_0.

Bridge numbers are created by a logic operation of bridging between the zero axis numbers and the real axes numbers. In this way, bridge numbers of two different types are created: the bridge numbers of a right type and the bridge numbers of a left type.

A soft number is defined as a pair of bridge numbers of opposite types but with the same components – the same zero axis number $x\bar{0}$ and the same real number *y*:

$x\bar{0}\dot{+}y = \{ x\bar{0} \perp y; y \perp x\bar{0}\}$,

where *x, y* are any real numbers, ⊥ is a sign of bridging, and $\dot{+}$ is a sign of unifying two bridge numbers in one soft number (the values of these bridge numbers are different, according to the Non-commutativity Axiom [6]).

In the set **SN** of all soft numbers, two special subsets are to be noted: the subset of all soft numbers with *x*=0, and the subset of all soft numbers with *y*=0. The first subset is isomorphic to the set of all real numbers, while the second subset is isomorphic to the set of all zero axis numbers (soft zeros). This isomorphism allows seeing numbers on the real and zero axes as soft numbers, with *x*=0, and y=0, correspondingly.

The structure of a soft number inspires the following way of its presentation on the SNS: For any two bridge numbers forming a soft number, the bridge number of a right type with a real component to the right of the bridge sign is presented by a point in the right part of the SNS; the bridge number of the left type is presented in the left part of the SNS, by a point symmetric to the first one about the zero axis. Thus, a soft number is represented as a symmetric pair of points, with the zero axis serving as the axis of symmetry. (Below, when the notion of a symmetric pair of points is used, it is always assumed that the zero axis is serving as the axis of symmetry.)

The following figure describes the soft coordinate number system. In the middle is the zero line with the multiplications $a\bar{0}$. The 1 axis is on the two sides, right and left. Two parameters are defined for any point on the SNS: a height *A* and a width *B* (Figure 1):

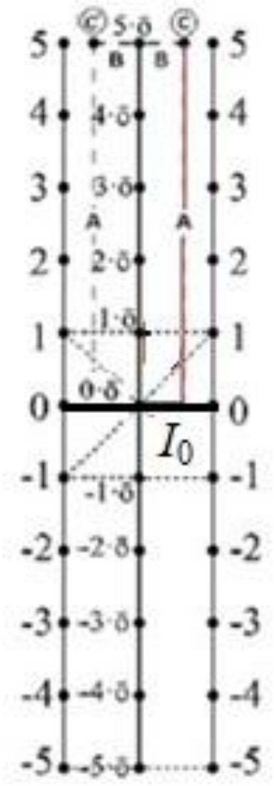

Figure 1 Figure 1

The height *A* is a vertical distance from the point to the horizontal segment I_0, taken with a plus sign if the point is above this segment, and with a minus sign if the point is below it, so that on the segment I_0 there is: *A*=0. Thus, the range of *A* is: $-\infty < A < \infty$.

The width *B* of a point on the SNS is its horizontal distance to the zero axis. The range of *B* is: $0 \leq B \leq 1$. The extreme values are reached on the zero axis (*B*=0) and on the real axes (*B*=1). Therefore, in following considerations, the zero axis is also called the 0-axis and the right real axis is called the 1-axis.

If two points on the SNS are symmetric about the zero axis, they have the same height and the same width (Figure 1). Therefore, to define a presentation of soft numbers $x\bar{0}\dotplus y$ by symmetric pairs of points on the SNS, we have to define a correspondence between these numbers and the pairs of real numbers (*A*, *B*), where the range of *A* is **R**, and the range of *B* is a closed interval [0,1]. Further on, we identify a pair of symmetric points on the SNS with a pair of its parameters (*A*,*B*).

Let us define:

$$\mathbf{SN} = \{x\bar{0}\dotplus y : x \in \mathbf{R}, y \in \mathbf{R}\}$$

− the set of all soft numbers;

$$\mathbf{SN}^* = \{x\bar{0}\dotplus y: \quad 0 \leq xy, x^2 + y^2 \neq 0\}$$

− the subset of all soft numbers for which at least one of the real factors *x, y* is not zero, and if they both are non-zero, they are both positive or both negative.

$\mathbf{SP} = \{(A.B): A \in \mathbf{R}, B \in [0,1]\}$

−the set of all symmetric pairs of points on the SNS, including extreme cases: two coinciding points on the zero axis (*B*=0), two symmetric points on the real axes (*B*=1) ;

$$\mathbf{SP}^{*}=\{(A.B): A\neq 0, B\in[0.1]\}$$

− the set **SP** without symmetric pairs of points on the segment I_0;

The investigation in our previous paper [6] showed that there is a one-to-one correspondence between the sets **SN** and **SP*** defined by the equations:

$$A=x+y,$$

$$B=\frac{y}{x+y}.$$

This pair of equations describes the way of presenting on the SNS the soft numbers $x\bar{0}\dotplus y$ from the partial set SN*. The soft numbers with x=0, $y\neq 0$ are presented on the real axes (B=1), and the soft numbers with y=0, $x\neq 0$ are presented on the zero axis (B=0). The soft numbers with x>0, y>0 are presented in the positive part of the SNS, and the soft numbers with x<0, y<0 in its negative part. The horizontal segment separating the parts (A=0) is not used for the presentation.

We sometimes refer to this method of presenting soft numbers by points as a 'convex presentation', or a 'convex method of presentation' of soft numbers on the SNS. The source of the attribute 'convex' in these names is the so-called 'convex combination' of vectors with coefficients (1-B), B, which was used in the development of this method. It may also be called 'a linear rational presentation', because it is defined with the help of the linear function and a quotient of the linear functions of two variables.

In this paper we investigate the behavior of the convex presentation of the sequences of soft numbers created recursively by soft functions.

In [6] we defined an extension of a real differentiable function $f(x)$ over **R** to a soft function over **SN** by the following formula:

$$\bar{f}(x\bar{0}\dotplus y)=xf'(y)\bar{0}\dotplus f(y).$$

In addition, a soft function can be defined directly and independently, using operations on soft numbers, which we defined and investigated in our previous paper [6].

Further on, we denote soft functions with the same symbols as real ones: *f, g*, etc.

Let *f* be a given soft function defined over the set **SN** of soft numbers. By a recursion rule:

$$C_0=X_0\bar{0}\dotplus Y_0\,,$$

$$C_k=f(C_{k-1}),\ k=1,2,3,\dots,$$

where C_0 is any given soft number, the infinite sequence of soft numbers C_k is created. It is called 'a trajectory of *f* in the set SN, starting at C_0 '. The collection of all such trajectories for a given soft function *f* with various starting soft numbers C_0 is called a dynamical system, or 'dynamics of the function *f* in the set of soft numbers **SN**'.

If the sequences of soft numbers are conceived as sequences of points on the SNS, we can examine the dynamics of *f* in a soft numbers coordinate system; such an examination is the subject of this paper.

We are interested in the following questions about dynamic (recursive) sequences presented as sequences of points on the SNS:

1. Does the sequence of the points tend to infinity or is it bounded?

2. Do the sequences of points tend to the 0-axis or to the 1-axis?

3. Do the kinds of geometrical shapes, known in the classic theory of dynamical systems, appear in dynamics with soft numbers, and in what form?

The answers to the first two questions are sought after and described in Section 2.2, and the answer to the third question is presented in Section 3. Section 4 contains the conclusion from the results of sections 2, 3, and plans for future research.

2.2 Dynamics with soft numbers

Given a differentiable real function *f*, defined on **R**, we can extend it to the set of all soft numbers **SN** and then apply it again and again, starting with any chosen and fixed soft number C_0 and continuing with its successors**.** In this way, we generate an infinite sequence of soft numbers. We wish to investigate the behavior of the infinite sequence $C_0, C_1, \dots, C_n, \dots$, where:

$$C_0 = X_0\bar{0}\dot{+}Y_0\,,$$

$$C_1 = f(C_0) = f(\,X_0\bar{0}\dot{+}Y_0)$$

$$= X_0 f'(Y_0)\bar{0}\dot{+}f(Y_0),$$

$$C_2 = f(C_1)$$

$$= X_0 f'(Y_0) f'(f(Y_0))\bar{0}\dot{+}f(f(Y_0))$$

and so on.

Definition: Let a real function$f(x)$, defined on R, be given. The composed functions obtained from it by the recursion:

$$f_1(x) = f(x), f_k(x) = f\big(f_{k-1}(x)\big) \quad (k = 2,3, \dots),$$

are called iterations of $f(x)$.

Lemma 1: Let $f(x)$ be any differentiable real function on **R**, and let $f(C)$ denote its soft extension on **SN**. If:

$$C_0 = X_0\bar{0}\dot{+}Y_0\,,$$

$$C_k = f(C_{k-1}), \qquad k = 1,2,3, \dots,$$

then:

$$C_k = X_0 f_k{}'(Y_0)\bar{0}\dot{+}f_k(Y_0), \quad k = 1,2,3, \dots .$$

Proof: We will check this formula by induction. In the case of *k* =1, the formula is correct:

$$C_1 = X_0 f'(Y_0)\bar{0}\dot{+}f(Y_0).$$

Now we assume its validity for some natural *k*:

$$C_k = X_0 f_k{}'(Y_0)\bar{0}\dot{+}f_k(Y_0).$$

By the recursive definition of the sequence C_k , we have:

$$C_{k+1} = f(C_k) = f\left(X_0 f_k{}'(Y_0)\bar{0} \dotplus f_k(Y_0)\right) = X_0 f_k{}'(Y_0) f'\left(f_k(Y_0)\right)\bar{0} \dotplus f_{k+1}(Y_0) = X_0 f_{k+1}{}'(Y_0)\bar{0} \dotplus f_{k+1}(Y_0).$$

We obtain that the formula is correct for k+1. By the induction principle, it is correct for every natural k. ■

Lemma 2: Let functions $f(x), f(C)$ be as defined in Lemma 1.

If the sequence of real numbers:

$$D_k = \frac{f_k{}'(Y_0)}{f_k(Y_0)} \qquad (k = 1,2,3, \dots)$$

tends to 0, then the sequence of points on the SNS, which are the right parts of the 'convex' images of the soft numbers C_k , defined in Lemma 1, tends to the 1-axis. If the sequence D_k tends to infinity and $X_0 > 0$, then the sequence of these points on the SNS tends to the 0-axis.

Proof: If the 'convex' method of the presentation of soft numbers as points on the SNS is used, then the distance of the points presenting the soft numbers C_k on the SNS to the zero axis is given by the equation:

$$B_k = \frac{Y_k}{X_k + Y_k} = \frac{1}{\frac{X_k}{Y_k} + 1} =$$

$$= \frac{1}{\frac{X_0 f_k{}'(Y_0)}{f_k(Y_0)} + 1} = \frac{1}{X_0 D_k + 1} \ .$$

The Lemma's statements are immediately obtained from the final expression for B_k in the above development. ■

Remark 1: In the previous considerations, it was supposed that a real function *f* is defined over the whole set **R** of real numbers. The obtained results may be generalized for a real function defined in any non-empty set *E* of real numbers, if the function values also belong to *E*. One example of such a function is given and treated further (Example 6).

Remark 2: In the following examples, the 'convex' presentation of soft numbers on the SNS is used, when the condition of its existence is fulfilled. In such a case, a sequence of soft numbers is conceived as a sequence of points in the right part of the SNS. The location of a point on the SNS is stated by its height *A* and width *B*. Therefore, the investigation of the behavior of point sequences on the SNS is reduced to finding and examining the sequences of real numbers A_k, B_k, that state the point's location on the SNS.

The functions *f* in the examples 1-5 are defined directly as operations in **SN**, while in examples 6-8 the functions are defined as extensions of the corresponding real functions.

Example 1: Let *a* be any given real non-zero number. Let us define

$$f(C) = aC \ \text{ for any soft number } C,$$

$C_0 = X_0\bar{0} \dotplus Y_0$,

$C_1 = f(C_0) = aX_0\bar{0} \dotplus aY_0$,

$$C_k = f(C_{k-1}) = a^k X_0\bar{0} \dotplus a^k Y \ .$$

Consequently,

$$A_k = a^k X_0 + a^k Y_0 = a^k(X_0 + Y_0),$$

$B_k = \frac{a^k Y_0}{a^k(X_0+Y_0)} = \frac{Y_0}{X_0+Y_0} = \frac{1}{1+\frac{X_0}{Y_0}}$.

By the last equation, the presentation of the soft numbers C_k as points on the SNS exists if $X_0 Y_0 > 0$, or $X_0 = 0, Y_0 \neq 0$. In these cases, all the points are of equal distance to the zero axis, i.e., they are positioned on one vertical line. When $|a| > 1$, the points tend to infinity, and when $|a| < 1$, the points tend to the limit point $A = 0, B = \frac{Y_0}{X_0+Y_0}$.

Example 2: For any given real $b \neq 0$ and for any soft number *C*, let us define:

$f(C) = C + b,$

$C_0 = X_0\bar{0} \dotplus Y_0,$

$C_1 = X_0\bar{0} \dotplus Y_0 + b,$

$C_k = X_0\bar{0} \dotplus Y_0 + kb.$

Consequently,

$$A_k = X_0 + Y_0 + kb,$$

$B_k = \frac{Y_0+kb}{X_0+Y_0+kb}$.

Suppose $Y_0 > 0, b > 0$. If $X_0 > 0$, the sequence of points presenting C_k on the SNS exists and tends to infinity and to the 1-axis. If $X_0 = 0$, then all the points are located on the 1-axis, tending to infinity as well. If $X_0 < 0$, the sequence of points on the SNS does not exist.

Example 3: Let *a, b* be any given real non-zero numbers, and *C* any soft number. We define:

$f(C) = aC + b,$

$C_0 = X_0\bar{0} \dotplus Y_0,$

$C_1 = aX_0\bar{0} \dotplus aY_0 + b,$

$C_2 = a^2X_0\bar{0} \dotplus a^2Y_0 + ab + b,$

$C_3 = a^3X_0\bar{0} \dotplus a^3Y_0 + a^2b + ab + b,$

$C_k = a^kX_0\bar{0} \dotplus a^kY_0 + a^{k-1}b + \cdots + ab + b.$

If $a \neq 1$, then

$C_k = a^kX_0\bar{0} \dotplus a^kY_0 + b\left(\frac{a^k-1}{a-1}\right).$

Hence,

$A_k = a^kX_0 + a^kY_0 + b\left(\frac{a^k-1}{a-1}\right),$

$$B_k = \frac{a^k Y_0 + b\left(\frac{a^k - 1}{a-1}\right)}{a^k X_0 + a^k Y_0 + b\left(\frac{a^k-1}{a-1}\right)} \ .$$

We suppose that X_0, Y_0, a, b are real numbers such that the condition $0 \le B_k \le 1$ is satisfied, and the point presentation of the numbers C_k on the SNS exists. In particular, if $a > 1$, and the values of b, X_0, Y_0, are positive, such a presentation does exist, and its points tend to infinity and to the vertical line:

$$B = \frac{Y_0 + \frac{b}{a-1}}{X_0 + Y_0 + \frac{b}{a-1}}.$$

For 0<a<1 and the same condition relative to b, X_0, Y_0, the points tend to the limit point $A = \frac{b}{1-a}$, $B = 1$.

Example 4: For any soft number *C*, let us define:

$f(C) = C^2 = (X\bar{0} \dotplus Y)^2 = 2XY\bar{0} \dotplus Y^2$.

` If $C_0 = X_0\bar{0} \dotplus Y_0$, the recursive sequence, created by $f(C)$, is:

$C_1 = 2X_0Y_0\bar{0} \dotplus Y_0^2$,

$C_2 = 4X_0Y_0^3\bar{0} \dotplus Y_0^4$,

$C_k = 2^k X_0 Y_0^{2^k - 1}\bar{0} \dotplus Y_0^{2^k}$.

Therefore,

$A_k = 2^k X_0 Y_0^{2^k-1} + Y_0^{2^k}$,

$$B_k = \frac{Y_0^{2^k}}{2^k X_0 Y_0^{2^k-1} + Y_0^{2^k}} = \frac{1}{2^k \frac{X_0}{Y_0} + 1}.$$

Conclusions: If $X_0 Y_0 > 0$, then when *k* tends to infinity, the distance of the points to the 0-axis tends to 0.

If $X_0 = 0$, $Y_0 \neq 0$, then $B_k = 1$, which means that all the points are on the 1-line. In both cases above, if $|Y_0| > 1$, the points tend to infinity; if $|Y_0| < 1$, the point sequence tends to the limit point *A*=0, *B*=1.

The case $X_0 = 0$, $|Y_0| = 1$ leads to the point *A*=1, *B*=1, which presents a fixed point in the dynamical system of the soft function

$f(C) = C^2$.

The above conclusions are reflected in Figure 5 in Section 3.

Example 5: We generalize the Example 4 function to any natural power *n*:

$f(C) = C^n = (X\bar{0} \dotplus Y)^n = nXY^{n-1}\bar{0} \dotplus Y^n$.

In this case the development, similar to the one in the previous example, gives:

$A_k = n^k X_0 Y_0^{n^k-1} + Y_0^{n^k}$,

$$B_k = \frac{Y_0^{n^k}}{n^k X_0 Y_0^{n^k-1} + Y_0^{n^k}} = \frac{1}{n^k \dfrac{X_0}{Y_0} + 1} \ .$$

Conclusions: For any natural power n>1, and $X_0 Y_0 > 0$, the sequence of points presenting the soft numbers C_k on the SNS tends to the 0- axis. If $X_0 > 0, Y_0 > 1$, the sequence simultaneously tends to infinity, and if $X_0 = 0$, $0 < Y_0 < 1$, the sequence is located on the 1-line and has a limit point A=0, B=1.

Example 6: A soft power function of a soft variable C, for any positive real power a, is defined as an extension of a real power function:

$$f(x) = x^a,$$

which involves:

$f(C) = C^a = (X\bar{0}\dot{+}Y)^a = aXY^{a-1}\bar{0}\ \dot{+}Y^a.$

It is to be noted that the real function $f(x) = x^a$ with a real power a is defined only for non-negative values of x. As its own values are also non-negative, there exists an infinite sequence of its iterations (see Remark 1 after the Lemma 2 Proof). These iterations are

$f_k = x^{a^k}$, k=1, 2, 3, ...

The points on the SNS that present the soft numbers C_k, which are created recursively by the function $f(C)$ for k=1, 2, 3, ..., starting at

$C_0 = X_0\bar{0}\dot{+}Y_0$, have the following heights and widths:

$$A_k = X_0 f_k{}'(Y_0) + f_k(Y_0) = a^k X_0 Y_0^{a^k-1} + Y_0^{a^k},$$

$$B_k = \frac{f_k(Y_0)}{X_0 f_k{}'(Y_0) + f_k(Y_0)} = \frac{Y_0^{a^k}}{a^k X_0 Y_0^{a^k-1} + Y_0^{a^k}} = \frac{1}{a^k \dfrac{X_0}{Y_0} + 1} \ .$$

(Remember that in the above equations, the value of Y_0 must be non-negative, as prescribed by the definition of the real power function.)

Lemma 3: Let X_0, Y_0, a be any given positive numbers.

(1) If $a > 1$, then the sequence of points on the SNS, presenting the soft numbers C_k created as defined above, tends to the 0-axis.

(2) If $0 < a < 1$, then the point sequence tends to the 1-axis.

Proof: The statements of the Lemma 3 are immediately obtained from the final expression for B_k in the development up to its formulation. ■

Example 7: Here we define a soft exponent $f(C) = e^C$, where $C = X\bar{0}\dot{+}Y$, as an extension of a real function $f(x) = e^x$, by the following formula:

$$f(C) = e^c = Xe^Y\bar{0}\ \dot{+}e^Y \ .$$

Given an initial soft number C_0, this function recursively creates an infinite sequence of soft numbers, as shown below:

$C_0 = X_0\bar{0}\dotplus Y_0$,

$C_1 = X_0 e^{Y_0}\bar{0}\dotplus e^{Y_0}$,

$C_2 = X_0 e^{Y_0} e^{e^{Y_0}}\bar{0}\dotplus e^{e^{Y_0}}$,

$$C_3 = X_0 e^{Y_0} e^{e^{Y_0}} e^{e^{e^{Y_0}}}\bar{0}\dotplus e^{e^{e^{Y_0}}} ,$$

and so on. The continuation of the iterative process leads, by Lemma 1, to the following sequence of soft numbers:

$$C_k = X_k\bar{0}\dotplus Y_k = X_0\, f_k{}'(Y_0)\bar{0}\dotplus f_k(Y_0), \qquad k = 1,2,3, \dots,$$

where $f_k(x)$ is the k-th iteration of the function $f(x) = f_1(x) = e^x$. This iteration may be written in the following form:

$f_k(x) = e \uparrow e \uparrow \cdots \uparrow e \uparrow x$, with k letters 'e' in the record.

From this presentation, by chain rule, we obtain:

$$f_k{}'(x) = f_k(x) f_{k-1}(x) f_{k-2}(x) \dots f_1(x).$$

We can now see that if k tends to infinity, both $f_k{}'(Y_0)$ and $f_k(Y_0)$, and their quotient:

$$D_k = \frac{f_k{}'(Y_0)}{f_k(Y_0)} = f_{k-1}(Y_0) f_{k-2}(Y_0) \dots f_1(Y_0)$$

tend to infinity for any Y_0. Let us now examine three cases for X_0:

(1) $X_0 > 0$. In this case, by Lemma 2, the sequence of points on the SNS presenting the sequence of the soft numbers C_k tends to the 0-axis.

(2) $X_0 = 0$. In this case all points are on the 1-axis.

(3) $X_0 < 0$. The sequence of points presenting the soft numbers C_k on the SNS does not exist.

It is to be noted that in both cases (1), (2), the sequence of heights of the points on the SNS:

$$A_k = X_k + Y_k = X_0 f_k{}'(Y_0) + f_k(Y_0)$$

tends to infinity, for any value of Y_0.

Conclusions: The points in the dynamics of the soft exponential function, defined on the SNS by the 'convex' method, do exist if X_0 is a non-negative real number. In this case they always tend to infinity, either tending simultaneously to the 0-axis ($X_0 > 0$) or being located on the 1-axis ($X_0 = 0$).

Figure 7 in Section 3 reflects these conclusions.

Example 8: Here we define a soft function $f(C) = \cos(C)$, where $C = X\bar{0}\dotplus Y$ is any soft number, as an extension of a real function $f(x) = \cos x$, by the following formula:

$$f(C) = \cos(C) = -X\sin(Y)\bar{0}\dotplus \cos(Y) .$$

Given an initial soft number C_0, this function recursively creates an infinite sequence of soft numbers:

$C_0 = X_0\bar{0}\dotplus Y_0$,

$C_1 = -X_0\sin(Y_0)\bar{0}\dotplus\cos(Y_0)$,

$C_2 = X_0\sin(Y_0)\sin(\cos(Y_0))\bar{0}\dotplus\cos(\cos(Y_0))$,

and so on. The continuation of the iterative process leads to the infinite sequence of soft numbers:

$$C_k = X_k\bar{0}\dotplus Y_k,$$

where:

$Y_k = \cos(\cos(\cos(\ldots(\cos(Y_0))))\ldots)$ with k names 'cos' in the record,

$X_k = (-1)^k X_0\sin(Y_0)\sin(Y_1)\sin(Y_2)\ldots\sin(Y_{k-1}),\ \ k = 1,2,3,\ldots$.

It is proved in the real analysis that for any initial Y_0, the sequence Y_k tends to a constant irrational number q=0.739… . This fact can be checked numerically. If in a functional calculator any real number is chosen and then the key 'cos' is pressed many times, the result is stabilized and shows the first digits of q.

If $X_0 = 0$ or $Y_0 = 0$, then $X_k = 0$ for all k.

Otherwise, when k tends to infinity, X_k is a product containing an increasing number of factors with absolute values not exceeding

sin (0.8)<o.8<1, and therefore the values of X_k tend to 0.

In any case, the heights $A_k = X_k + Y_k$ of the points presenting the soft numbers C_k on the SNS tend to 0+q=q, and therefore any 'convex' point presentation of the sequence of soft numbers created recursively by

f(*C*)= cos(*C*) , is bounded.

If $X_0 = 0$ or $Y_0 = 0$, then such a sequence has a full point presentation on the SNS, resting on the 1-line. Otherwise the presentation is only partial, because the signs of X_k alternate and the condition $X_kY_k > 0$, necessary for the 'convex' point presentation of soft numbers in the inner space of the SNS, is satisfied only for a part of k (even or odd).

In any case, any 'convex' point presentation on the SNS, full or partial, of any dynamic sequence of cos(*C*) rests on the 1-line or tends to it, because the widths $B_k = \frac{Y_k}{X_k+Y_k}$ of the points equal 1 ($X_k = 0$) or tend to $\frac{q}{0+q} = 1$.

Resuming the above considerations, it is to be noted that any dynamic sequence created by *f*(*C*)= cos(*C*) on the SNS has a limit point *A*=*q*, *B*=1.

Figure 6 in Section 3 reflects the findings of Example 8.

3.Results

3.1 Soft dynamics geometry

The Mandelbrot set [7] is defined in the following way: For any complex number *c* we define a sequence by a recursive instruction:

$$a_1(c) = c,$$

$$a_{n+1}(c) = (a_n(c))^{\,2} + c$$

(n=1,2,3,...).

The infinite sequence $a_n(c)$ can be bounded or unbounded depending on the starting number *c*. The Mandelbrot set (Figure 2) is defined as the set of all complex numbers *c* on the complex plane, for which a sequence $a_n(c)$ is bounded.

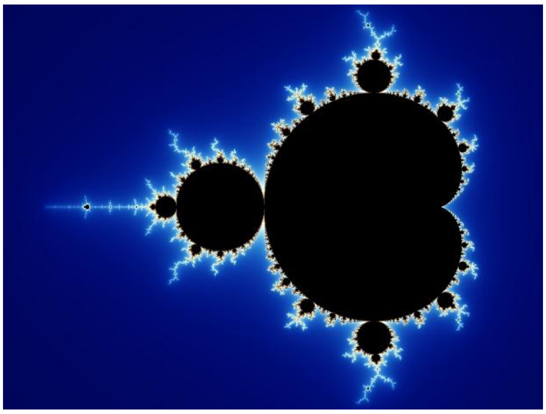

Figure 2: The Mandelbrot set

We wished to investigate the dynamics of the same function on the SNS (soft numbers strip):

$f(c) = c^2 + c,$

where *c* is any soft number. For this function, and with the help of a computer simulation, we obtained the following picture (Figure 3), which is quite different from the picture for a complex c:

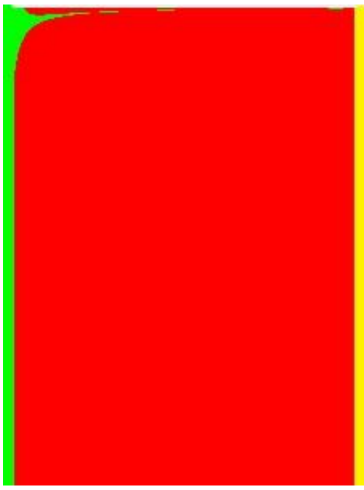

Figure 3: The soft dynamics set of $f(c) = c^2 + c$

The notion of a 'soft dynamics set' of a given soft function *f*, used in the description of Figure 3 and of other figures further on in this section, refers to the set of soft numbers *c*, classified according to the properties of the sequences created by the function *f*, by a recursion starting at *c*. Such sequences are called recursive or dynamic sequences of the function *f*, while all of them together constitute its dynamical system or dynamics.

The properties of dynamic sequences are connected to their point presentation on the SNS by the 'convex' method' and are marked by colors, as described in Table 1 below:

Table 1: The 6 domains of dynamics with soft numbers

The sequence is	Tending to the 0-line	Tending to the 1-line	Enclosed between the 0- and 1- lines
Bounded			
Not bounded			

According to this table, any point in the red domain in Figure 3 presents a soft number that initiates a sequence of soft numbers created recursively by the function $f(c) = c^2 + c$, for which a sequence of point images on the SNS found by the 'convex' method is unbounded and tends to the 0-line. A point in the green part of the picture has a similar meaning with one difference: the point sequence on the SNS is bounded, as is any point on the 0-line. On the other hand, any point on the positive part of the right 1-line initiates an unbounded recursive sequence, which however does not tend to the 0-line, remaining entirely on the 1-line, and therefore it must be yellow.

The following pictures show the soft dynamics sets of some basic soft functions, restricted to the positive right part of the SNS. A picture with a number of colors shows a number of sets corresponding to these colors. A one-color picture shows only one set.

Figure 4: The soft dynamics sets of *f(c)=c*

The explanation of Figure 4 is as follows:

The soft function $\boldsymbol{f(c) = c}$ ('the identity function') has any soft number as a start of a trajectory of a special extreme kind: a fixed point. Thus, the dynamical system of this function coincides with the whole set of soft numbers. The point convex presentations of some of them on the SNS fill the space between the 0-line and the 1-line and these lines themselves. By the meaning of the colors stated in Table 1, the color of the points between these lines has to be white; on the 0-line – green; and on the 1-line – blue.

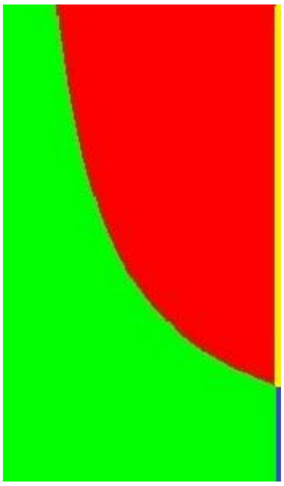

Figure 5: The soft dynamics sets of $f(c) = c^2$

Clarifications for Figure 5 may be found in Example 4 in Section 2.

The following pictures show the results of computer simulations and present the dynamics of several soft functions defined as extensions of the corresponding real functions.

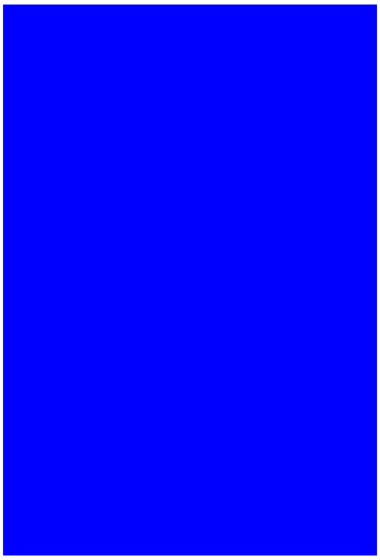

Figure 6: The soft dynamics set of *f(c*)=cos(*c*)

The special one-colored picture in Figure 6 reflects the facts discovered in Example 8 in Section 2.

Figure 7: The soft dynamics sets of $f(c) = e^c$

In Example 7 in Section 2, it is shown that all recursive sequences created by the soft exponential function tend to infinity. In addition, they all tend to the 0-axis, except those that started at a number on the 1-line and belong to it. All these facts are reflected in the picture shown in Figure 7. The red color covers the whole area of the convex presentation on the SNS, except the 1-line, which has to be yellow.

The pictures of the soft dynamic sets of some additional functions are shown in Figures 8-10 without an explanation of their forms. Their theoretical basis may be found in the corresponding examples in Section 2.

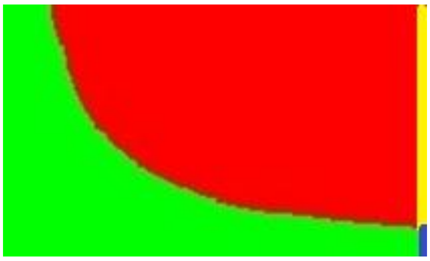

Figure 8: The soft dynamics sets of $f(c) = c^3$

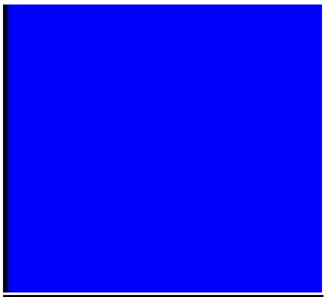

Figure 9: The soft dynamics set of $f(c) = c^{0.5}$

(the 0-line does not belong to the set, as the function is not defined on it)

Figure 10: The soft dynamics set of $f(c) = c + 1$

The soft dynamics of the functions in Figures 11, 12 have not yet been studied theoretically. The pictures are a result of a computer experiment.

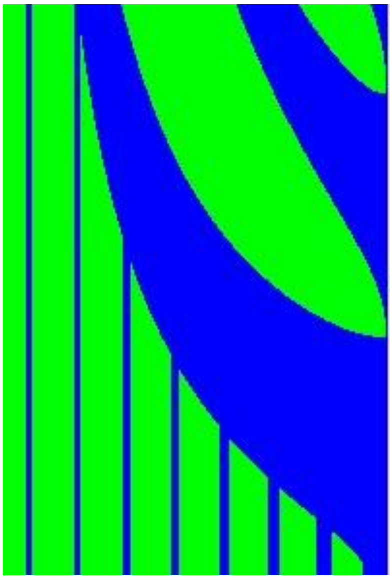

Figure 11: The soft dynamics sets of sin(c)

An experimental two-color picture. All sequences are bounded. The 0-line is entirely green, the 1-line is blue.

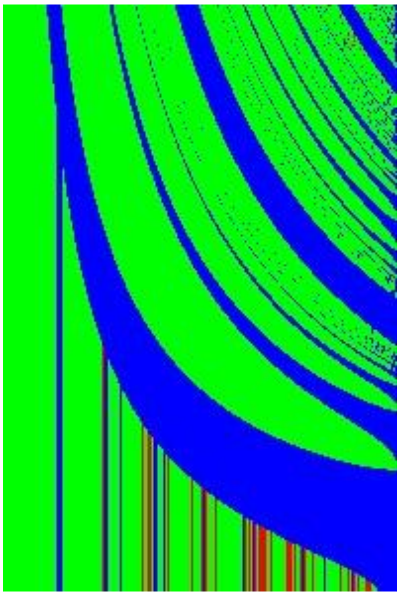

Figure 12: The soft dynamics sets of $c * \cos(c)$

An experimental picture with a number of colors, including the colors of unbounded sequences, but not on the axes. The 0-line is totally green, the 1-line is blue.

3.2 Some remarks concerning the results

The richness of appearance of the soft dynamics sets shown in Figures 3-12 opens one's mind towards a further investigation of the meaning of soft dynamics. One of the topics in this direction is finding soft dynamics sets for functions which cannot create an infinite recursive sequence starting at any soft number in their domain. One such function is tan(c). This function, for example, is defined for *c*=arctan(π/2), but already the second iteration of tan(*c*) does not exist for this value of c, and there is yet an infinite set of such numbers.

Such a situation makes a principal distinction between soft functions treated in this section and soft tangents, so that investigating this function dynamics may be a real challenge.

4. Conclusion and further research

In this paper, we presented the richness of the geometrical nature of the dynamics in a soft numbers coordinate system.

In the future, we aim to define the dimensions of the soft dynamics geometrical shapes. In addition, we wish to extend the idea of a soft number to a soft operation. For example, the operation denoted @ could be both addition or multiplication, so that 4@6={10,24}, with a given probability that the operation will be addition or multiplication.

David Bohm believed in non-locality and searched for a mathematical theory to serve as the foundation for relativity theory and quantum theory. We wish to research the relevance of Soft Logic for this purpose.

Finally, we wish to research the technological aspect of Soft Logic. Can we present the noise of certain electrical devices by using the multiplication of $\bar{0}$ with real numbers? Is the zero axis a mathematical presentation of non-local transmission? We hope to find answers, at least partial ones, to these questions.

Conflicts of Interest

There are no conflicts of interest

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