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Some Structural Results on Prime Graphs

Ibtesam Ali Alrowily

Al-Jouf University, Sakakah, Saudi Arabia

earowily@ju.edu.sa

Abstract

Given a graph G = (V,E), a subset M of V is a module [17] (or an interval [10] or an autonomous [11] or a clan [8] or a homogeneous set [7]) of G provided that x ∼ M for each vertex x outside M. So V,φ and {x}, where x ∈ V, are modules of G, called trivial modules. The graph G is indecomposable [16] if all the modules of G are trivial. Otherwise, we say that G is decomposable **.** The prime graph G is an indecomposable graph with at least four vertices. Let G and H be two graphs. Let If G has no induced subgraph isomorphic to H, then we say that G is H-free. In this paper, we will prove the next theorem

Theorem 2.1 If G is a $\{P_5,\overline{P_5},bull\}$-free graph, then exactly one of the following assertions holds.

(i) $|V(G)| \leq 2$

(ii) G is isomorphic to C_5.

(iii) There is $n \geq 2$ such that G is isomorphic to G_{2n} or $\overline{G_{2n}}$.

(iv) G is decomposable.

Keywords: Graphs, Prime graphs, Module, and Decomposition.

1.Introduction

In this paper, we will present some concepts and tools involved in combinations in the study of graphs, and more precisely, in the study of prime graphs. The quotient of a graph, one decomposes its vertex set into modules which are vertex subsets compatible with the graph. Note that the notion of modules is a generalization of the notion of interval of the usual total order on the set of real numbers. A graph with at least three vertices is decomposable if it has a quotient with fewer vertices which is not reduced to a singleton. Otherwise, the graph is prime. The fundamental Theorem of decomposition (Gallia 1967[16]) says that any finite graph admits a canonical quotient which is a complete graph or an empty graph or a prime graph. It follows that the main difficulty in many problems lies in the study of prime graphs. A book and several papers on the prime graphs and their prime subgraphs have then appeared (1, 2, 3, 4, 5, 6, 7, 10, 12, 14, 15, 17,18). On the other hand, in many studies on the classes of graphs defined by some forbidden, the prime graphs play a major role (4,5,9)

2. Some Notions on Graphs and Their Modules

In this section, first we fix our conventions on graphs. Second, we present the notion and the basic properties of the modules of a given graph. Third, we recall the basic structural results on the prime subgraphs of a given prime graph. Finally, we use these results to obtain some examples of prime graphs that will be used in the following.

- A graph G is a pair (*V* (*G*), *E*(*G*)) consisting of a finite vertex set *V* (*G*) and an edge set *E*(*G*) such that *E*(*G*) is a subset of the set of the 2-element subsets of *V* (*G*). We denote an edge {*u*,*v*} by *uv*.

- Two distinct vertices *u* and *v* are adjacent if *uv* ∈ *E*(*G*); otherwise *u* and *v* are nonadjacent.

- The set of neighbors of a vertex u, denoted by $N_G(u)$, is the set of vertices which are adjacent to u, and the degree of u, denoted by $d_G(u)$, equals $|N_G(u)|$.

- For a vertex subset X of a graph G, the subgraph of G induced by X is the graph $G[X]$ whose vertex set is X such that two vertices are adjacent in $G[X]$ if they are adjacent in G. For a vertex subset X of G, the subgraph of G induced by $V(G)-X$ is denoted by $G-X$. For a vertex v of G, the subgraph $G-\{v\}$ is denoted by $G-v$.

- An isomorphism from a graph G onto a graph H is a bijection f from $V(G)$ onto $V(H)$ such that for any two vertices u and v of G, u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. Two graphs G and H are isomorphic if there is an isomorphism from G onto H, in which case we write $G \cong H$.

- For a positive integer n, the *complete graph* K_n is the graph defined on $\{1,\dots,n\}$ in which any two distinct vertices are adjacent.

- For a positive integer n, the *path* P_n is the graph whose vertex set is $\{1,\dots,n\}$ such that two distinct vertices are adjacent if and only if they are consecutive.

- For an integer n, with $k \geq 3$, the *cycle* C_n is the graph whose vertex set is $\{1,\dots,n\}$ and the edge set is $\{\{i,i+1\} : 1 \leq i \leq n-1\} \cup \{\{1,n\}\}$.

- For each integer n, with $n \geq 4$, the graph Q_n is defined on $\{1,\dots,n\}$ by: $Q_n[\{1,\dots,n-2\}] = P_{n-2}$ and $E(Q_n) = E(P_{n-2}) \cup \{\{n-1,i\} : 1 \leq i \leq n-3\} \cup \{\{n-1,n\}\}$.

- For each integer n with $n \geq 2$, the graph G_{2n} is defined on $V = \{0,1,\dots,2n-1\}$ as follows. For $x \neq y \in V$, ($\{x,y\}$ is an edge of G_{2n}) if and only if ($\exists 0 \leq i \leq j \leq n-1 : \{x,y\} = \{2i,2j+1\}$).

- A graph G is *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, the pair $\{V_1,V_2\}$ is called a *bipartition* of the vertex set V of G. For example, for each integer n with $n \geq 2$, the graph G_{2n} is bipartite by the partition $\{\{2i : 0 \leq i \leq n-1\},\{2j+1 : 0 \leq j \leq n-1\}\}$.

- A *complete bipartite* graph is a graph that has its vertex set partitioned into two subsets with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

- With each graph G = (V,E), associate its complement $\overline{G} = (V,\overline{E})$ defined as follows: given $x \neq y \in V$, $\{x,y\} \in \overline{E}$ if $\{x,y\} \notin E$.

- Given a graph $G = (V, E)$, we define an equivalence relation C on V in the following way. For each $x \in V$, xCx, and for any $x \neq y \in V$, xCy if there are vertices $x = x_0,\dots,x_n = y$ such that $\{x_i, x_{i+1}\} \in E$ for $0 \leq i \leq n-1$. The equivalence classes of C are called the connected components of G. A vertex x of G is isolated if $\{x\}$ constitutes a *connected component* of G. The graph G is connected if $V = \varphi$ or V is the unique connected component of G.

- A tree is a connected acyclic (with no cycles) graph.

- Let G and H be two graphs.

- Let If G has no induced subgraph isomorphic to H, then we say that G is H-free. If G is not H-free, then G contains H, and a copy of H in G is an induced subgraph of G isomorphic to H.

- For a family *F* of graphs, we say that *G* is *F*-free if *G* is *F*-free for every $F \in F$.
- If the graph *G* is K_3-free, then *G* is called a triangle-free graph.
- A vertex subset of a graph is a stable set if its elements are pairwise nonadjacent.
- A vertex subset of a graph is a clique if its elements are pairwise adjacent.

2.2 Modules and their basic properties. Prime graphs

In this section, we review the well-known properties of the modules of a graph. Then we recall the Theorem of the partition induced by a prime proper subgraph.

2.2.1 Modules and their basic properties

Definition 2.1 Given *a graph* $G = (V, E)$, *a subset M of V is a module* [17] *(or an interval* [10] *or an autonomous* [11] *or a clan* [8] *or a homogeneous set* [7] *) of G provided that* $x \sim M$ *for each vertex x outside M.*

Proposition 2.1 *Let* $G = (V, E)$ *be a graph and let* M *be the set of modules of G.*

1. V, φ, and $\{x\}$, where $x \in V$, are modules of *G*, called trivial modules.
2. If $X, Y \in$ M, then $X \cap Y \in$ M.
3. If $X, Y \in$ M and if $X \cap Y \neq \varphi$, then $X \cup Y \in$ M.
4. If $X, Y \in$ M and if the set $X - Y$ is not empty, then $Y - X \in$ M.
5. If $W \subseteq V$ and if $X \in$ M, then $X \cap W$ is a module of the subgraph $G[W]$.

Definition 2.2 A graph G is indecomposable [16] if all the modules of G are trivial. Otherwise we say that G is decomposable.

Definition 2.3 A prime graph G is an indecomposable graph with at least four vertices.

Remark 2.1 • A graph G is prime if and only if its complement is prime. The connected components of a graph G are modules of G.

2.2.2 Existence of a small prime subgraph in a prime graph

By the following result, D. P. Sumner [18] shows that each prime graph has a small prime subgraph.

Proposition 2.2 *Given a prime graph* $G = (V, E)$, *there is a* 4-*vertex subset X such that* $G[X]$ *is prime.*

Thus any prime graph has an induced P_4. To construct larger prime subgraphs, we

consider the following definition.

Definition 2.4 *Given a graph* $G = (V, E)$ *and a proper subset X of V such that* $G[X]$ *is prime, we consider the following subsets of* $V - X$.

- $Ext(X)$ *is the family of the elements x of* $V - X$ *such that* $G[X \cup \{x\}]$ *is prime;*
- $\langle X \rangle$ *is the family of the elements x of* $V - X$ *such that X is a module of* $G[X \cup \{x\}]$;

- *For each $u \in X$, $X(u)$ is the family of the elements x of $V - X$ such that $\{u,x\}$ is a module of $G[X \cup \{x\}]$.*

The family of the nonempty elements of the set $\{Ext(X), \langle X \rangle\} \cup \{X(u);\ u \in X\}$ is denoted by p_X.

The following definition will be used only in our last chapter.

Definition 2.5 *Let $G = (V, E)$ be a graph and X be a proper subset X of V such that $G[X]$ is prime.*

1. The set $\langle X \rangle$ is divided into X^- and X^+ as follows.

- *X^- is the set of the elements x of $\langle X \rangle$ such that $x \ldots X$;*
- *X^+ is the set of the elements x of $\langle X \rangle$ such that $x \leftrightarrow X$ ".*

2. For each $u \in X$, $X(u)$ is divided into $X^-(u)$ and $X^+(u)$ as follows.

- *$X^-(u)$ is the set of the elements x of $X(u)$ such that $\{u,x\} \notin E$;*
- *$X^+(u)$ is the set of the elements x of $X(u)$ such that $\{u,x\} \in E$.*

3. The family of the nonempty elements of the set $\{Ext(X), X^-, X^+\} \cup \{X^-(u) : u \in X\} \cup \{X^+(u) : u \in X\}$ is denoted by q_X.
In [8], A. Ehrenfeucht and G. Rozenberg obtained the following theorem.

Theorem 1.1 *Given a graph $G = (V,E)$ and a proper subset X of V such that $G[X]$ is*

prime, the family p_X realizes a partition of $V - X$. Moreover, the following assertions are satisfied.

1. *Given $x \neq y \in Ext(X)$, if $G[X \cup \{x,y\}]$ is decomposable, then $\{x,y\}$ is a module of $G[X \cup \{x,y\}]$.*
2. *Given $x \in X(u)$ and $y \in V - (X \cup X(u))$, where $u \in X$, if $G[X \cup \{x,y\}]$ is decomposable, then $\{u,x\}$ is a module of $G[X \cup \{x,y\}]$.*
3. *Given $x \in \langle X \rangle$ and $y \in V - (X \cup \langle X \rangle)$, if $G[X \cup \{x,y\}]$ is decomposable, then $X \cup \{y\}$ is a module of $G[X \cup \{x,y\}]$.*
   In [8], using Theorem 1.1, A. Ehrenfeucht and G. Rozenberg obtained the following upward hereditary property.

Corollary 2.1 *Let $G = (V, E)$ be a graph and X be a subset of V such that $G[X]$ is prime. If G is prime and $|V - X| \geq 2$, then there exist $x \neq y \in V - X$ such that $G[X \cup \{x,y\}]$ is prime.* The following downward hereditary property is an immediate consequence of Proposition 1.2 and Corollary 1.1

Corollary 2.2 *Given an n-vertex prime graph G with $n \geq 5$, there is a vertex subset Y such that $|Y| \in \{n-1, n-2\}$ and $G[Y]$ is prime.* This corollary was improved by the following result of J.H. Schmerl and W.T. Trotter [16].

Theorem 2.2 *Given an n-vertex prime graph G with $n \geq 7$, there is an $(n-2)$-vertex subset Y such that $G[Y]$ is prime.*

For the proof of Theorem 1.2, J.H. Schmerl and W.T. Trotter introduced and studied the following concept.

Definition 2.6 *Let G be a prime graph. A vertex v of G is critical if the subgraph $G - v$ is decomposable. The graph G is critical prime if all its vertices are critical. The description obtained in [16] is the following.*

Theorem 2.3 *Up to isomorphism, the prime critical graphs are G_{2n} and $G_{\underline{2n}}$, where $n \geq 2$.*

2.2.3 Some examples of prime graphs

In this subsection, using Theorem 1.1, we give some examples of prime graphs.

Example 2.1 *For each integer k with $k \geq 4$, the path P_k is prime. We proceed by induction on $k \geq 4$. Since P_4 is prime, consider an integer $k \geq 4$ and assume that P_k is prime. So that the graph $P_{k+1}[X] = P_k$ is prime by induction hypothesis, where $X = \{1,2,\dots,k\}$. In addition, in P_{k+1}, $k + 1 \leftrightarrow k$ and $k + 1 \dots k - 1$. Therefore, $k + 1 \notin \langle X \rangle$. On the other hand, in P_{k+1}, we have $i \leftrightarrow (i-1)$ and $(k + 1)\dots(i-1)$ for each $i \in \{2,\dots,k\}$. It follows that $(k + 1) \notin X(2) \cup \dots \cup X(k)$. Since $1 \leftrightarrow 2$ and $(k + 1)\dots 2$, $(k + 1) \notin X(1)$. Thus $(k + 1) \notin X(u)$ for each $u \in X$. Therefore, Theorem 1.1 implies that $k + 1 \in Ext(X)$, and hence Pk+1 is prime because Pk+1 = Pk+1 $[X \cup \{k + 1\}]$.*

Example2.2 *For each integer k with $k \geq 5$, the cycle C_k is prime.*

Let $X = \{1,\dots,k\}$. Since $C_k[X] = P_{k-1}$, Example 1.1 implies that $C_k[X]$ is prime. Moreover, since $N_{Ck}\{k\} = \{1,k - 1\}$, it is easy to verify that $k \notin \langle X \rangle$ and $k \notin X(u)$ for each $u \in X$. Thus Theorem 1.1 implies that $k+1 \in Ext(X)$, and hence C_k is prime because $C_k= C_k[X \cup \{k\}]$.

Example 2.3 *For each integer k with $k \geq 4$, the graph Q_k is prime.*

Let $X = \{2,3,4,5\}$. The subgraph $Q_5[X]$ is prime because it is isomorphic to P_4.

Moreover, since $1 \leftrightarrow 2$ and $1\dots 3$, $1 \notin \langle X \rangle$. Since $1 \leftrightarrow 4$ and $3\dots 4$, $1 \notin X(3)$. Furthermore,

$1 \notin X(2)$ *because* $1\dots 3$ *and* $2 \leftrightarrow 3$. *By interchanging 3 (resp. 2) and 5 (resp. 4), we obtain* $1 \notin X(4) \cup X(5)$. *Consequently,* $1 \notin \langle X \rangle \cup X(2) \cup \dots \cup X(5)$. *Thus Theorem 1.1 implies that* $1 \in Ext(X)$, *and hence* Q_5 *is prime because* $Q_5 = Q_5[X \cup \{1\}]$. *We may assume that* $k \geq 6$ *in such a way that* $Q_k[X] = P_{k-2}$ *is prime, where* $X = \{1,\dots,k-2\}$. *Since* $i\dots k$ *for* $i \in X$, $k \in \langle X \rangle$. *On the other hand, since* $(k - 1) \leftrightarrow 1$ *and* $(k - 1)\dots(k - 2)$, $(k-1) \notin \langle X \rangle$. Since $(k - 1) \leftrightarrow k$ and $k\dots X$, $k \nleftrightarrow (X \cup \{k-1\})$, and hence $X \cup \{k - 1\}$ is not a module of $Qk[X \cup \{k - 1,k\}]$. Thus Theorem 1.1 implies that Qk is prime because $Qk= Qk[X \cup \{k - 1,k\}]$.

3.{P_5, $\overline{P5}$, *BULL*}-FREE GRAPHS

In this section, using the basic results on prime graphs, which are recalled in Chapter

1, and Theorems 1.2 and 1.3, we give a new proof of the following result of J.L. Fouquet [9].

Theorem 2.1 *If G is a $\{P_5,\overline{P_5},bull\}$-free graph, then exactly one of the following assertions holds.*

(v) $|V(G)| \leq 2$

(vi) *G is isomorphic to C_5.*

(vii) *There is $n \geq 2$ such that G is isomorphic to $\overline{G_{2n}}$ or G_{2n}.*

(viii) *G is decomposable.*

This result was obtained by J.L. Fouquet in the following paper:

"J.L. Fouquet, A decomposition for a class of (P_5,P_5)- free graphs, Discrete Mathematics 121, (1993) 75-83 "

Notice that, in [5], using a recent result of M. Chudnovsky and P. Seymour [6], M.

Chudnovsky and P. Maceli gave a shorter proof of this result. First, we obtain the following new Lemma.

Lemma 2.1 *Let $G = (V,E)$ be a $\{P_5,\overline{P_5}\}$-free graph, and X be a vertex subset such that $G[X]$ is isomorphic to C_5. The following assertion holds*

(i) *$Ext(X)$ is empty.*

(ii) *For any distinct vertices u and v outside X, the subgraph $G[X \cup \{u,v\}]$ is decom- posable.*

Proof. (i) Let $X = \{v_1,v_2,v_3,v_4,v_5\}$ such that $(v_1,v_2,v_3,v_4,v_5,v_1)$ is an induced C_5 of G.

On the contrary, suppose that there is a vertex v_6 in $Ext(X)$, and hence, the subgraph $H = G[X \cup \{v_6\}]$ is prime. Considering the complement of G, we may assume that $d_H(v_6) \leq 2$. Since $H$is prime, $d_H(v_6) \neq 0$, and hence, $d_H(v_6) \in \{1,2\}$.

First, assume that $d_H(v_6) = 1$. We may assume that $N_H(v_6) = \{v_1\}$, and hence, (v_6,v_1,v_2,v_3,v_4) is an induced P_5 of G; which contradicts the fact G is a P_5−free graph.

Second, assume that $d_H(v_6) = 2$. Since $v_6 \notin \cup_{1\leq i\leq 5} X(v_i)$, $N_H(v_6)$ is a pair of two consecutive vertices in the cycle $(v_1,v_2,v_3,v_4,v_5,v_1)$. Thus, we may assume that $N_H(v_6)$ =

$\{v_1,v_2\}$, and hence (v_6,v_2,v_3,v_4,v_5) is an induced P_5 of G; which contradicts the fact that G is a P_5-free graph.

(ii) To the contrary, suppose that there are $u \neq v \in V \setminus X$ such that $G[X\cup\{u,v\}]$ is prime. By the first assertion, $Ext(X)$ is empty. If $\{u,v\} \subseteq \langle X\rangle$, then X is a non-trivial module of $G[X \cup \{u,v\}]$; which contradicts the fact that $G[X \cup \{u,v\}]$ is prime. Moreover if there is $x \in X$ such that $\{u,v\} \subseteq \cup X(u)$, then $\{u,v,x\}$ is a non-trivial module of $G[X \cup \{u,v\}]$; which contradicts the fact that $G[X \cup\{u,v\}]$ is prime. Therefore, by Theorem 1.1, we have to consider only the following three cases below.

First, assume that there are $v_i \neq v_j \in X$ such that $u \in X(v_i)$ and $v \in X(v_j)$. We may assume that either $(i,j) = (1,2)$ or $(i,j) = (1,3)$. In the first case, Theorem 1.1 implies that $uv \notin E(G)$, and hence (v,v_3,v_4,v_5,u) is an induced P_5 of G; which contradicts the fact that G is a P_5-free graph. In the second one, Theorem 1.1 implies that $UV\in E(G)$, and hence G[{u,v,v2,v4,v5}] is isomorphic to $\overline{P5}$; which contradicts our assumption.

Second, assume that here is $v_i \in X$ such that $u \in \langle X\rangle$ and $\overline{v} \in X(v_i)$. We may assume that $u \in \langle X\rangle$ and $v \in X(v_1)$. Thus Theorem 1.1 implies that $uv\in E(G)$, and hence (u,v,v_2,v_3,v_4) is an induced P_5 of G; which contradicts our assumption.

Finally, assume that here is $v_i \in X$ such that $u \in \langle X\rangle+$ and $v \in X(v_i)$. We may assume that $u \in \langle X\rangle^+$ and $v \in X(v_1)$. Thus Theorem 1.1 implies that $uv \notin E(G)$, and hence $G[\{u,v,v_5,v_4,v_2\}]$ is isomorphic to P_5; which contradicts our assumption. ■

Second, using Lemma 2.1, we deduce the following result of J.L. Fouquet [9].

Corollary 2.1 *If G is a prime $\{P_5,\overline{P5}\}$-free graph which contains C_5, then G is isomorphic to C_5.*

Proof. Let X be a vertex subset such that $G[X]$ is isomorphic to C_5. By the second assertion of Lemma 2.1, there is no 7-element vertex subset Y including X such that $G[Y]$ is prime. Therefore, Corollary 1.1 implies that $|V(G)| \leq 6$. Clearly, the first assertion of Lemma 2.1 implies that $|V(G)| \neq 6$, and hence $|V(G)| = 5$, and G is isomorphic to C_5.

Lemma 2.2 *If G is a 5-vertex prime graph, then G is isomorphic to one element of $\{P_5,P_5,C_5,Q_5\}$*

Proof. Let G be a 5-vertex prime graph. By Proposition 1.2, there is a vertex subset X such that $G[X]$ is isomorphic to P_4. Set $V(G) = \{v_1,v_2,v_3,v_4,v_5\}$, where $X = \{v_1,v_2,v_3,v_4\}$, and (v_1,v_2,v_3,v_4) is an induced P_4 of G. Considering the complement of G, we may assume that $d_G(v_5) \in \{1,2\}$. Since $v_5 \notin (X(v_1) \cup X(v_2) \cup X(v_3) \cup X(v_4)$, we may assume that

$N_G(v_5) \in \{\{v_1\},\{v_1,v_4\},\{v_2,v_3\}\}$

- If $N_G(v_5) = \{v_1\}$, then (v_5,v_1,v_2,v_3,v_{45}) is an included P_5 of G, and hence $G \cong P_5$.
- If $N_G(v_5) = \{v_1,v_4\}$, then $(v_5,v_1,v_2,v_3,v_4,v_5)$ is an included $C_5$of G, and hence $G \cong C_5$.
- If $N_G(v_5) = \{v_2,v_3\}$,then $G \cong Q_5$. ■
- The following lemma is well known.

Lemma 2.3 *A prime graph G is a critical prime if and only if it has no prime* 5-*vertex subgraph.*

Proof. First, consider an n-vertex critical prime graph $G = (V,E)$, and to the contrary, suppose that there is a 5-element vertex subset X such that $G[X]$ is prime. By Proposition

1.2, there is a vertex subset Y such that $G[Y]$ is isomorphic to P_4, and hence $G[Y]$ is prime. If n is even (resp. odd), then by applying several times Corollary 1.1, we obtain an $(n-1)$ element vertex subset Z including X (resp. including Y) such that the subgraph $G[Z]$ is prime; which contradicts the fact that G is a critical prime graph.

Second, consider an n-vertex prime graph G with no prime 5-vertex subgraph.

If $n = 4$, then G is critical prime because all the 3-vertex graphs are decomposable.

Thus in the sequel, we assume that $n \geq 5$. Since G has no prime 5-vertex subgraph, $n \geq 6$.

If $n = 6$, then $G-u$ is decomposable for each vertex u, and hence G is critical prime. Thus, we may assume that $n \geq 7$. By Theorem 1.2, there is a prime $(n-2)$-vertex subgraph, and hence $n \geq 8$.

If n is odd, then by applying several times Theorem 1.2, we obtain a prime 5-vertex subgraph of G; which contradicts our assumption. Therefore, n is even. To the contrary, suppose that G is not critical prime. Thus, there is a vertex v such that $G - v$ is prime. Since $|V - v| = n - 1 \geq 7$ and $(n - 1)$ is odd, by applying several times Theorem 1.2, we obtain a prime 5-vertex subgraph of $G-v$; which contradicts the fact that G has no prime 5-vertex subgraph. ■ The following corollary is an immediate consequence of Lemmas 2.2 and 2.3.

Corollary 2.2 *If G is a prime* $\{P_5, \overline{P5}, C_5,$ *bull*$\}$*-free graph, then the graph G is prime critical.*

Remark 2.1 *Theorem 2.1 is an easy consequence of Corollary 2.1, Corollary 2.2, and the description of the prime critical graphs given by Theorem 1.3. On the other hand, in* [5], *using a recent result of M. Chudnovsky and P. Seymour* [6], *which is not recalled in our document, M. Chudnovsky and P. Maceli gave another proof of Theorem 2.1.*

Bibliography

1. A. Ehrenfeucht, T. Harju and G. Rozenberg, *The Theory of 2-Structures. A Framework for Decomposition and Transformation of Graphs*, World Scientific, Singapore (1999).
2. I. Boudabbous and P. Ille, *Critical and infinite directed graphs*, Discrete Math. 307 (2007) 2415-2428.
3. Y. Boudabbous and P. Ille, Indecomposability graphs and critical vertices of an indecomposable graph, Discrete Math. 309 (2009) 2839-2846.

4. A. Brandst¨adt and D. Kratsch, *On the structure of (P_5, gem)-free graphs*, Discrete Applied Mathematics 145, (2005) 155-166.

5. M. Chudnovsky and P. Maceli, *Simplicial vertices in graphs with no induced four-edge path or four-edge antipath, and the H6-conjecture*, Journal of Graph Theory, Vol 76, Issue 4, (2014) 249261.

6. M. Chudnovsky and P. Seymour, *Growing without cloning*, SIAM. Discrete Math., vol. 26, No. 2 (2012) 860-880.

7. A. Courier and M. Habib, *An efficient algorithm to recognize prime undirected graphs*, in: Graph-Theoretic Concepts in Computer Science, Proceedings of the 18th Internat. The workshop, WG'92, Wiesbaden-Naurod, Germany, June 1992, in: E. W. Mayr(Ed.), Lecture Notes in Computer Science, Vol. 657, Springer, Berlin pp. 212-224 (1993).

8. A. Ehrenfeucht and G. Rozenberg, Primitivity is hereditary for 2-structures, fundamental study, Theoret. Comput. Sci. 3 (70), 343-358 (1990).

9. J.L. Fouquet, *Adecompostion for a class of* (P_5,P_5)*-free graphs*, Discrete Mathematics 121 (1993) 75-83.

10. R. Fra¨ıss´e, *L'intervalleenth´eorie des relations, sesg´en´eralisations, filtreintervallaire et clˆotured'une relation*, in: M. Pouzet and D. Richard eds., Order, Description and Roles, North-Holland, Amsterdam, (1984) 313- 342.

11. T. Gallai, *TransitivorientierbareGraphen*, Acta. Math. Acad. Sci. Hungar., 18 (1967) 25-66.

12. P. Ille, *Indecomposable graphs*, Discrete Math. 173 (1997) 71-78.

13. P. Ille, *A new proof of the main theorem of the paper: P. Ille, Indecomposable graphs, Discrete Math. 173 (1997) 71-78*, Homepage of Pierre Ille.

14. P. Ille, *A characterization of the Indecomposable and Infinite Graphs*, Glob. J. Pure Appl. Math. volume 1, Number 3, 272-285 (2005).

15. M. Pouzet and I. Zaguia, *On minimal prime graphs and posets*, Order Vol. 26 No. 4, (2009) 357-375.

16. J.H. Schmerl and W.T. Trotter, *Critically indecomposable ordered sets, graphs, tournaments and other binary relation structures*, Discrete Math. 113, 191-205 (1993).

17. J. Spinrad, *P4-trees and substitution decomposition*, Discrete Appl. Math. 39 (1992) 263-291.

18. D.P. Sumner, *Graphs indecomposable with respect to X-join*, Discrete Math. 6, (1973) 281-298