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The Trisection of an Arbitrary Angle: A Condensed Classical Geometric Solution

Arthur Clair Rediske

General Electric Company: Retired, Olympia, Washington, USA

art1rediske@gmail.com

Abstract

This paper presents a short version of an elegant geometric solution of angle trisection that was published by this author on 2018-04-30 in Volume: 14 Issue: 02 of the Journal of Advances in Mathematics.

The style of writing for the above paper was based on how teaching geometry was taught in high schools from 1940 to 1942. Proofs of a problem consisted of a statement that was followed by a valid reason why the statement was made. If the proof was many lines in length, the teacher wanted the students to show each step. The students were not allowed to skip a step or steps to reach the final line of the proof.

This short version was generated when a copy of the above paper was reviewed by a retired school teacher, who suggested the proof of the trisection of an arbitrary angle could be shortened.

The exposed methods of proof have not changed from the Euclidean postulates of classical geometry.

Keywords: Angle Trisection; An Arbitrary Angle; An Angle; Compass; Unmarked Straightedge; Classical Geometry.

Introduction

Angle trisection is one of the three classic problem of compass and straightedge constructions of ancient Greek mathematics [1]. It concerns construction of an angle equal to one-third of a given arbitrary angle, using only two tools: an unmarked straightedge and an unmarked compass. The problem is stated to be generally impossible to solve, as shown by (Pierre Laurent Wantzel, 1837) [2]. The generic definition of the angle trisection problem is based on the assumption that: the trisection was to be exact according to plane geometry rules [3] and [4]. The rules stated that a point had no dimension, a line had no width. When a line intersected another line a point was created. A compass could be used to measure a distance and then be used to create another distance equal to the first distance. A compass could be used to draw an arc or a circle. The straight edge could have no marks or indents. Quantum analysis of the angle trisection proof of impossibility expose that the proof pose a serious geometrical misconception. The proof does not take into account, and the above stated governing conditions in the sense that it is purely algebraic. It assumes that, for one to trisect an angle, the size of the angle has to be known. This is a very wrong perception. Euclidean geometry is typically classical. No measurement and arithmetic is allowed. It is based on logical reasoning, and the governing proofs are purely geometrical. Through the year 2017, several researchers; [5], and [6], have published a high profile scientific refutes against the angle trisection impossibility statement. [9] and [14] provides a clear description of Euclidean geometry, interpretation of the governing rules, and the differences between Euclidean geometry and Solid geometry, together with the other types of geometry, well-constructed. A clear account of Euclidean and non-Euclidean geometry is found in the treatise; *Euclid's elements* translated and interpreted by different mathematics generations into various fashions as in [11]. This paper corresponds to this consideration, in presenting an alternative classical geometric solution for the stated old age problem of Greeks' mathematics. It is equally considered that, as demonstrated in [1], [2], [13], the angle trisection impossibility proof is not geometrical, but, algebraic, an approach completely prohibited in classical geometric constructions. The provided proofs is virtuously classical based on proportion of some quantities such as angles and triangles.

Those commensurate imply quantities of similar magnitude, otherwise, no equality. This approach is in harmony with the consideration that all classical geometrical problems should be sought geometrically [10]. In his *Book III of La Géométrie* [10], Descartes gave an account on the roots of cubic and quartic equations. He considers polynomials with integer coefficients. It is stated that, if there is an integer root, that gives a numerical solution to the problem, but, if there are no integral roots, the solutions must be constructed geometrically. Sections 1.1 and 1.2 discuss the geometrical foundations fundamentally followed in this work, and characterization of the methodology, respectively.

Materials and Methods

This section presents the proposed methodology. The presented contents depend upon the discussion in sections 1.1 through 1.5. It is reflected that if an arc were drawn, using the apex of an unknown angle opening, to intersect the two rays forming the angle, the arc could be divided into four equal parts; then, one part of the four could be subtracted, leaving three congruent parts. If the three congruent parts could be moved closer to an exact position relative to the apex of the angle, three congruent triangles could be created, and therefore, the original unknown angle would be divided into three equal parts.

1.1 Geometry Foundations

The provided constructions is theoretically established based on the following geometrical facts:

1. Mirror Image-each side of a fiducial or reflector line is the reflection of the other side.
2. Corresponding line segments and angles of the mirror image are congruent.
3. Reflexive property - any segment or angle is congruent to itself.
4. A quadrilateral has four straight sides.
5. A parallelogram is a quadrilateral that has two pairs of parallel sides.
6. One pair of opposite sides are parallel and equal in length.
7. If a traverse intersects two parallel lines, alternate interior angles are congruent.

1.2 Notations

1. $\angle$ Angle notation
2. $\overline{AB}$ Used to denote a straight line segment and a length
3. $\widehat{DE}$ and $\widehat{a-a}$ Used to denote a curve
4. $\cong$ Congruent
5. Δ Used to denote a triangle
6. $\parallel$ Used to denote parallelism

1.3 Characterizations

The elementary proof of the proposed methodology is based on some geometrical quantities such as triangles and other quadrilaterals characterized by the following properties. This depicts the classical geometric validity of the methods in accordance to Euclidean ways of proofing geometrical analogies. It is considered that a simple (non-intersecting) quadrilateral is a parallelogram if and only if any one of the following statements is true:

1. Two pairs of opposite sides are equal in length.

2. Two pairs of opposite angles are equal in measure.
3. The diagonals bisect each other.
4. One pair of opposite sides is parallel and equal in length.
5. Adjacent angles are supplementary.
6. Each diagonal divides the quadrilateral into two congruent triangles.
7. The sum of the squares of the sides equals the sum of the squares of the diagonals. (This is the parallelogram law.)

1.4 Important Note on Rectangle Properties

Among the elemental objects to be produced in the construction is a rectangle. For justification, a rectangle must be a parallelogram having the following properties, which will help in the results verification:

1. A pair of sides both parallel and congruent.
2. Contains at least one right angle.
3. Diagonals are congruent and bisect each other.

2.0 Hypothesis

The logic behind the proposed angle trisection solution has the foundation based on: "*I was listening to a course on calculus, and one statement stuck with me. The teacher said*, 'Don't forget to subtract the starting point.' *That statement provided the key to complete my solution of how to trisect an unknown angle using only a straight edge, a compass, and following all the rules of geometry*." The solution takes an angle of unknown degrees, but no greater than 90 degrees, splits it into four congruent angles inside the original angle, then removes one of the smaller angles, leaving the remaining three congruent angles. Next, the three congruent angles have three congruent bases attached to them to form three congruent isosceles triangles, as illustrated from section 3.0. Then, as the base of each triangle, whose lengths do not change, is moved closer to the apex of the main angle, this shortens each side of all three triangles and widens the apex angle of each. There is only one place where the following properties are met:

1. The endpoints of the bases of these three triangles touch each other from left to right.
2. The left-most and right-most bases intercept the two sides of the original unknown angle.
3. Four lines form the sides of the three congruent isosceles triangles.
4. The apexes of these three triangles all touch the apex of the original unknown angle.
5. The outer lines of the left- and right-most triangles are the same as the lines of the original unknown angle.

In other words, this repositioning results in an increase in the apex angle of each congruent isosceles triangle. It can now be proven that these three isosceles triangles are congruent-i.e. they are identical in size-thereby proving the mathematical puzzle of trisecting an angle following the rules of the puzzle.

Results and Discussion

A. Results

3.1 Author's "Solution" to Trisection of an Angle

The solution overview is, as discussed in section 2.0. However, for angles over 90 degrees, one has to divide them in half and then trisect one of the halves. Two of these added together will be equal to one of the angles in the original larger angle.

3.1.1 Trisection of Angles Less than 90° (An Application of Art's "Solution")

The solution is examined in the trisection of an angle less than 90°. Consider the following construction steps and the generated results:

1. Draw arbitrary angle. Proof can be for any angle (see Figure 1).

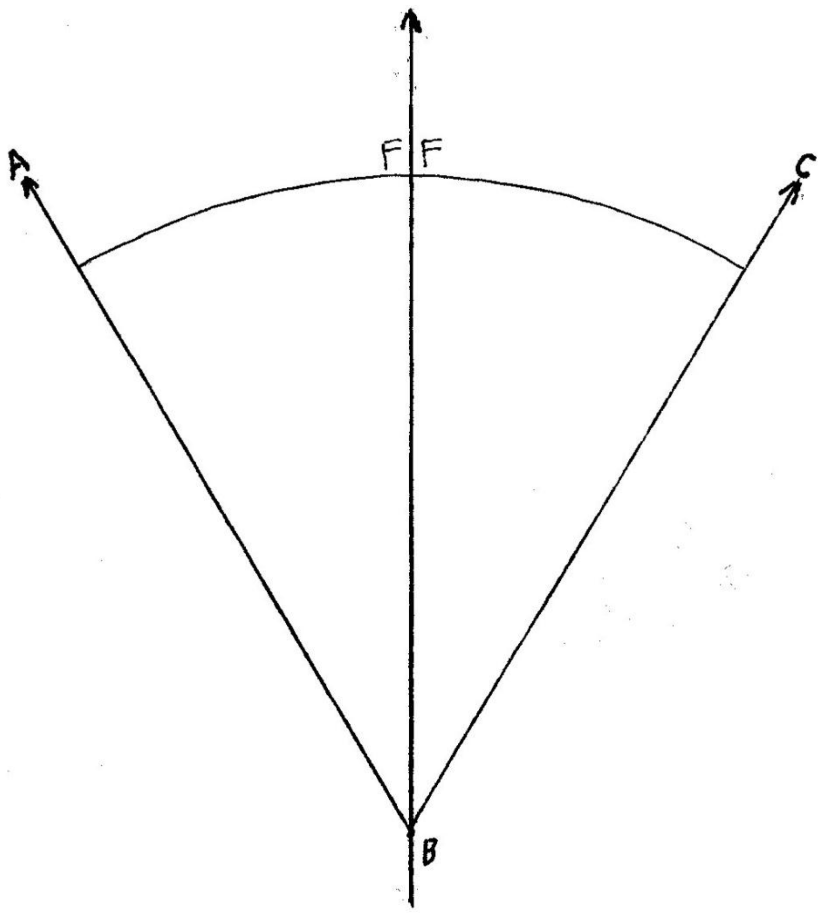

Figure 1: Arbitrary angle ABC

2. Draw rays BA and BC (see Figure 2).

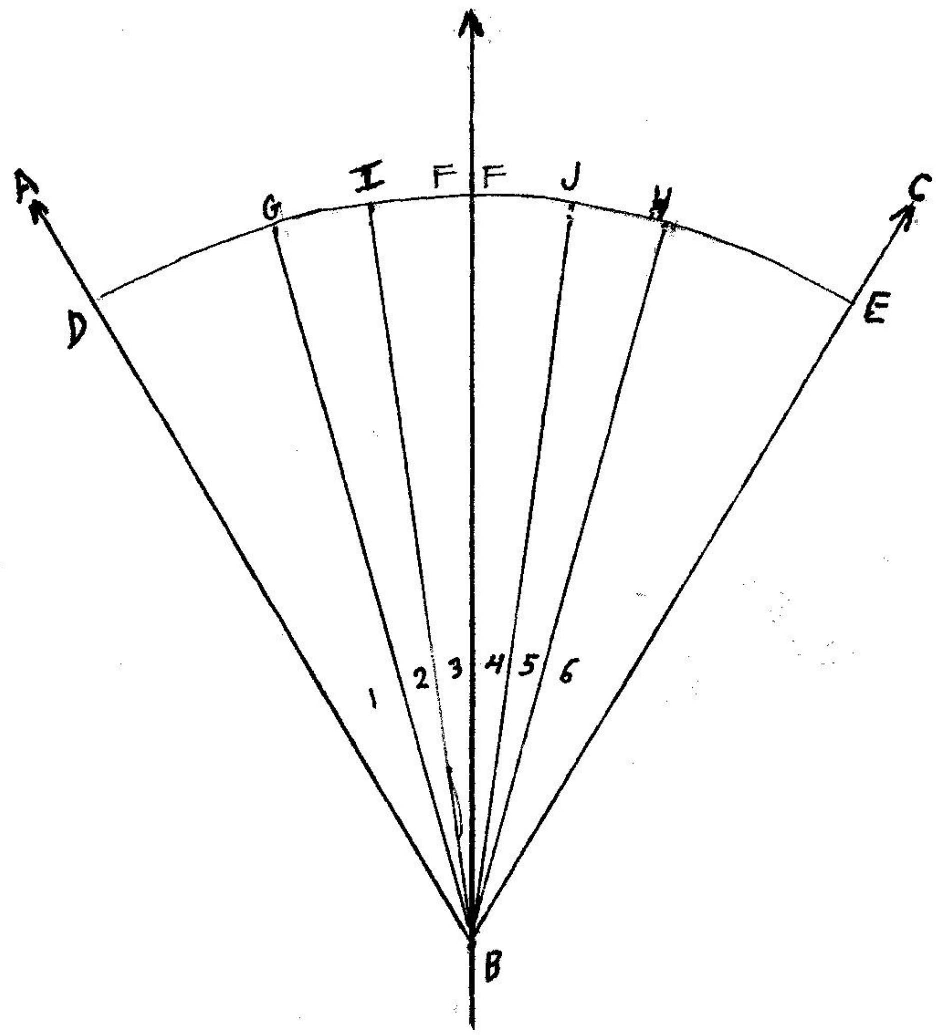

Figure 2: Bisecting angle ABC

3. Bisect ∠ABC to create four ≅ angles.

4. Bisect two central angles.

5. Number angles left to right, 1, 2, 3, 4, 5, and 6.

6. Draw $\widehat{DE}$ with center B and point D on $\overline{AB}$ and point E on $\overline{BC}$.

7. Draw fiducial line from B to the center of the $\widehat{DE}$ at point FF (see Figure 3).

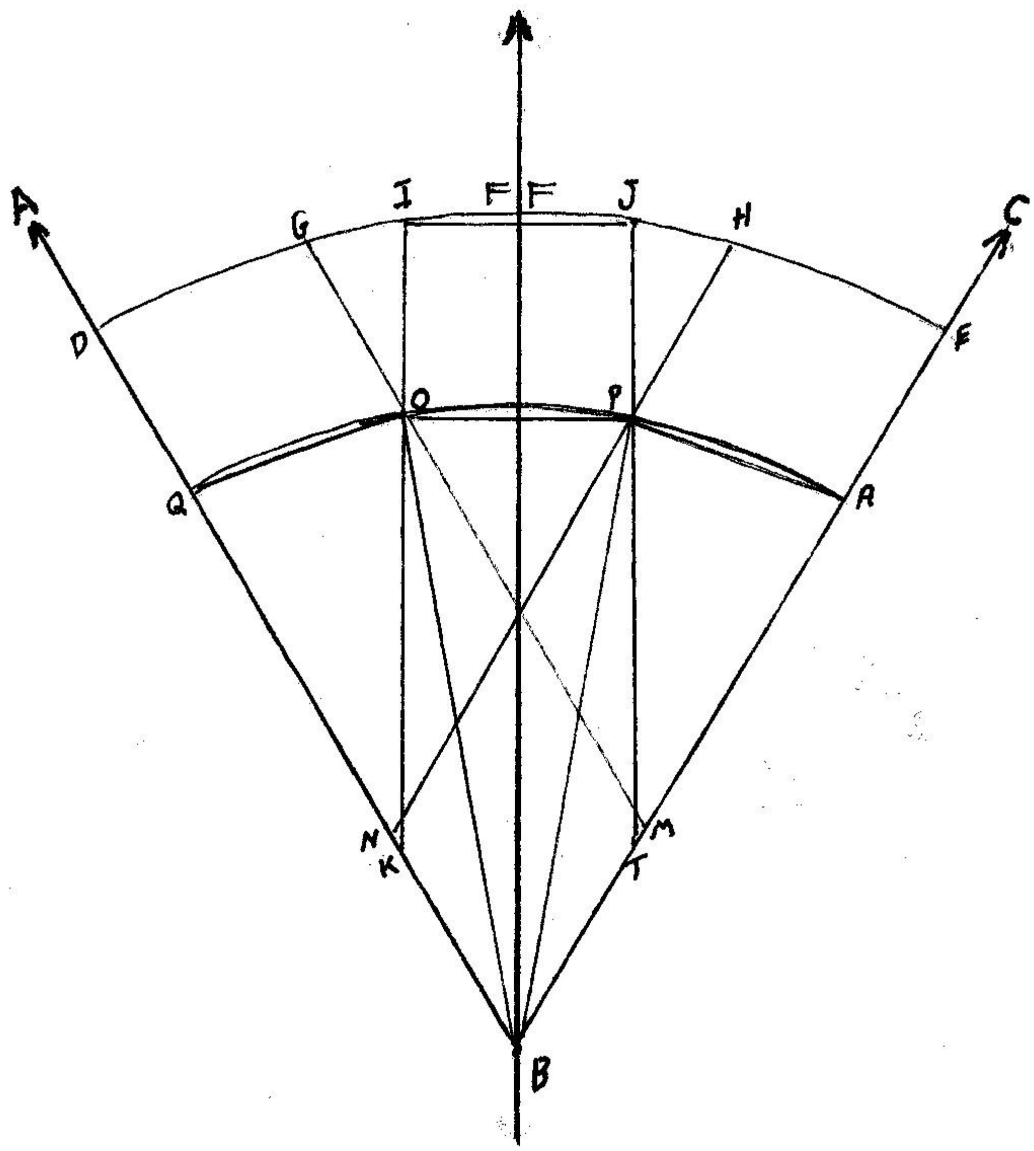

Figure 3: Constructing lines I-K and J-T parallel to fiducial mirror lines FF-B

At point, G construct a line ∥ to $\overline{DB}$, to intersect $\overline{BE}$ at point M.

8. At point, H construct a line ∥ to $\overline{BE}$, to intersect $\overline{BA}$ at point N.

9. Draw chord I-J.

10. Construct $\overline{IK}$ and $\overline{JT}$ parallel to fiducial $\overline{FFB}$.

11. At point I erect a perpendicular to $\overline{IJ}$ to intersect $\overline{BD}$ at point K.

12. At point, J erect a perpendicular to $\overline{IJ}$ to intersect $\overline{BE}$ at point T.

13. Where $\overline{IK}$ crosses $\overline{GM}$, point O is created.

14. Open compass to $\overline{BO}$. Draw $\widehat{QOPR}$.

15. Draw chords O-P, Q-O, and P-R.

16. $\overline{OP}$ is 1/3 of $\widehat{GR}$. Therefore, $\overline{QO}$ and $\overline{PR}$ are each 1/3 of $\widehat{QR}$.

17. Fiducial $\overline{BFF}$ makes $\overline{QO}$ and $\overline{PR}$ $\cong$.

18. Therefore, the three $\cong$ triangles have $\cong$ apex angles.

19. Therefore, the original ∠ABC is trisected.

B. Discussion

Throughout the presented methodology, it has been shown that the angle trisection problem is solvable using the proposed method of construction. The representation of results in this work is from figure (1) to figure (3). Figures (1) through (3) present the obtained construction results, based on Euclidean rigor of classical geometry. All the work constructions are performed classically, using compass-straightedge techniques as set in [11]. Figure (1) presents the basic geometric construction, involving definition of quantities such as angles, and the characteristic properties. Figures (2) and (3) exhibit the application of the generic Euclid postulates, implying the practical aspect of the method. The very significant consideration employed in this work is the logic that the problem of trisecting an arbitrary angle could be solved if a particular algorithm resolves the trisection of two known angles, without altering the construction principal. The classical geometric trisection of any angle greater than 90° in magnitude, as stated earlier, could be performed by dividing the angle in half and then trisecting one of the halves using the proposed approach. Two of those angles added together will be equal to one of the angles in the original large angle.

Conclusions

This paper has presented a multistep classical geometric algorithm for solving the ancient Greek's problem of angle trisection. The offered method shows that, in fact, angle trisection is possible, and has shown that this mathematical problem is geometrically solvable, by providing an elegant solution governed by the classical rules of Euclidean geometry [10]. The involved analogs concern a significant look at elemental geometric quantities (elemental lengths, curves, and geometric figures), and their geometrical relationship on a plane. The obtained results justify that the angle trisection impossibility statement, together with the presented non-Euclidean solutions [11], [12], [13], has no geometric rationality. This paper, therefore, concludes that the angle trisection impossibility statement is not geometrically valid and that the angle trisection problem is resolvable following the provided method.

Acknowledgments

I wish to again acknowledge my high school algebra teacher, Miss Sloan (Yakima High School, Yakima, Washington, 1941), for her help in making me understand the basic mathematical concepts. I also wish to thank Carma Budsberg, who, after reading my first proof, said: "I am a firm believer." Carma Budsberg holds a Masters degree in mathematics and taught various branches of mathematics for twenty-four years. She acknowledges that my proof is based not on numbers but on proportions. I also appreciate the support of my blog master, Maitri Sojourner, for making my solution accessible to a global audience, and for the final editing of this paper. I wish to again thank Mr. M. Alex Kimuya for his assistance in re-stating the classic mathematical puzzle and P.L. Wantzel's 'impossibility' claim; and for Mr. Kimuya's explanation that Wantzel's claim is algebraically-based, while my proof is based strictly on Euclidean geometry.

References

1. C. B. Boyer. 1991. *A History of Mathematics, 2nd Ed.*, John Wiley & Sons, Inc, (1991), p.64.

2. P.L. Wantzel. 1837. Recherches sur les moyens de reconnaitre si un problème de géométrie peut se résoudre avec la règle et le compass. Journal de Mathematiques pures et appliques, (1837), Vol. 2, pp.366-372.

3. Kimuya .M. Alex, Josephine Mutembei, The Cube Duplication Solution (A Compass-straightedge (Ruler) Construction), International Journal of Mathematics Trends and Technology (IJMTT) – Volume 50 Number (5 October 2017).

4. University of Toronto. 1997. The Three Impossible Constructions of Geometry, <mathnet@math.toronto.edu>, 14 Aug, (1997).

5. Kimuya .M. Alex, Josephine Mutembei, The Angle Trisection Solution (A Compass-Straightedge (Ruler) Construction), Journal of Advances in Mathematics, (2017).

6. Kimuya, M. Alex. 2017 The Possibility of Angle trisection (A Compass Straightedge Construction), Journal of Mathematics and System Science, Vol.7, (January 25, 2017), P. 25-42. doi: 10.17265/2159-5291/2017.01.003.

7. W.S. Anglin. 1994. Mathematics: A Concise History and Philosophy, Springer-Verlag, New York, (1994), pp. 75-80.

8. Rediske, Arthur Clair. The Trisection of an Arbitrary Angle. Journal of Advances in Mathematics. Vol. 14, No. 2 (May 31, 2018).

9. J. Castellanos. 1994-2007. What is Non-Euclidean Geometry?, <https://www.cs.unm.edu/~joel/NonEuclid/noneuclidean.html>, (1994)-(2007).

6. Burton, M. David. 1999. *A History of Mathematics: An Introduction, 4th Ed.*, McGraw-Hill, p. 116, (1999).

10. R. Descartes. 1664. *La Géométrie*, C. Angot, Paris, (1664) (first published in 1637).

11. R. Fitzpatrick. 1883–1885. Euclid's Elements of Geometry, from Euclidis Elementa, edidit et Latine interpretatus est I.L. Heiberg, in aedibus, B.G. Teubneri.

12. H. Florentino Ĺatortue. 2017. The Solution to the Impossible Problem, Library of Congress Control Number: 2017905297, NY, Printed in the United States of America, (2017).

13. C.A.Hart, and D. D. Feldman. 1912, 2013. Plane and Solid Geometry, American Book Company, Chicago, (1912), (2013), p.341-342.

14. Kimuya, M. Alex. 2017. Personal communication.

Preparation of Figures

All figures of the proof itself (Figures 1-3) were originally hand-drawn by the author, using only a straight-edge and a compass. Per the rules of the mathematical puzzle, no marks on the straight-edge were used, or were any degree marks or marks of any kind used on the compass to make the drawings of the proof itself (Figures 1-3). The drawings were then scanned to create PDF files for inclusion in the manuscript. The PDF source files for the figures are available upon request. Figure 4 in Annex 2 was taken from the internet (see bibliography [14]).

Supplementary Materials

ANNEX 1: The Mistake Made in Proving the Angle Trisection Impossibility Statement (Wantzel, 1837)

There is much to be found on the internet, in university libraries, and other resources on the angle trisection problem and its solutions. Various authors have taken different approaches to the problem. They have all exhibited degrees of misconception in their attempts to solve the problem. Among the most influential accounts on the angle trisection problem is P.L.Wantzel's proof of the "impossibility," which states, "the trisection of an angle corresponds to resolving a certain cubic equation, whose solution cannot be sought geometrically." In his proof, Wantzel used ideas from abstract algebra in an attempt to demonstrate that the problem cannot be solved, as the governing cubic equation cannot be factorized geometrically. This paper is aimed at refuting P.L Wantzel's claim by exposing a purely classical construction of trisecting any angle. As in the case of angle trisection, consider the following relation (sourced from [3], showing that no algebraic proof is correct in justifying a geometric impossibility, and so is the angle trisection proof of impossibility (see Figure 4):

Assumed that √2 is irrational as it is algebraically believed. Therefore, suppose that √2 is rational, then, √2 can be written as

$\sqrt{2} = a/b$, where both a and b are whole numbers with no common factor. It follows that;

$2 = (a/b)$^2

Therefore, $2(b^2) = (a^2)$

Clearly, following the algebraic fact that;

a. The product of two odd numbers is odd

b. The product between an odd number and an even number is even

c. The product between two even numbers is even

It can be deduced that if both a and b have no common factor, then the relation $2(b^2) = (a^2)$ poses a contradiction, since the product $2(b$^2) is even, because of the number 2. This implies that a itself must be even. Therefore, it is not possible to construct the factor √2, which is classified as irrational.

ANNEX 2: Geometric construct of √2

From Figure (4), it is evident that the factor √2 is geometrically constructible. The fact that the value of √2 cannot be algebraically determined, the geometric interpretation of this case is that the factor √2 is a multiplicative factor, which propagate along with any such two-dimensional figure of the kind shown in Figure (4). Thus, in this case, the impossibility proof presented in section 1.4 is false in the sense that, it provides proof to a statement, and not a construction. This is in harmony with Euclid's geometrical proof of the Pythagoras theorem governed by the equation $c^2 = a^2 + b^2$, that it is geometrically possible to construct the diagonals of a square, and that it is possible to double the content (area) of a square. The algebraic consideration of this case remain an impossibility, which is not geometrically justified. One can, therefore, make a similar conclusion to the angle trisection impossibility, which is established based on algebra, that it is not geometrically valid to term the angle trisection problem as impossible without a geometric proof of the claim. Figure (4) show the construction of the factor √2.

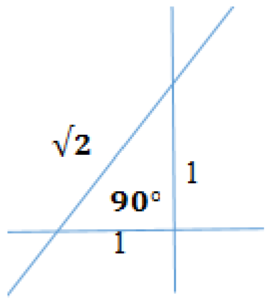

Figure 4: Geometrical construction of the irrational number √2 [14]

ANNEX 3: Classical Geometric Reasons Governing the Construction

1. Two rays intersect to form an angle.
2. (Segment XA used later in proof.)
3. An arc may be drawn with a compass set to a known or unknown opening with the sharp point on a given point, and the arc length may or may not be given.
4. Mirror image – Each side of the fiducial line is the reflection of the other side.
5. Corresponding line segments and angles of the mirror image are congruent.
6. A mirror image is of the same size as the original object.
7. Reflexive Property – Any segment or angle is congruent to itself.
8. A quadrilateral has four straight sides.
9. A parallelogram has two pairs of parallel sides; opposite sides are congruent.
10. A parallelogram has four sides, if one pair of opposite sides are congruent, then the other pair of opposite sides are congruent.
11. A rectangle has all the attributes of a parallelogram, and at least one angle must be a right angle.
12. If a traverse intersects two parallel lines, alternate interior angles are congruent.
13. Open a compass to the length of a segment, then using that amount of opening, mark off another segment of the same length.
14. A compass can be used to copy an angle.
15. A compass can be used to bisect angles and segments.
16. A compass can be used to construct the perpendicular bisector of a segment.
17. A compass can be used to construct a line perpendicular to a given line through a point on a given line.
18. A compass can be used to construct a line perpendicular to a given line through a point, not on the given line.

19. A collinear line may have 3 or more points.
20. Construction.
21. Three sides congruent (SSS).
22. CPCTC – Corresponding parts of congruent triangles are congruent.
23. Both were drawn with the same compass opening.
24. An arc can have only one chord.
25. Congruent arcs have congruent chords.
26. Subtraction property of equality.
27. Addition property of equality.
28. Two points determine a line or segment.
29. Two lines are parallel if they are cut by a transversal such that two corresponding angles are congruent.
30. Two lines parallel to a third line are parallel to each other.
31. By definition.
32. Reflection line.
33. If a radius is perpendicular to a chord, then it bisects the chord.
34. If a radius bisects a chord, then it is perpendicular to the chord.
35. If two chords of a circle are equidistant from the center of the circle, then they are congruent.
36. If two chords of a circle are congruent, then they are equidistant from the center.

Proofs

Author will comply with the requirement that corrected proofs must be returned to the publisher within two to three days of receipt and understands that the publisher will do everything possible to ensure prompt publication.

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As stated above, this research and publication project has been totally self-funded (private). No financially supporting bodies, such as research councils, public or private foundations, and etcetera, have been involved in any part of this project (writing, editing, approving, and etcetera).

Author's Biography with Photo

Arthur Clair Rediske is 96 years old. He has worked in various companies, including: General Electric, Douglas United Nuclear, Kaiser Engineering, and Vitro, Bechtel Corp. He worked as a Field Engineer in a welding lab. He worked as a certified inspector for electrical and instrument installations. Mr. Rediske was a certified inspector for electrical and instrument installations for "N" nuclear reactor and worked on all phases of instrumentation for "N" reactor startup. He has worked with expert manual TIG arc welders, and with master, welders using automated welding machines. He served as past president of the Richland, WA branch of the ISA, Instrument Society of America. Interestingly, Mr. Rediske invented a method of measuring the number of rotations an arrow makes as it travels to a target. Mr. Rediske has been designing and making sterling silver jewelry since 1940.

Arthur C. Rediske

(1970's and 2019)

Publications

Mr. Rediske, while employed at Bechtel Corporation in their weld lab at Hanford, Washington, USA, wrote the first calibration and maintenance manual for the Astro-arc automatic pipe welding system. This manual, commissioned by the Atomic Energy Commission, did not exist prior to Mr. Rediske's effort. Mr. Rediske also wrote a calibration and maintenance manual for tubing welders used at "N" nuclear reactor, Hanford, WA. He has published several articles in the Instrument Society of America (ISA) journal, ISA December 1962, Hybrid Instruments, Hanford Library HW-SA-2674 Hanford Library HW-54-2724. He has also published articles in the ISA magazine that are available in the Hanford Library in Richland, Washington. He has also published articles in woodworking and archery magazines.