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Lie Symmetry Solutions of Sawada-Kotera Equation

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Abstract

In this article, the Lie Symmetry Analysis is applied in finding the symmetry solutions of the fifth-order Sawada-Kotera equation of the form $u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx} = 0$. The technique is among the most powerful approaches currently used to achieve precise solutions of the partial differential equations that are nonlinear. We systematically show the procedure to obtain the solution which is achieved by developing infinitesimal transformation, prolongations, infinitesimal generators, and invariant transformations/solutions hence symmetry solutions of the fifth-order Sawada-Kotera equation.

Keywords: Lie symmetry analysis. Sawada-Kotera equation. Symmetry groups. Prolongations. Invariant solutions. Power series solutions. Symmetry solutions.

1. Introduction

Recently, many phenomena in physical, engineering and mathematical fields are expressed in terms of Partial Differential Equations and Ordinary Differential Equations. Among the many approaches that have been recommended for obtaining the precise solutions of the given differential equations is Lie symmetry analysis, which provides a very effective procedure.

The other methods include Jacobi elliptic function expansion technique as discussed by Liu *et al.* [1], the Hirota's bilinear transformation scheme as demonstrated by Hirota [2], Backlund transformation technique as illustrated by Rogers and Shadwick [3], homotopy analysis technique as discussed by Liao [4], variational iteration scheme as demonstrated by Noor and Mohyud [5].

Recently, Lie symmetry analysis method is effectively used in the study of several differential equations to obtain solutions such as Burgers equation as discussed by Oduor [6], nonlinear evolution equations having advanced order nonlinearity as discussed by Khongorzul *et al.* [7], nonlinear beam equation as deliberated by [8,9], nonlinear ordinary differential equation as discussed by Aminer [10], a scheme of nonlinear partial differential equation as discussed by Andronikos *et al.* [11], nonlinear Helmholtz equation as illustrated by Sakkaravarthi [12], reaction-diffusion equation as demonstrated by Yildirim and Pinar [13] and many more.

In this article, we apply Lie symmetry analysis method to develop prolongations, symmetry groups, invariant solutions, exact solutions and symmetry solutions of Sawada-Kotera equation of the form

$$u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx} = 0 \qquad (1.1)$$

in which $u = u(x,t)$.

The outline of this article is as follows. In section 2, we present the methods of using Lie symmetry analysis. Section 3 presents symmetry solutions of equation (1.1). Finally, section 4 gives the conclusion and remarks.

2. Methods

2.1 Lie symmetry analysis

We let the generator G of (1.1) to be of the form

$$G=\alpha(x,t,u)\frac{\partial}{\partial x}+\xi(x,t,u)\frac{\partial}{\partial t}+\rho(x,t,u)\frac{\partial}{\partial u} \qquad (1.2)$$

All the coefficient functions α,ξ and ρ are found so that all the equivalent one-parameter Lie group of transformations becomes

$x^{*}=X(x,t,u;\varepsilon),\ t^{*}=T(x,t,u;\varepsilon),\ u^{*}=U(x,t,u;\varepsilon)$ for a symmetry group of equation (1.1)

For the symmetry state to be satisfied by (1.1), we then have

$$G^{[5]}\left[u_t+45u^2u_x+15u_xu_{xx}+15uu_{xxx}+u_{xxxxx}\right]=0 \qquad (1.3)$$

Where $G^{[5]}$ is the fifth prolongation/extension of (1.2) given by

$$G^{[5]}=\alpha\frac{\partial}{\partial x}+\xi\frac{\partial}{\partial t}+\rho\frac{\partial}{\partial u}+\rho^{t}\frac{\partial}{\partial u_t}+\rho^{x}\frac{\partial}{\partial u_x}+\rho^{tt}\frac{\partial}{\partial u_{tt}}+\rho^{tx}\frac{\partial}{\partial u_{tx}}+\rho^{xx}\frac{\partial}{\partial u_{xx}}+\rho^{ttt}\frac{\partial}{\partial u_{ttt}}+\rho^{ttx}\frac{\partial}{\partial u_{ttx}}$$
$$+\rho^{txx}\frac{\partial}{\partial u_{txx}}+\rho^{xxx}\frac{\partial}{\partial u_{xxx}}+\rho^{tttt}\frac{\partial}{\partial u_{tttt}}+\rho^{tttx}\frac{\partial}{\partial u_{tttx}}+\rho^{ttxx}\frac{\partial}{\partial u_{ttxx}}+\rho^{txxx}\frac{\partial}{\partial u_{txxx}}+\rho^{xxxx}\frac{\partial}{\partial u_{xxxx}}$$
$$+\rho^{ttttt}\frac{\partial}{\partial u_{ttttt}}+\rho^{ttttx}\frac{\partial}{\partial u_{ttttx}}+\rho^{tttxx}\frac{\partial}{\partial u_{tttxx}}+\rho^{ttxxx}\frac{\partial}{\partial u_{ttxxx}}+\rho^{txxxx}\frac{\partial}{\partial u_{txxxx}}+\rho^{xxxxx}\frac{\partial}{\partial u_{xxxxx}} \qquad (1.4)$$

Substituting (1.4) into (1.3) we get

$$[\alpha\frac{\partial}{\partial x}+\xi\frac{\partial}{\partial t}+\rho\frac{\partial}{\partial u}+\rho^{t}\frac{\partial}{\partial u_t}+\rho^{x}\frac{\partial}{\partial u_x}+\rho^{tt}\frac{\partial}{\partial u_{tt}}+\rho^{tx}\frac{\partial}{\partial u_{tx}}+\rho^{xx}\frac{\partial}{\partial u_{xx}}+\rho^{ttt}\frac{\partial}{\partial u_{ttt}}+\rho^{ttx}\frac{\partial}{\partial u_{ttx}}$$
$$+\rho^{txx}\frac{\partial}{\partial u_{txx}}+\rho^{xxx}\frac{\partial}{\partial u_{xxx}}+\rho^{tttt}\frac{\partial}{\partial u_{tttt}}+\rho^{tttx}\frac{\partial}{\partial u_{tttx}}+\rho^{ttxx}\frac{\partial}{\partial u_{ttxx}}+\rho^{txxx}\frac{\partial}{\partial u_{txxx}}+\rho^{xxxx}\frac{\partial}{\partial u_{xxxx}}$$
$$+\rho^{ttttt}\frac{\partial}{\partial u_{ttttt}}+\rho^{ttttx}\frac{\partial}{\partial u_{ttttx}}+\rho^{ttxxx}\frac{\partial}{\partial u_{ttxxx}}+\rho^{ttxxx}\frac{\partial}{\partial u_{ttxxx}}+\rho^{txxxx}\frac{\partial}{\partial u_{txxxx}}+\rho^{xxxxx}\frac{\partial}{\partial u_{xxxxx}}]$$
$$\left[u_t+45u^2u_x+15u_xu_{xx}+15uu_{xxx}+u_{xxxxx}\right]=0$$

The infinitesimal condition above reduces to

$$\rho^{t}+90\rho u_{x}+45\rho^{x}u^{2}+15\lambda^{x}u_{xx}+15\rho^{xx}u_{x}+15\rho u_{xxx}+15\rho^{xxx}u+\rho^{xxxxx}=0 \tag{1.5}$$

With ρ^{t}, ρ^{x}, ρ^{xx}, ρ^{xxx} and ρ^{xxxxx} explicitly defined.

We then equate the coefficients of several polynomials in the first to the fifth partial derivatives of u . This is satisfied when u_t is replaced by $-45u^2u_x-15u_xu_{xx}-15uu_{xxx}-u_{xxxxx}$ whenever it occurs in the equation hence obtaining the determining equations for obtaining the Lie groups of (1.1)

The solutions from the determining equations become

$$\alpha=c_1+\frac{1}{5}c_3x \tag{1.6}$$

$$\xi=c_2+tc_3 \tag{1.7}$$

$$\rho=-\frac{2}{5}uc_3 \tag{1.8}$$

Thus the infinitesimal symmetries/generators of (1.1) are as given below

$$w_1=\frac{\partial}{\partial x}$$

$$w_2=\frac{\partial}{\partial t}$$

$$w_3=x\frac{\partial}{\partial x}+5t\frac{\partial}{\partial t}-2u\frac{\partial}{\partial u}$$

Lie groups admitted by infinitesimal generators are obtained by solving the corresponding Lie equations through exponentiation. This leads to

$$w_1=\frac{\partial}{\partial x};G_1(\varepsilon):X(x,t,u;\varepsilon)\rightarrow X_1(x+\varepsilon,t,u)$$

$$w_2=\frac{\partial}{\partial t};G_2(\varepsilon):X(x,t,u;\varepsilon)\rightarrow X_1(x,t+\varepsilon,u)$$

$$w_3=x\frac{\partial}{\partial x}+5t\frac{\partial}{\partial t}-2u\frac{\partial}{\partial u};G_3(\varepsilon):X(x,t,u;\varepsilon)\rightarrow X_3\left(xe^{\varepsilon},te^{5\varepsilon},ue^{-2\varepsilon}\right)$$

2.2 Invariant solutions and exact power series solutions of (1.1)

A group invariant solution is obtained when a group of transformations maps a solution into itself. The invariant solution of equation (1.1) under the one-parameter group of generator V can be obtained by calculating two independent invariants $N_1 = k(x,t)$ and $N_2 = \mu(x,t,u)$ by solving the equation

$$N(J) \equiv \alpha(x,t,u)\frac{\partial N}{\partial x} + \xi(x,t,u)\frac{\partial N}{\partial t} + \rho(x,t,u)\frac{\partial N}{\partial u} = 0 \tag{1.9}$$

Or its system of characteristics

$$\frac{dx}{\alpha(x,t,u)} = \frac{dt}{\xi(x,t,u)} = \frac{du}{\rho(x,t,u)} \tag{2.0}$$

Here we consider the group transformations that arise from all the generators of (1.1)

We then allocate one of the invariants as a function of the other as given below

$$\mu = \phi(k) \tag{2.1}$$

We then substitute for μ, in (2.1) to get an ordinary differential equation for the function $\phi(k)$ of one variable. By doing this, we decrease the figure of independent variables by one.

We now show the list of generators (X_i) and their equivalent Invariant Solutions (*u*)

<u>Case 1</u>

For the infinitesimal generator, $w_1 = \frac{\partial}{\partial x}$ we get $u = \phi(t)$. When we substitute it into equation (1.1) we get the trivial solution to be

$$u = \phi(t) = k \tag{2.2}$$

<u>Case 2</u>

For the generator $w_2 = \frac{\partial}{\partial t}$ we have $u = \phi(x)$. When it is substituted into equation (1.1), the equation is reduced into the following ordinary differential equation

$$45\phi^2\phi' + 15\phi'\phi'' + 15\phi\phi''' + \phi^{(5)} = 0 \tag{2.3}$$

Where $\phi' = \frac{d\phi}{d\mu}$

Case 3

For the generator, $w_3 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$ we have $u = t^{-\frac{2}{5}}\phi(\mu)$ where $\mu = xt^{-\frac{1}{5}}$. Substituting into equation (1.1), will reduce the equation into the following ordinary differential equation

$$-\frac{2}{5}\phi - \frac{1}{5}\mu\phi' + 45\phi^2\phi' + 15\phi'\phi'' + 15\phi\phi''' + \phi^{(5)} = 0 \qquad (2.4)$$

Where $\phi' = \frac{d\phi}{d\mu}$

The exact solutions of some ODEs or some PDEs that are lower in order than the original PDE can be obtained as discussed by Galaktionov and Svirshchevskii [14]. Thus in our case, the exact solutions of the fifth-order Sawada-Kotera equation are found from the obtained ODE's. Even though there are so many methods to be applied in solving differential equations such as reduction by quadratures, and so many others but it is not always easy for simple semilinear PDE's.

Therefore, the analytic solutions of the reduced equations are found through the power series technique since this scheme is used in solving all kinds of differential equations including those with non-constant coefficients, as discussed elsewhere in [15–18]. Thus the exact analytic solutions obtained are the solutions of the original PDE.

In this case, we deliberate equations (2.3) and (2.4).

In view of (2.3) we have

$$45\phi^2\phi' + 15\phi'\phi'' + 15\phi\phi''' + \phi^{(5)} = 0$$

We obtain its solution by use of a power series method

$$\phi(\beta) = \sum_{a=0}^{\infty} c_a \beta^a \qquad (2.5)$$

Substituting (2.5) into (2.3) we get

$$\begin{aligned} &120c_5 + \sum_{a=1}^{\infty}(a+1)(a+2)(a+3)(a+4)(a+5)c_{a+5}\mu^a + 90c_0c_3 + \\ &\quad 15\sum_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_z c_{a-z+2}\mu^a + 30c_1c_2 + \\ &\quad 15\sum_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2} + 45c_0^2c_1 + 45\sum_{z=0}^{a}\sum_{i=0}^{z}(a-z+1)c_i c_{z-i} c_{a-z+1} \end{aligned} \qquad (2.6)$$

Setting the a=0 we get

$$120c_5 + 90c_0c_3 + 30c_1c_2 + 45c_0^2c_1 = 0$$

On simplifying we get

$$c_5 = -\frac{1}{120}\left(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1\right) \tag{2.7}$$

For $a \geq 1$ we obtain

$$c_{a+5} = \frac{-1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\sum_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3} +$$
$$15\sum_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2} + 45\sum_{z=0}^{a}\sum_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}] \tag{2.8}$$

For a=0, 1, 2...

When a=1 we have

$$c_6 = \frac{-1}{720}\left(360c_0c_4 + 180c_1c_3 + 90c_0^2c_2 + 60c_2^2 + 90c_0c_1^2\right) \tag{2.9}$$

When a=2 we have

$$c_7 = \frac{-1}{2520}\left(900c_0c_5 + 540c_1c_4 + 135c_0^2c_3 + 360c_2c_3 + 270c_0c_1c_2 + 45c_1^3\right) \tag{3.0}$$

Thus the power series solution (2.5) into (2.3) is an exact analytic solution which can be given as

$$\phi(\beta) = c_0 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + \sum_{a=1}^{\infty}c_{a+5}\mu^{a+5}$$

$$= c_0 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 - \frac{1}{120}\left(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1\right)\mu^5 -$$
$$\sum_{a=1}^{\infty}\frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\sum_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3} +$$
$$15\sum_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2} + 45\sum_{z=0}^{a}\sum_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}]\ \mu^{a+5} \tag{3.1}$$

Now the exact power series solution of (1.1) is obtained to be

$$u(x,t) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \sum_{a=1}^{\infty}c_{a+5}x^{a+5}$$

$$=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4-\frac{1}{120}\left(90c_0c_3+30c_1c_2+45c_0^2c_1\right)x^5-$$
$$\sum_{a=1}^{\infty}\frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\sum_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3}+$$
$$15\sum_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2}+45\sum_{z=0}^{a}\sum_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}]\ x^{a+5} \qquad (3.2)$$

Where $c_i\,(i=0,1,2,3,4)$ are arbitrary constants.

Also, we find a solution of equation (2.4) in a power series of the method (2.5). Substituting into (2.4), and comparing coefficients, we obtain

$$120c_5+\sum_{a=1}^{\infty}(a+1)(a+2)(a+3)(a+4)(a+5)c_{a+5}\mu^a+90c_0c_3+$$
$$15\sum_{d=0}^{a}(a-d+1)(a-d+2)(a-d+3)c_dc_{a-d+3}\mu^a+30c_1c_2+$$
$$15\sum_{d=0}^{a}(a-d+1)(a-d+2)(d+1)c_{d+1}c_{a-d+2}+45c_0^2c_1+45\sum_{d=0}^{a}\sum_{i=0}^{d}(a-d+1)c_ic_{d-i}c_{a-d+1}$$
$$-\frac{2}{5}c_a-\frac{1}{5}ac_a=0 \qquad (3.3)$$

Setting the coefficients for $a=0$ we obtain

$$120c_5+90c_0c_3+30c_1c_2+45c_0^2c_1-\frac{2}{5}c_0=0$$

which is simplified to give

$$c_5=-\frac{1}{120}\left(90c_0c_3+30c_1c_2+45c_0^2c_1-\frac{2}{5}c_0\right) \qquad (3.4)$$

For $a\geq 1$, we obtain

$$c_{a+5}=\frac{-1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\sum_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3}+$$

$$15\sum_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2}+45\sum_{z=0}^{a}\sum_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1}-\frac{2}{5}c_a-\frac{1}{5}ac_a] \qquad (3.5)$$

For the values of a=0, 1, 2...

When a=1 we have

$$c_6 = \frac{-1}{720}\left(360c_0c_4 + 180c_1c_3 + 90c_0^2c_2 + 60c_2^2 + 90c_0c_1^2 - \frac{3}{5}c_1\right) \tag{3.6}$$

When a=2 we have

$$c_7 = \frac{-1}{2520}\left(900c_0c_5 + 540c_1c_4 + 135c_0^2c_3 + 360c_2c_3 + 270c_0c_1c_2 + 45c_1^3 - \frac{4}{5}c_2\right) \tag{3.7}$$

Therefore the power series solution of equation (2.4) is given as

$$\phi(\beta) = c_0 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + \sum_{a=1}^{\infty} c_{a+5}\mu^{a+5}$$

$$= c_0 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 - \frac{1}{120}\left(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1 - \frac{2}{5}c_0\right)\mu^5 -$$

$$\sum_{a=1}^{\infty}\frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\sum_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3} +$$

$$15\sum_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2} + 45\sum_{z=0}^{a}\sum_{i=0}^{z}(a-z+1)c_ic_{z-i}c_{a-z+1} - \frac{2}{5}c_a - \frac{1}{5}ac_a]\mu^{a+5}$$

Hence the exact analytic solution to equation (1.1) is

$$u(x,t) = c_0t^{-\frac{2}{5}} + c_1xt^{-\frac{3}{5}} + c_2x^2t^{-\frac{4}{5}} + c_3x^3t^{-1} + c_4x^4t^{-\frac{6}{5}} - \frac{1}{120}\left(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1 - \frac{2}{5}c_0\right)x^5t^{-\frac{7}{5}}$$

$$-\sum_{a=1}^{\infty}\frac{1}{(a+1)(a+2)(a+3)(a+4)(a+5)}[15\sum_{z=0}^{a}(a-z+1)(a-z+2)(a-z+3)c_zc_{a-z+3} +$$

$$15\sum_{z=0}^{a}(a-z+1)(a-z+2)(z+1)c_{z+1}c_{a-z+2} + 45\sum_{z=0}^{a}\ \sum_{i=0}^{z}(a-d+1)c_ic_{z-i}c_{a-z+1} - \frac{2}{5}c_a - \frac{1}{5}ac_a]x^{a+5}t^{-\frac{a+7}{5}}$$

(3.8)

3.0 Results and Discussions on Symmetry solutions

Symmetry transformations convert known solutions into new solutions. Considering group transformations that arise from the infinitesimal generators

$$w_1 = \frac{\partial}{\partial x}, w_2 = \frac{\partial}{\partial t}, w_3 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$$

known to be

$$G_1 : X\left(x,t,u;\varepsilon\right) \to X_1\left(x+\varepsilon,t,u\right)$$

$$G_2 : X\left(x,t,u;\varepsilon\right) \to X_2\left(x,t+\varepsilon,u\right)$$

$$G_3 : X\left(x,t,u;\varepsilon\right) \to X_3\left(xe^{\varepsilon},te^{5\varepsilon},ue^{-2\varepsilon}\right)$$

Since $u = u\left(x,t\right)$ is a known solution of equation (1.1), so is

$$G_3\left(\varepsilon\right) f\left(x,t\right) = f\left(xe^{-\varepsilon},te^{-5\varepsilon}\right)e^{-2\varepsilon}$$

We consider the group G_3. Thus the new symmetry transformed solution under G_3 becomes

$$u = f\left(xe^{\varepsilon},te^{5\varepsilon}\right).e^{-2\varepsilon} \tag{3.9}$$

Whenever a known solution of (1.1) is given as $u = u\left(x,t\right)$

3.1 Solution 1

Considering the invariant result of (1.1), $u = c$. Substituting $u = c$ into equation (3.9) we obtain

$$u = ce^{-2\varepsilon}$$

3.2 Solution 2

Inserting the exact solution

$$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 - \frac{1}{120}\left(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1\right)x^5 - \left(b^*\right)x^{a+5}$$

as a known solution of equation (1.1) in which

$$b^* = \sum_{a=1}^{\infty}\frac{1}{\left(a+1\right)\left(a+2\right)\left(a+3\right)\left(a+4\right)\left(a+5\right)}[15\sum_{z=0}^{a}\left(a-z+1\right)\left(a-z+2\right)\left(a-z+3\right)c_zc_{a-z+3} +$$
$$15\sum_{z=0}^{a}\left(a-z+1\right)\left(a-z+2\right)\left(z+1\right)c_{z+1}c_{a-z+2} + 45\sum_{z=0}^{n}\sum_{i=0}^{z}\left(a-z+1\right)c_ic_{z-i}c_{a-z+1}]$$

into (3.9) then we obtain

$$u = [c_0 + c_1xe^{\varepsilon} + c_2\left(xe^{\varepsilon}\right)^2 + c_3\left(xe^{\varepsilon}\right)^3 + c_4\left(xe^{\varepsilon}\right)^4 - \frac{1}{120}\left(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1\right)\left(xe^{\varepsilon}\right)^5 -$$
$$\left(b*\right)\left(xe^{\left(a\varepsilon+5\varepsilon\right)}\right)].e^{-2\varepsilon}$$

3.3 Solution 3

Substituting the exact solution

$$u = c_0 t^{-\frac{2}{5}} + c_1 x t^{-\frac{3}{5}} + c_2 x^2 t^{-\frac{4}{5}} + c_3 x^3 t^{-1} + c_4 x^4 t^{-\frac{6}{5}} - \frac{1}{120}\left(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1 - \frac{2}{5}c_0\right)x^5 t^{-\frac{7}{5}} - (b*)\left(x^{a+5} t^{-\frac{a+7}{5}}\right)$$

as a known solution of equation (1.1) and a^* taken as stated above into equation (3.9) we obtain

$$u = [c_0\left(te^{5\varepsilon}\right)^{-\frac{2}{5}} + c_1 xe^{\varepsilon}\left(te^{5\varepsilon}\right)^{-\frac{3}{5}} + c_2\left(xe^{\varepsilon}\right)^2\left(te^{5\varepsilon}\right)^{-\frac{4}{5}} + c_3\left(xe^{\varepsilon}\right)^3\left(te^{5\varepsilon}\right)^{-1} + c_4\left(xe^{\varepsilon}\right)^4\left(te^{5\varepsilon}\right)^{-\frac{6}{5}} - \frac{1}{120}\left(90c_0c_3 + 30c_1c_2 + 45c_0^2c_1 - \frac{2}{5}c_0\right)\left(xe^{\varepsilon}\right)^5\left(te^{5\varepsilon}\right)^{-\frac{7}{5}} - (b*)\left(xe^{\varepsilon}\right)^{a+5}\left(te^{5\varepsilon}\right)^{-\frac{a+7}{5}}].e^{-2\varepsilon}$$

4.0 Conclusion and Remarks

In this article, the symmetry solutions of the fifth-order Sawada-Kotera equation have been acquired by means of Lie symmetry analysis method. Also, deliberation on the group-invariant solutions and the exact analytic solutions to the equation through the power series technique has been done. Our obtained symmetry solutions demonstrate that Lie symmetry analysis technique is a candid and best mathematical tool to obtain analytical results of highly nonlinear PDE's. Furthermore, this technique can be more efficiently used to study other nonlinear PDE's, which are often used in mathematical fields, engineering fields, and physical sciences.

Conflicts of Interest

There are no conflicts of interest

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