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Convergence of the Collatz Sequence

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Abstract

For any natural number was created the supplement sequence, that is convergent together with the original Collatz sequence. The numerical parameter - index was defined, that is the same for both sequences. This new method provides the following results:

1. All natural numbers were distributed into six different classes;
2. The properties of the index were found for the different classes;
3. For any natural number was constructed the bounded sequence of increasing numbers, that is convergent together with the regular Collatz sequence.

Keywords: Collatz Sequence

Introduction

For any natural number the Collatz sequence defined by the rule (Ref.1):

$$a_{n+1}=\begin{cases}\frac{1}{2}a_n, & if\ a_n\ is\ even\\ 3a_n+1, & if\ a_n\ is\ odd\end{cases}$$

Thus, for N=11, the corresponding sequence is

11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, and the sequence will repeat itself. The number N is convergent if the corresponding sequence contains 1.

For the natural number $\mathrm{N}=2^p\cdot(2k+1)$ $(p=0,1,2,3,...;\ k=0,1,2,3,...)$ were defined as the <u>index</u> $\delta(\mathrm{N})$ by the following (Ref.2):

1. $\delta(1)=0$

2. $$\delta(\mathrm{N})=\begin{cases}\delta(3\mathrm{N}+1)+1, & if\ \mathrm{N}\ is\ odd\\ \delta\left(\frac{1}{2}\mathrm{N}\right)+1, & if\ \mathrm{N}\ is\ even\end{cases}$$

For example, if N=11, $\delta(11)=14$, which is the number of steps number 11 goes to 1.

Materials and Methods: Abstract proofs

Results and Discussion:

1. All natural numbers were distributed into six different classes;
2. The properties of the index were found for the different classes;
3. For any natural number was constructed the bounded sequence of increasing numbers, that is convergent together with the regular Collatz sequence.

Main Text:

The Supplement Sequences.

For the original sequence corresponding to number N, we may write different supplement sequences, that are convergent together with the original sequence, and all such sequences have the same index.

Example. For the sequence corresponding to N = 26, we may construct another sequence by the rule:

$$a_{n+1} = \begin{cases} \dfrac{a_n - 1}{2} & if \ \ a_n \ \ is \ odd \\ a_n \cdot \dfrac{3}{2} + 1 & if \ \ a_n \ \ is \ even \end{cases}$$

and we received two "relative" sequences:

$$26 \xrightarrow{\div 2} 13 \xrightarrow{\times 3+1} 40 \xrightarrow{\div 2} 20 \xrightarrow{\div 2} 10 \xrightarrow{\div 2} 5 \xrightarrow{\times 3+1} 16 \xrightarrow{\div 2} 8 \ldots$$

$$25 \xrightarrow{-1 \div 2} 12 \xrightarrow{\times \frac{3}{2}+1} \longrightarrow 19 \xrightarrow{-1 \div 2} 9 \xrightarrow{-1 \div 2} 4 \xrightarrow{\times \frac{3}{2}+1} \longrightarrow 7 \ldots$$

or we may construct other sequences

$$24 \xrightarrow{\div 2-1} 11 \xrightarrow{+1 \times \frac{3}{2}} \longrightarrow 18 \xrightarrow{\div 2-1} 8 \xrightarrow{\div 2-1} 3 \xrightarrow{+1 \times \frac{3}{2}} \longrightarrow 6 \xrightarrow{\div 2-1} 2 \xrightarrow{\div 2-1} 0$$

$$23 \xrightarrow{-3 \div 2} 10 \xrightarrow{\times \frac{3}{2}+2} \longrightarrow 17 \xrightarrow{-3 \div 2} 7 \xrightarrow{-3 \div 2} 2 \xrightarrow{\times \frac{3}{2}+2} \longrightarrow 5 \xrightarrow{-3 \div 2} 1 \xrightarrow{-3 \div 2} -1$$

The reader may construct different supplement sequences, by finding an appropriate rule,

such that all those sequences are convergent with the same index.

Basic Supplement Sequence.

In this article, we will consider the supplement sequence defined by the rule (Ref.2):

$$a_{n+1} = \begin{cases} \dfrac{a_n + 1}{2} & if \ \ a_n \ \ is \ odd \\ a_n \cdot \dfrac{3}{2} & if \ \ a_n \ \ is \ even \end{cases}$$

Example: For the number 26 the original and supplement sequences can be written

$$26 \xrightarrow{\div 2} 13 \xrightarrow{\times 3+1} 40 \xrightarrow{\div 2} 20 \xrightarrow{\div 2} 10 \xrightarrow{\div 2} 5 \xrightarrow{\times 3+1} 16 \xrightarrow{\div 2} 8 \xrightarrow{\div 2} 4 \xrightarrow{\div 2} 2 \xrightarrow{\div 2} 1$$

$$27 \xrightarrow{+1\div 2} 14 \xrightarrow{\times\frac{3}{2}} \longrightarrow 21 \xrightarrow{+1\div 2} 11 \xrightarrow{+1\div 2} 6 \xrightarrow{\times\frac{3}{2}} \longrightarrow 9 \xrightarrow{+1\div 2} 5 \xrightarrow{+1\div 2} 3 \xrightarrow{+1\div 2} 2$$

We see that numbers in the second raw exceed corresponding numbers in the first raw by one. For the new supplement sequence we define the operations:

$$A(a_n) = \frac{a_n + 1}{2} \;\; if \;\; a_n \;\; is\ odd \;\; and \;\; B(a_n) = a_n \cdot \frac{3}{2} \;\; if \;\; a_n \;\; is\ even,$$

and inverse operations $A^{-1}(a_n) = a_n \cdot 2 - 1,\;\; B^{-1}(a_n) = a_n \cdot \frac{2}{3}$.

Theorem 3.1 If the natural number N converges to 1 with index δ (the number of steps in the original sequence), then the corresponding supplement sequence for the number N+1 converges to 2 with the same index, where the operation A is equivalent to one step, and the operation B is equivalent to two steps. Numbers in supplement sequence exceed corresponding numbers in original sequence exactly by 1.

Proof. a) If a_n is odd, then in the original sequence $a_n \xrightarrow{\times 3+1} 3 \cdot a_n + 1 \xrightarrow{\div 2} \frac{3a_n + 1}{2}$.

For the supplement sequence, $a_n + 1$ is even, and $a_n + 1 \xrightarrow{B} \frac{(a_n + 1)\cdot 3}{2} = \frac{3 \cdot a_n + 1}{2} + 1$.

b) If a_n is even, then in the original sequence $a_n \xrightarrow{\div 2} \frac{a_n}{2}$, and the corresponding supplement sequence $a_n + 1 \xrightarrow{A} \frac{a_n + 1 + 1}{2} = \frac{a_n}{2} + 1$.

Corollary. The operation A reduces the index of a number by 1 and the operation B reduces index by 2, whereas inverse operations A^{-1} and B^{-1} increase the index by 1 and by 2 correspondingly.

Theorem 3.2. For the even number N = 4k+2 (k=0, 1, 2, 3,...), $\delta(4k+2) = \delta(3k+2) + 3$.

Proof. $4k + 2 \xrightarrow{B} 6k + 3 \xrightarrow{A} 3k + 2$, and both operations reduce index totally by 3.

Corollary. From Theorem 3.2 we see that the even number 4k+2 is possible to reduce to the smaller number 3k+2. Any odd number can be reduced in one step using operation A. Therefore, to prove that any natural number converges, it is enough to show this only for the numbers which are divisible by 4.

Properties of the index.

We separate all natural numbers by the six groups:

Group 1 $N = 2^n(4k+1)$, n is odd, $n \geq 3$, k=0,1,2,…

and $N = 2^n(4k+3)$, n is even, $n \geq 2$, k=0,1,2,…;

Group 2 $N = 2(4k+1)$, k=0,1,2,...;

Group 3 $N = 4k+3$, k=0,1,2,...;

Group 4 $N = 2^n(4k+1)$, n is even, $n \geq 2$,

and $N = 2^n(4k+3)$, n is odd, $n \geq 3$, k=0,1,2,...;

Group 5 $N = 2(4k+3)$, k=0,1,2,...

Group 6 $N = 4k+1$, k=0,1,2,...

Lemma 4.1. $\frac{3^n+1}{2}\begin{cases} is \text{ even, if n is odd} \\ \text{is odd, if n is even} \end{cases}$.

We recognize that because 3^n+1 is divisible by 4 if n is odd, and 3^n+1 is divisible by 2 but not by 4 if n is even.

Theorem 4.1. If n is even, $n \geq 2$, then for the numbers from group 4,

$$\delta\left[2^n(4k+1)\right] = \delta\left[2^{n-1}(4k+1)\right] + 1.$$

Proof. Using lemma 4.1, we obtain

$$2^n(4k+1) \xrightarrow{B^n} 3^n(4k+1) \xrightarrow{A} 3^n \cdot 2k + \frac{3^n+1}{2} \xrightarrow{A} 3^n \cdot k + \frac{3^n+3}{4};$$

$$2^{n-1}(4k+1) \xrightarrow{B^{n-1}} 3^{n-1}(4k+1) \xrightarrow{A} 3^{n-1} \cdot 2k + \frac{3^{n-1}+1}{2} \xrightarrow{B} 3^n \cdot k + \frac{3^n+3}{4}.$$

We see that $\delta\left[2^n(4k+1)\right] = \delta\left[3^n \cdot k + \frac{3^n+3}{4}\right] + (n+2)$, and

$\delta\left[2^{n-1}(4k+1)\right] = \delta\left[3^n \cdot k + \frac{3^n+3}{4}\right] + (n+1)$, which gives the proof of the Theorem 4.1.

Theorem 4.1 shows that numbers from group 4 are possible to reduce to twice smaller numbers with an index lower by one. Table 4.1 gives a list of such "good numbers."

k \ n	0	1	2	3	4	5	6	7	8	9	10	...
2	4	20	36	52	68	84	100	116	132	148	164	...
4	16	80	144	208	272	336	400	464	528	592	656	...
6	64	320	576	832	1088	1344	1600	1856	2112	2368	2624	...
8	256	1280	2304	3328	4352	5376	6400	7424	8448	9472	10,496	...
...												

Theorem 4.2. If n is odd (n=1,3,5,...), then for the numbers from group 4,

$$\delta\left[2^n\left(4k+3\right)\right]=\delta\left[2^{n-1}\left(4k+3\right)\right]+1.$$

Proof. $2^n\left(4k+3\right)\xrightarrow{B^n}3^n\left(4k+3\right)\xrightarrow{A}3^n\cdot 2k+\dfrac{3^{n+1}+1}{2}\xrightarrow{A}3^n\cdot k+\dfrac{3^{n+1}+3}{4}$;

$$2^{n-1}\left(4k+3\right)\xrightarrow{B^{n-1}}3^{n-1}\left(4k+3\right)\xrightarrow{A}3^{n-1}\cdot 2k+\frac{3^n+1}{2}\xrightarrow{B}3^n\cdot k+\frac{3^{n+1}+3}{4}.$$

The last entries are the same, and $\delta\left[2^n\left(4k+3\right)\right]=\delta\left[3^n\cdot k+\dfrac{3^{n+1}+3}{4}\right]+\left(n+2\right)$, whereas

$\delta\left[2^{n-1}\left(4k+3\right)\right]=\delta\left[3^n\cdot k+\dfrac{3^{n+1}+3}{4}\right]+\left(n+1\right)$, and we get a proof of Theorem 4.2.

Table 4.2 gives some numbers from group 4 that is possible to reduce to twice lower numbers

k \ n	0	1	2	3	4	5	6	7	8	9	10	...
3	24	56	88	120	152	184	216	248	280	312	344	...
5	96	224	352	480	608	736	864	992	1120	1248	1376	...
7	384	896	1408	1920	2432	2944	3456	3968	4480	4992	5504	...
...												

Theorem 4.3 For the numbers from groups 1, 2, and 3, $\delta\left(N\right)=\delta\left(3N\right)+1$.

Proof Using Theorems 4.1 and 4.2, we have $\underset{\delta}{N} \xrightarrow{\times 2} \underset{\delta+1}{2N} \xrightarrow{B} \underset{(\delta+1)-2}{3N}$.

Theorem 4.4 If N is even, then $\delta(N) = \delta(3N-1)+1$.

Proof $\underset{\delta}{N} \xrightarrow{B} \underset{\delta-2}{\frac{3}{2}N} \xrightarrow{A^{-1}} \underbrace{3N-1}_{(\delta-2)+1}$.

Example 4.1

	Group 1	index	Group 2	index	Group 3	index
N	240	52	178	31	359	32
3N	720	51	534	30	1077	31
3N-1	719	51	533	30	1076	31

Theorem 4.5 For the numbers from groups 1, 2, and 3, $\delta(N) = \delta(4N-1)-2$.

Proof $\underset{\delta}{N} \xrightarrow{\times 2} \underset{\delta+1}{2N} \xrightarrow{A^{-1}} \underbrace{4N-1}_{(\delta+1)+1}$.

Theorem 4.6 If N is even, then $\delta(N) = \delta(4N-2)-2$.

Proof $\underset{\delta}{N} \xrightarrow{A^{-1}} \underbrace{2N-1}_{\delta+1} \xrightarrow{\times 2} \underbrace{4N-2}_{(\delta+1)+1}$.

Theorem 4.7 For any natural number N, $\delta(N) = \delta(4N-3)-2$.

Proof $\underset{\delta}{N} \xrightarrow{A^{-1}} \underbrace{2N-1}_{\delta+1} \xrightarrow{A^{-1}} \underbrace{4N-3}_{(\delta+1)+1}$.

Example 4.2

	Group 3	index	Group 5	index	Group 6	index
N	295	117	182	18	357	32
4N-1	1179	119				
4N-2			726	20		
4N-3	1177	119	725	20	1425	34

Theorem 4.8 For the numbers from groups 1, 2, and 3, $\delta(N)=\delta(6N-1)$.

Proof $\underset{\delta}{N} \xrightarrow{\times 3} \underset{\delta-1}{3N} \xrightarrow{A^{-1}} \underbrace{6N-1}_{(\delta-1)+1}.$

Theorem 4.9 If N is divisible by 4, then $\delta(N)=\delta(6N-2)$.

Proof $\underset{\delta}{N} \xrightarrow{\times 3-1} \underbrace{3N-1}_{\delta-1} \xrightarrow{\times 2} \underbrace{6N-2}_{(\delta-1)+1}.$

Theorem 4.10 If N is even, then $\delta(N)=\delta(6N-3)$.

Proof $\underset{\delta}{N} \xrightarrow{\times 3-1} \underbrace{3N-1}_{\delta-1} \xrightarrow{A^{-1}} \underbrace{6N-3}_{(\delta-1)+1}.$

Example 4.3

	Group 2	index	Group 4	index	Group 5	index
N	146	116	272	42	150	23
6N-1	875	116				
6N-2			1630	42		
6N-3	873	116	1629	42	897	23

Theorem 4.11 For the numbers from groups 1, 2, and 3, $\delta(N)=\delta(8N-2)-3=\delta(8N-3)-3$.

Proof a) $\underset{\delta}{N} \xrightarrow{\times 2} \underset{\delta+1}{2N} \xrightarrow{\times 4-2} \underbrace{8N-2}_{(\delta+1)+2};$

b) $\underset{\delta}{N} \xrightarrow{\times 2} \underset{\delta+1}{2N} \xrightarrow{\times 4-3} \underbrace{8N-3}_{(\delta+1)+2}.$

Theorem 4.12 If N is even, then $\delta(N)=\delta(8N-5)-3$.

Proof $\underset{\delta}{N} \xrightarrow{\times 4-2} \underbrace{4N-2}_{\delta+2} \xrightarrow{A^{-1}} \underbrace{8N-5}_{(\delta+2)+1}.$

Example 4.4

	Group 2	index	Group 3	index	Group 5	index
N	202	18	323	99	286	104

8N-2	1614	21	2582	102		
8N-3	1613	21	2581	102		
8N-5	1611	21			2283	107

Theorem 4.13 For the numbers from groups 1, 2, and 3, $\delta(N)=\delta(9N-1)+2.$

Proof $\underset{\delta}{N} \xrightarrow{\times 3} \underset{\delta-1}{3N} \xrightarrow{\times 3-1} \underbrace{9N-1}_{(\delta-1)-1}.$

Theorem 4.14 If N is divisible by 4, then $\delta(N)=\delta(9N-3)+2.$

Proof $\underset{\delta}{N} \xrightarrow{\times 3-1} \underbrace{3N-1}_{\delta-1} \xrightarrow{\times 3} \underbrace{9N-3}_{(\delta-1)-1}.$

Example 4.5

	Group 1	index	Group 2	index	Group 4	index
N	368	45	282	42	612	38
9N-1	3311	43	2537	40		
9N-3	3309	43			5505	36

Theorem 4.15 For the numbers from groups 1 and 2, $\delta(N)=\delta(12N-2)-1.$

Proof $\underset{\delta}{N} \xrightarrow{\times 3} \underset{\delta-1}{3N} \xrightarrow{\times 4-2} \underbrace{12N-2}_{(\delta-1)+2}.$

Theorem 4.16 For the numbers from groups 1,2, and 3, $\delta(N)=\delta(12N-3)-1.$

Proof $\underset{\delta}{N} \xrightarrow{\times 6-1} \underbrace{6N-1}_{\delta} \xrightarrow{A^{-1}} \underbrace{12N-3}_{\delta+1}.$

Theorem 4.17 If N is divisible by 4, then $\delta(N)=\delta(12N-5)-1.$

Proof $\underset{\delta}{N} \xrightarrow{\times 6-2} \underbrace{6N-2}_{\delta} \xrightarrow{A^{-1}} \underbrace{12N-5}_{\delta+1}.$

Theorem 4.18 If N is even, then $\delta(N)=\delta(12N-7)-1.$

Proof $\underset{\delta}{N} \xrightarrow{\times 6-3} \underbrace{6N-3}_{\delta} \xrightarrow{A^{-1}} \underbrace{12N-7}_{\delta+1}.$

Example 4.6

	Group 2	index	Group 3	index	Group 5	index
N	882	116	419	40	326	24
12N-2	10582	117				
12N-3	10581	117	5025	41		
12N-7	10577	117			3905	25

Theorem 4.19 For the numbers from group 1, $\delta(N)=\delta(18N-2)+1=\delta(27N-3)+3.$

Proof a) $\underset{\delta}{N}\xrightarrow{\times 3}\underset{\delta-1}{3N}\xrightarrow{\times 6-2}\underbrace{18N-2}_{\delta-1};$

b) $\underset{\delta}{N}\xrightarrow{\times 3}\underset{\delta-1}{3N}\xrightarrow{\times 9-3}\underbrace{27N-3}_{(\delta-1)-2}.$

Theorem 4.20 For the numbers from groups 1 and 2, $\delta(N)=\delta(18N-3)+1.$

Proof $\underset{\delta}{N}\xrightarrow{\times 3}\underset{\delta-1}{3N}\xrightarrow{\times 6-3}\underbrace{18N-3}_{\delta-1}.$

Theorem 4.21 If N is divisible by 4, then $\delta(N)=\delta(18N-7)+1.$

Proof $\underset{\delta}{N}\xrightarrow{\times 3-1}\underset{\delta-1}{3N}\xrightarrow{\times 6-1}\underbrace{18N-7}_{\delta-1}.$

Example 4.7

	Group 1	index	Group 1	index	Group 2	index
N	296	55	460	128	306	37
18N-2	5326	54	8278	127		
18N-3	5325	54	8277	127	5505	36
18N-7	5321	54	8273	127		
27N-3	7989	52	12,420	125		

Theorem 4.22 For the numbers from groups 1,2, and 3,

$$\delta(N)=\delta(16N-5)-4=\delta(16N-7)-4.$$

Proof a) $\underset{\delta}{N} \xrightarrow{\times 8-2} \underbrace{8N-2}_{\delta+3} \xrightarrow{A^{-1}} \underbrace{16N-5}_{(\delta+3)+1};$

b) $\underset{\delta}{N} \xrightarrow{\times 8-3} \underbrace{8N-3}_{\delta+3} \xrightarrow{A^{-1}} \underbrace{16N-7}_{(\delta+3)+1}.$

Theorem 4.23 If N is even, then $\delta(N)=\delta(16N-10)-4=\delta(16N-11)-4.$

Proof a) $\underset{\delta}{N} \xrightarrow{\times 4-2} \underbrace{4N-2}_{\delta+2} \xrightarrow{\times 4-2} \underbrace{16N-10}_{(\delta+2)+2};$

b) $\underset{\delta}{N} \xrightarrow{\times 4-2} \underbrace{4N-2}_{\delta+2} \xrightarrow{\times 4-3} \underbrace{16N-11}_{(\delta+2)+2}.$

Example 4.8

	Group 3	index	Group 4	index	Group 5	index
N	623	87	472	128	614	38
16N-5	9963	91				
16N-7	9961	91				
16N-10			7542	132	9814	42
16N-11			7541	132	9813	42

Theorem 4.24 For the numbers from groups 1 and 2, $\delta(N)=\delta(24N-5)-2.$

Proof $\underset{\delta}{N} \xrightarrow{\times 12-2} \underbrace{12N-2}_{\delta+1} \xrightarrow{A^{-1}} \underbrace{24N-5}_{(\delta+1)+1}.$

Theorem 4.25 For the numbers from groups 1,2, and 3, $\delta(N)=\delta(24N-7)-2.$

Proof $\underset{\delta}{N} \xrightarrow{\times 12-3} \underbrace{12N-3}_{\delta+1} \xrightarrow{A^{-1}} \underbrace{24N-7}_{(\delta+1)+1}.$

Theorem 4.26 If N is divisible by 4, then $\delta(N)=\delta(24N-11)-2.$

Proof $\underset{\delta}{N} \xrightarrow{\times 12-5} \underbrace{12N-5}_{\delta+1} \xrightarrow{A^{-1}} \underbrace{24N-11}_{(\delta+1)+1}.$

Theorem 4.27 If N is even, then $\delta(N)=\delta(24N-15)-2.$

Proof $\underset{\delta}{N} \xrightarrow{\times 12-7} \underbrace{12N-7}_{\delta+1} \xrightarrow{A^{-1}} \underbrace{24N-15}_{(\delta+1)+1}$.

Example 4.9

	Group 1	index	Group 4	index	Group 5	index
N	412	133	376	45	406	27
24N-5	9883	135				
24N-7	9881	135				
24N-11	9877	135	9013	47		
24N-15	9873	135	9009	47	9729	29

Theorem 4.28 For the numbers from groups 1,2, and 3,

$$\delta(N) = \delta(32N-10) - 5 = \delta(32N-11) - 5.$$

Proof a) $\underset{\delta}{N} \xrightarrow{\times 2} \underset{\delta+1}{2N} \xrightarrow{\times 4-2} \underbrace{8N-2}_{(\delta+1)+2} \xrightarrow{\times 4-2} \underbrace{32N-10}_{(\delta+3)+2}$;

b) $\underset{\delta}{N} \xrightarrow{\times 2} \underset{\delta+1}{2N} \xrightarrow{\times 4-2} \underbrace{8N-2}_{(\delta+1)+2} \xrightarrow{\times 4-3} \underbrace{32N-11}_{(\delta+3)+2}$.

Theorem 4.29 For the numbers from groups 1,2, and 3, $\delta(N) = \delta(32N-15) - 5$.

Proof a) $\underset{\delta}{N} \xrightarrow{\times 16-7} \underbrace{16N-7}_{\delta+4} \xrightarrow{A^{-1}} \underbrace{32N-15}_{(\delta+4)+1}$.

Theorem 4.30 If N is even, then $\delta(N) = \delta(32N-21) - 5 = \delta(32N-23) - 5$.

Proof a) $\underset{\delta}{N} \xrightarrow{\times 16-10} \underbrace{16N-10}_{\delta+4} \xrightarrow{A^{-1}} \underbrace{32N-21}_{(\delta+4)+1}$;

b) $\underset{\delta}{N} \xrightarrow{\times 16-11} \underbrace{16N-11}_{\delta+4} \xrightarrow{A^{-1}} \underbrace{32N-23}_{(\delta+4)+1}$.

Example 4.10

	Group 2	index	Group 3	index	Group 5	index
N	282	42	275	91	278	16
32N-15	9009	47	8785	96		

32N-21	9003	47			8875	21
32N-23	9001	47			8873	21

Fundamental Theorem.

We will try to prove that any natural number N is convergent to the number 2 (Ref.3,4).

More precisely, we will show that using Theorems proved in 4°, any natural number can be reduced to a smaller number. As we mentioned in 3°, we need to consider numbers that are divisible by 4.

For each natural number N, we will construct the sequence of increasing numbers (not consecutive) having the same index as the original number N (Ref.5).

<u>Proposition 5.1</u> If N is odd, $N \geq 3$, then $\delta(4N+2)=\delta(4N+1)$.

<u>Proof</u> $4N+2 \xrightarrow{B} 6N+3 \xrightarrow{A} 3N+2 \xrightarrow{A} \frac{3N+3}{2};$

$$4N+1 \xrightarrow{A} 2N+1 \xrightarrow{A} N+1 \xrightarrow{B} \frac{(N+1)\cdot 3}{2}.$$

The last entries are the same after 4 steps.

Here are some examples of the pair of numbers with equal index:

<u>Example 5.1</u>

N	21	index	739	index	1873	index	2391	index
4N+1	85	9	2957	22	7493	88	9565	122
4N+2	86	9	2958	22	7494	88	9566	122

<u>Theorem 5.1</u> $\delta\left[2^{p}\cdot(odd\ \text{number})+1\right]=\delta\left[2^{p}\cdot(odd\ \text{number})+1+2^{p-2}\right],\ \ \text{p}\geq 2,$

$(\text{odd number})=3,5,7,\ldots.$

<u>Proof.</u> The number in the left side $2^{p}\cdot(odd\ \text{number})+1 \xrightarrow{A^{p-2}} 2^{2}\cdot(odd\ \text{number})+1;$

the number on the right side $2^{p}\cdot(odd\ \text{number})+1+2^{p-2} \xrightarrow{A^{p-2}} 2^{2}\cdot(odd\ \text{number})+1+2^{0},$

and by Proposition 5.1, $\delta\left[4\cdot(odd\ \text{number})+1\right]=\delta\left[4\cdot(odd\ \text{number})+2\right].$

Corollary. If p is an even number, then $\delta\left[2^p\cdot(odd\ \text{number})+1\right]=\delta\left[2^p\cdot(odd\ \text{number})+1+2^{p-2}+2^{p-4}+...+2^2+1\right]$;

and if p is an odd number, then

$$\delta\left[2^p\cdot(odd\ \text{number})+1\right]=\delta\left[2^p\cdot(odd\ \text{number})+1+2^{p-2}+2^{p-4}+...+2^3+2\right].$$

Proof. The right side in Theorem 5.1 is

$\delta\left[2^p(odd\ \text{number})+1+2^{p-2}\right]=\delta\left\{2^{p-2}\left[4(odd\ \text{number})+1\right]+1\right\}$, and by applying

Theorem 5.1 again, we get $\delta\left[2^p(odd\ \text{number})+1+2^{p-2}+2^{p-4}\right]$ and continue by induction, we get desire conclusion.

Definition. Define the following operations:

$$P(N)=6N-3,\ \ Q(N)=6N-1,\ \ R\left[2^p(odd\ number)+1\right]=2^p(odd\ number)+1+2^{p-2}.$$

Now we will use those operations to construct for the natural number N corresponding sequence of increasing numbers having the same index.

Example 5.2 For N=180, using operations $P,Q,\ and\ R,$ we get the sequence

$$180\xrightarrow{P}1077\ \left(2^2\cdot 269+1\right)\xrightarrow{+2^0}1078\xrightarrow{P}6465\ \left(2^6\cdot 101+1\right)\xrightarrow{+2^4+2^2+2^0}6486$$

$$\xrightarrow{P}38{,}913\left(2^{11}\cdot 19+1\right)\xrightarrow{+2^9+2^7+2^5+2^3+2^1}39595\xrightarrow{Q}237{,}569\ \left(2^{13}\cdot 29+1\right)$$

$$\xrightarrow{+2^{11}+2^9+2^7+2^5+2^3+2^1}240{,}299\xrightarrow{Q}1{,}441{,}793\ \left(2^{17}\cdot 11+1\right)\xrightarrow{+2^{15}+2^{13}+2^{11}+2^9+2^7+2^5+2^3+2^1}$$

$$1{,}485{,}483\ \xrightarrow{Q}8{,}912{,}897\ \left(2^{19}\cdot 17+1\right)\xrightarrow{+2^{17}+2^{15}+2^{13}+2^{11}+2^9+2^7+2^5+2^3+2^1}9{,}087{,}659$$

$$\xrightarrow{Q}54{,}525{,}953\ \left(2^{22}\cdot 13+1\right)\xrightarrow{+2^{20}+2^{18}+2^{16}+2^{14}+2^{12}+2^{10}+2^8+2^6+2^4+2^2+2^0}55{,}924{,}054$$

$$\xrightarrow{P}335{,}544{,}321\ \left(2^{26}\cdot 5+1\right)\xrightarrow{+2^{24}+2^{22}+2^{20}+2^{18}+2^{16}+2^{14}+2^{12}+2^{10}+2^8+2^6+2^4+2^2+2^0}357{,}913{,}942$$

$$\xrightarrow{P}2{,}147{,}483{,}649=2^{31}+1. \qquad (5.1)$$

Therefore, $\delta(180)=31.$

By the regular way, using operations A and B, we have:

$$180\xrightarrow{B}270\xrightarrow{B}405\xrightarrow{A}203\xrightarrow{A}102\xrightarrow{B}153\xrightarrow{A}77\xrightarrow{A}39\xrightarrow{A}20\xrightarrow{B}30$$

$$\xrightarrow{B}45\xrightarrow{A}23\xrightarrow{A}12\xrightarrow{B}18\xrightarrow{B}27\xrightarrow{A}14\xrightarrow{B}21\xrightarrow{A}11\xrightarrow{A}6\xrightarrow{B}9$$

$$\xrightarrow{A} 5 \xrightarrow{A} 3 \xrightarrow{A} 2. \qquad (5.2)$$

For the original Collatz Sequence for $N = 180 - 1 = 179$, we have

$$179 \to 538 \to 269 \to 808 \to 404 \to 202 \to 101 \to 304 \to 152 \to 76 \to 38 \to 19 \to 58 \to 29$$
$$\to 88 \to 44 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4$$
$$\to 2 \to 1. \qquad (5.3)$$

Let us consider the increasing numbers in our sequence (5.1):

$$1077 = 2^2 \cdot 269 + 1 \xrightarrow{A^2} 2^0 \cdot 269 + 1 = 270;$$

$$1078 \xrightarrow{B} 1617 = 2^2 \cdot 404 + 1 \xrightarrow{A^2} 2^0 \cdot 404 + 1 = 405;$$

$$6465 = 2^6 \cdot 101 + 1 \xrightarrow{A^6} 2^0 \cdot 101 + 1 = 102;$$

$$6486 \xrightarrow{B} 9729 = 2^6 \cdot 152 + 1 \xrightarrow{A^6} 2^0 \cdot 152 + 1 = 153;$$

$$38,913 = 2^{11} \cdot 19 + 1 \xrightarrow{A^{11}} 2^0 \cdot 19 + 1 = 20;$$

$$39,595 \xrightarrow{A} 19,798 \xrightarrow{B} 29,697 = 2^{10} \cdot 29 + 1 \xrightarrow{A^{10}} 2^0 \cdot 29 + 1 = 30;$$

$$237,569 = 2^{13} \cdot 29 + 1 \xrightarrow{A^{13}} 2^0 \cdot 29 + 1 = 30;$$

$$240,299 \xrightarrow{A} 120,150 \xrightarrow{B} 180,225 = 2^{12} \cdot 44 + 1 \xrightarrow{A^{12}} 2^0 \cdot 44 + 1 = 45;$$

$$1,441,793 = 2^{17} \cdot 11 + 1 \xrightarrow{A^{17}} 2^0 \cdot 11 + 1 = 12;$$

$$1,485,483 \xrightarrow{A} 742,742 \xrightarrow{B} 1,114,113 = 2^{16} \cdot 17 + 1 \xrightarrow{A^{16}} 2^0 \cdot 17 + 1 = 18;$$

$$8,912,897 = 2^{19} \cdot 17 + 1 \xrightarrow{A^{19}} 2^0 \cdot 17 + 1 = 18;$$

$$9,087,659 \xrightarrow{A} 4,543,830 \xrightarrow{B} 6,815,745 = 2^{18} \cdot 26 + 1 \xrightarrow{A^{18}} 2^0 \cdot 26 + 1 = 27;$$

$$54,525,953 = 2^{22} \cdot 13 + 1 \xrightarrow{A^{22}} 2^0 \cdot 13 + 1 = 14;$$

$$55,924,054 \xrightarrow{B} 83,886,081 = 2^{22} \cdot 20 + 1 \xrightarrow{A^{22}} 2^0 \cdot 20 + 1 = 21;$$

$$335,544,321 = 2^{26} \cdot 5 + 1 \xrightarrow{A^{26}} 2^0 \cdot 5 + 1 = 6;$$

$$357,913,942 \xrightarrow{B} 536,870,913 = 2^{26} \cdot 8 + 1 \xrightarrow{A^{26}} 2^0 \cdot 8 + 1 = 9;$$

$$2{,}147{,}483{,}649 = 2^{31} + 1 \xrightarrow{A^{31}} 2^0 + 1 = 2. \qquad (5.4)$$

We may see relations between sequences 5.1, 5.2, and 5.3. Sequence 5.3 converges to number 1 in 31steps. Sequence 5.2 converges to number 2 in 31 steps (operation A gives one step, but operation B is equivalent to 2 steps). So, both sequences have the same index $\delta = 31$, and we may see the relation between corresponding numbers of two sequences. Constructed sequence 5.1 contains increasing numbers, and all those numbers have the same index by Theorems 4.8, 4.10, and 5.1. Index of the last number of the sequence 5.1 is $\delta\left(2^{31}+1\right) = 31$.

The number $2^{31}+1$ is the greatest natural number that has index 31. Further increasing this number changes index, $2^{31}\cdot 1+1 \xrightarrow{R} 2^{31}\cdot 1+1+2^{29} = 2^{29}\cdot 5+1 \xrightarrow{A^{29}} 2^0\cdot 5+1 = 6$, therefore, $\delta\left(2^{31}+1+2^{29}\right) = 29+\delta(6) = 29+5 = 34$. That's why in the Theorem 5.1, (odd number) should be greater than 1. By the formulas 5.4 we recognize, that the increasing numbers in the constructed sequence 5.1 are related to corresponding numbers of the sequence 5.2. Collecting information in 5°, we may claim

Theorem 5.2 (Fundamental). For any natural number N > 2, we may construct the sequence, using operations P, Q, R, which satisfy the following conditions:

a) sequence is increasing and bounded,

b) all numbers in the sequence have the same index,

c) the biggest number in the sequence is 2^n+1, where $n = \delta(N)$.

Conclusions:

This proves the Collatz Conjecture for five out of six classes.

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