ISSN 2347-1921

On the growth of system of entire homogenous polynomials of several complex variables

MushtaqShakirA.Hussein(1) and Aseel Hameed Abed Sadaa(2)
(1)Department of Mathematics, College of science, Mustansiriha University
(2) Department of Mathematics, College of Basic Education, Mustansiriha University

ABSTRACT

In this paper we study the growth of entire functions represented by homogenous polynomials of two complex variables. The characterizations of their order and type have been obtained.

Keywords: homogenous polynomials; order and type; lower order and lower type.

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol .11, No.9

www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

1-INTRODUCTION

If $\upsilon:C^2 \to R^+$, be a real – valued function such that the following conditions hold:-

(i) $\upsilon(z+z') \leq \upsilon(z)+\upsilon(z')$ $\forall z,z' \in C^2$.

(ii) $\upsilon(\lambda z) \leq |\lambda|\upsilon(z) \forall \lambda \in C$.

(iii) $\upsilon(z)=0 \leftrightarrow z=0$, Then υ is a norm .

Let $f(z_1,z_2)=\sum_{m,n=0}^{\infty} p_{m,n}(z_1,z_2)$ ………(*) be the Taylor series expansion of $f(z_1,z_2)$ in terms of homogeneous polynomials $p_{m,n}(z_1,z_2)$: $C^2 \to C$ of degree (m+n). We have:

$M(r_1,r_2)=sup_{\upsilon(z_t)\leq \upsilon}|f(z_1,z_2)|$, t=1,2. $\upsilon=\max(\upsilon_1,\upsilon_2)$,Is the maximum modulus of $f(z_1,z_2)$ $\forall \upsilon_1,\upsilon_2 \in R^+$ with respect to the norm υ .

Define:

$C_{m,n}=sup_{\upsilon(z_t)\leq 1}|p_{m,n}(z_1,z_2)|$

The order, lower order and typeof entire functions aredefined respectively by:-

$$\rho=\lim_{r_1,r_2\to\infty}\sup\frac{\log\log M(r_1,r_2)}{\log(r_1,r_2)}$$

$$\lambda=\lim_{r_1,r_2\to\infty}\inf\frac{\log\log M(r_1,r_2)}{\log(r_1,r_2)}$$

$$T=\lim_{r_1,r_2\to\infty}\sup\frac{\log M(r_1,r_2)}{(r_1^{\rho}+r_2^{\rho})}$$

$$t=\lim_{r_1,r_2\to\infty}\inf\frac{\log M(r_1,r_2)}{(r_1^{\rho}+r_2^{\rho})}$$

In [2], D. Kumar,and K.N. Arora proved the following results

$$\rho=\lim_{m+n\to\infty}\sup\frac{\log[(m+n)\alpha_{m,n}]}{\log|c_{m,n}|^{\frac{-1}{m+n}}} \quad (1.1)$$

$$e\rho T=\lim_{m+n\to\infty}\sup\frac{(m+n)\alpha_{m,n}}{(c_{m,n})^{\frac{-\rho}{m+n}}} \quad (1.2)$$

Where

$$\alpha_{m,n}=\frac{(m^m n^n)^{\frac{1}{m+n}}}{(m+n)} \text{ if } m,n\geq 1$$

$$= 0 \qquad \text{if } m.n=0$$

Analogously, the lower order and lowertypeare defined by:

$$\lambda=\lim_{m+n\to\infty}\inf\frac{\log[(m+n)\alpha_{m,n}]}{\log|c_{m,n}|^{\frac{-1}{m+n}}} \quad (1.3)$$

$$e\rho t=\lim_{m+n\to\infty}\inf\frac{(m+n)\alpha_{m,n}}{(c_{m,n})^{\frac{-\rho}{m+n}}} \quad (1.4)$$

In this paper we consider have system of entire functions as follows:

$$f_i(z_1,z_2)=\sum_{m,n=0}^{\infty} p_{m,n}^{(i)}(z_1,z_2) , i=1,2,\ldots,k . \quad (1.5)$$

Then we obtain some relations between the function represented by (*) and the system of entire homogenous polynomials (1.5) and study the relations between the coefficients in Taylor expansion of entire homogenous polynomials and their type. Also we continue the work of H.H. khan and R.Ali [3] , Where they generalized and improve the results of R.K.Srvivastava , vinod Kumar [5] and S.SDalal [1] .

2-Main Results.

Theorem 2.1

Let $f_i(z_1,z_2)=\sum_{m,n=0}^{\infty} p_{m,n}^{(i)}(z_1,z_2)$ where i=1,2,……,k be (k) entire homogeneous polynomials of finite regular growth $\rho_1,\rho_2,\ldots,\rho_k$ respectively and

$\alpha_{m,n}^{(i)} \sim \alpha_{m,n}$ (2.1)

In order these functions have the same order is that satisfy the following condition

$$\text{Log}\ \{\frac{|C_{m,n}^{(i)}|^{\frac{-1}{m+n}}}{|C_{m,n}^{(i-1)}|^{\frac{-1}{m+n}}}\}=o(\log\{(m+n)\alpha_{m,n}\})$$

Proof

Since f_i, i=1, 2,…., k have regular growth then

$$\lim_{m+n\to\infty}\sup\frac{\log[(m+n)\alpha_{m,n}^{(i)}]}{\log(C_{m,n}^{(i)})^{\frac{-1}{m+n}}}=\rho_i=\lambda_i=\lim_{m+n\to\infty}\inf\frac{\log[(m+n)\alpha_{m,n}^{(i)}]}{\log(C_{m,n}^{(i)})^{\frac{-1}{m+n}}}$$

Since the functions $f_i(z_1,z_2)$ for i=1,…,k have the same order then $\rho=\rho_i=\lambda_i=\lambda$ for i=1,2,….,k

Or $\lim_{m+n\to\infty}\sup\frac{\log|C_{m,n}^{(i)}|^{\frac{-1}{m+n}}}{\log[(m+n)\alpha_{m,n}^{(i)}]}=\frac{1}{\rho}$

$\lim_{m+n\to\infty}\sup\frac{\log|C_{m,n}^{(i-1)}|^{\frac{-1}{m+n}}}{\log[(m+n)\alpha_{m,n}^{(i-1)}]}=\frac{1}{\rho}$, using condition (2.1) then

$\frac{\log|C_{m,n}^{(i)}|^{\frac{-1}{m+n}}-\log|C_{m,n}^{(i-1)}|^{\frac{-1}{m+n}}}{\log[(m+n)\alpha_{m,n}]}=0$, Hence

$\text{Log}\ \{\frac{|C_{m,n}^{(i)}|^{\frac{-1}{m+n}}}{|C_{m,n}^{(i-1)}|^{\frac{-1}{m+n}}}\}=o(\log\{(m+n)\alpha_{m,n}\})$ as $m+n\to\infty$.

Let us prove the converse

Let $f_i(z_1,z_2)$, i=1,2,….,k have orders ρ_i , i=1,2,….,k respectively then by using condition (2.1) we have

$$\frac{1}{\rho_i}-\frac{1}{\rho_{i-1}}=\lim_{m+n\to\infty}\sup\frac{\log|C_{m,n}^{(i)}|^{\frac{-1}{m+n}}}{\log[(m+n)\alpha_{m,n}]}-\lim_{m+n\to\infty}\sup\frac{\log|C_{m,n}^{(i-1)}|^{\frac{-1}{m+n}}}{\log[(m+n)\alpha_{m,n})]}$$

$\lim_{m+n\to\infty}\sup\frac{\log\{|C_{m,n}^{(i)}|^{\frac{-1}{m+n}}/|C_{m,n}^{(i-1)}|^{\frac{-1}{m+n}}\}}{\log[(m+n)\alpha_{m,n}]}=0$, Then $\rho_i=\rho_{i-1}$.

Theorem 2.2

Letthe system of homogenous polynomials $f_i(z_1,z_2)$ for i=1,2,….,k , have the regular type and the same order such that

(1) $\lim_{m+n\to\infty}\sup|C_{m,n}^{(i)}|^{\frac{-1}{m+n}}=0$ for i=1,2,…..,k

(2.2)

(2) $\alpha_{m,n}^{(i)}\sim\alpha_{m,n}$ for i=1,2,…..,k

In order these functions have the same type is that satisfy the condition

$$\text{Log}\ \{\frac{|C_{m,n}^{(i-1)}|}{|C_{m,n}^{(i)}|}\}=o(m+n)\ \text{as}\ m+n\to\infty$$

Proof:-

The functions $f_i(z_1,z_2)$, i=1,2,….,k have the regular type therefore

ISSN 2347-1921

$\lim_{m+n\to\infty}\inf\frac{1}{e\rho}\frac{(m+n)\alpha_{m,n}^{(i)}}{(C_{m,n}^{(i)})^{\frac{-\rho}{m+n}}}$= t_i=T_i= $\lim_{m+n\to\infty}\sup\frac{1}{e\rho}\frac{(m+n)\alpha_{m,n}^{(i)}}{(C_{m,n}^{(i)})^{\frac{-\rho}{m+n}}}$

$\lim_{m+n\to\infty}\inf\frac{1}{e\rho}\frac{(m+n)\alpha_{m,n}^{(i-1)}}{(C_{m,n}^{(i-1)})^{\frac{-\rho}{m+n}}}$ = t_{i-1}=T_{i-1}= $\lim_{m+n\to\infty}\sup\frac{1}{e\rho}\frac{(m+n)\alpha_{m,n}^{(i-1)}}{(C_{m,n}^{(i-1)})^{\frac{-\rho}{m+n}}}$

Let the functions $f_i(z_1,z_2)$, i=1,2,…..,k have the same type that is to say

$\lim_{m+n\to\infty}\sup\frac{1}{e\rho}\frac{(m+n)\alpha_{m,n}^{(i)}}{(C_{m,n}^{(i)})^{\frac{-\rho}{m+n}}}$ =T= $\lim_{m+n\to\infty}\sup\frac{1}{e\rho}\frac{(m+n)\alpha_{m,n}^{(i-1)}}{(C_{m,n}^{(i-1)})^{\frac{-\rho}{m+n}}}$

Hence if we take into account the condition (2) in (2.2) then we have:

$\lim_{m+n\to\infty}\frac{\rho}{m+n}\{log|c_{m,n}^{(i-1)}|-log|c_{m,n}^{(i)}|\ \}$=0

At last

Log $\{\frac{|C_{m,n}^{(i-1)}|}{|C_{m,n}^{(i)}|}\}$ =o (m+n) as m+n →∞

Let us prove the converse.

Let the functions $f_i(z_1,z_2)$, i=1,2,….,k have type T_i and T_{i-1} respectively .

Then log T_i – log T_{i-1} = ρ $\lim_{m+n\to\infty}\frac{1}{m+n}$ log $\{\frac{|C_{m,n}^{(i-1)}|}{|C_{m,n}^{(i)}|}\}$ =0 . Hence T_{i-1}=T_i .

Theorem 2.3

Let each function of system (1.5) be an entire homogenous polynomials of order ρ_i ($0<\rho_i<\infty$) and type T_i ($0< T_i< \infty$) ,t_i ($0< t_i<\infty$), i=1,…,k respectively, if

(1) $\alpha_{m,n}^{(i)}\sim\alpha_{m,n}$ for i=1,2,…..,k

(2) $C_{m,n}\sim\prod_{i=1}^{k}(c_{m,n}^{(i)})^{r_i}$

(3) $\frac{(m+n)\alpha_{m,n}}{(c_{m,n})^{\frac{-\rho}{m+n}}}\sim\prod_{i=1}^{k}[\frac{(m+n)\alpha_{m,n}^{(i)}}{(c_{m,n}^{(i)})^{-\frac{\rho}{m+n}}}]^{r_i}$ where $r_i>0$, $\sum_{i=1}^{k}r_i$ = 1(2.3)

Then f (z_1, z_2) is entire function and $\prod_{i=1}^{k}(t_i)^{r_i}\le t\le T\le\prod_{i=1}^{k}(T_i)^{r_i}$, Where T, t are type and lower type of f (z_1, z_2) respectively.

Proof

It can be easily seen [4] that the necessary and sufficient condition for $f_i(z_1,z_2)$ to represent an entire polynomialsof two complex variables (z_1 , z_2)is that :

$\lim_{m+n\to\infty}(c_{m,n}^{(i)})^{\frac{1}{m+n}}$ =0 for i=1,2,…..,k

Also from conditions (1, 2) in (2.3) we get

$\lim_{m+n\to\infty}\sup(\frac{c_{m,n}}{\alpha_{m,n}})^{\frac{1}{m+n}}\le\prod_{i=1}^{k}\lim_{m+n\to\infty}\sup[(\frac{c_{m,n}^{(i)}}{\alpha_{m,n}^{(i)}})^{r_i}]^{\frac{1}{m+n}}$Hence f ($z_1$, z_2) is an entire polynomial .

From (1.2) we have

$\frac{1}{e\rho}[\frac{(m+n)\alpha_{m,n}^{(i)}}{(c_{m,n}^{(i)})^{-\frac{\rho}{m+n}}}]<T_i+\varepsilon$ for i=1,2,…..,k

Hence $\prod_{i=1}^{k}[\frac{1}{e\rho}\{\frac{(m+n)\alpha_{m,n}^{(i)}}{\left(c_{m,n}^{(i)}\right)^{-\frac{\rho}{m+n}}}\}]^{r_i} < \prod_{i=1}^{k}(T_i+\varepsilon)^{r_i}$ (2.4)

Similarly $\prod_{i=1}^{k}(t_i-\varepsilon)^{r_i} < \prod_{i=1}^{k}[\frac{1}{e\rho}\{\frac{(m+n)\alpha_{m,n}^{(i)}}{\left(c_{m,n}^{(i)}\right)^{-\frac{\rho}{m+n}}}\}]^{r_i} < \prod_{i=1}^{k}(T_i+\varepsilon)^{r_i}$

Taking into account the condition (3) in (2.3)

$$\prod_{i=1}^{k}(t_i-\varepsilon)^{r_i} < \prod_{i=1}^{k}[\frac{1}{e\rho}\{\frac{(m+n)\alpha_{m,n}}{(c_{m,n})^{-\frac{\rho}{m+n}}}\}]^{r_i} < \prod_{i=1}^{k}(T_i+\varepsilon)^{r_i}$$

Passing to limits as m+n →∞ we obtain

$\prod_{i=1}^{k}(t_i)^{r_i} \le t \le T \le \prod_{i=1}^{k}(T_i)^{r_i}$. Thus the theorem is proved.

Theorem 2.4

Let $f_1(z_1,z_2)=\sum_{m,n=0}^{\infty} p_{m,n}^{(1)}(z_1,z_2)$, $f_2(z_1,z_2)=\sum_{m,n=0}^{\infty} p_{m,n}^{(2)}(z_1,z_2)$ be two entire polynomials of finite non –zero orders

ρ_1,ρ_2 and finite non-zero types T_1,T_2 respectively then the function

$f(z_1,z_2)=\sum_{m,n=0}^{\infty} p_{m,n}(z_1,z_2)$ with

$$c_{m,n} \sim (a_{m,n}\,.\,b_{m,n})^{\frac{1}{2}},\quad \alpha_{m,n}^{(i)} \sim \alpha_{m,n} \tag{2.5}$$

Where $c_{m,n}=sup_{\upsilon(z_t)\le 1}\left|p_{m,n}(z_1,z_2)\right|$, $a_{m,n}=sup_{\upsilon(z_t)\le 1}\left|p_{m,n}^{(1)}(z_1,z_2)\right|$ and

$b_{m,n}=sup_{\upsilon(z_t)\le 1}\left|p_{m,n}^{(2)}(z_1,z_2)\right|$ is an entire function such that

$(\rho T)^{\frac{2}{\rho}} \le (\rho_1 T_1)^{\frac{1}{2}\rho_1}.(\rho_2 T_2)^{\frac{1}{2}\rho_2}$, Where ρ and T are order and type of $f(z_1,z_2)$ respectively and

$$2/\rho = \frac{1}{\rho_1} + \frac{1}{\rho_2}. \tag{2.6}$$

Proof

We can prove as the proof of theorem 2.3 that $f(z_1,z_2)$ is an entire function where

$c_{m,n} \sim (a_{m,n} b_{m,n})^{\frac{1}{2}}$, and $\alpha_{m,n}^{(i)} \sim \alpha_{m,n}$. Further, using (1.2) for the function $f_1(z_1,z_2)$, $f_2(z_1,z_2)$, we have

$$\lim_{m+n\to\infty} sup\,\{((m+n)\alpha_{m,n}^{(1)})^{\frac{1}{\rho_1}} * (a_{m,n})^{\frac{1}{m+n}}\}^{\rho_1} = e\rho_1 T_1 \tag{2.7}$$

$$\lim_{m+n\to\infty} sup\,\{(m+n)\alpha_{m,n}^{(2)})^{\frac{1}{\rho_2}} * (b_{m,n})^{\frac{1}{m+n}}\}^{\rho_2} = e\rho_2 T_2 \tag{2.8}$$

From (2.7) and (2.8), we get for an arbitrary ε>0.

$((m+n).\alpha_{m,n}^{(1)})^{\frac{1}{\rho_1}} * (a_{m,n})^{\frac{1}{m+n}} < (e\rho_1(T_1+\varepsilon))^{\frac{1}{\rho_1}}$,for $m+n>k_1$.

$((m+n).\alpha_{m,n}^{(2)})^{\frac{1}{\rho_2}} * (b_{m,n})^{\frac{1}{m+n}} < (e\rho_2(T_2+\varepsilon))^{\frac{1}{\rho_2}}$, for $m+n>k_2$.

Thus for $(m+n)>k=\max(k_1,k_2)$ and $2/\rho = \frac{1}{\rho_1} + \frac{1}{\rho_2}$, and using the condition $\alpha_{m,n}^{(i)} \sim \alpha_{m,n}$ we have

$$[((m+n)\alpha_{m,n})^{\frac{1}{\rho}}((a_{m,n}.b_{m,n})^{\frac{1}{2}})^{\frac{1}{m+n}}]^2 < (e\rho_1(T_1+\varepsilon))^{\frac{1}{\rho_1}}(e\rho_2(T_2+\varepsilon))^{\frac{1}{\rho_2}}$$

Therefore, if $c_{m,n} \sim (a_{m,n} b_{m,n})^{\frac{1}{2}}$, we have obtain

$$\lim_{m+n\to\infty} \sup[((m+n)\alpha_{m,n})^{\frac{1}{\rho}}(c_{m,n})^{\frac{1}{m+n}}] < (e\rho_1 T_1)^{\frac{1}{2}\rho_1}(e\rho_2 T_2)^{\frac{1}{2}\rho_2}$$

Or

$(e\rho T)^{\frac{1}{\rho}} \leq (e\ \rho_1\ T_1)^{\frac{1}{2}\rho_1}.\ (\ e\ \rho_2\ T_2)^{\frac{1}{2}\rho_2}$,Where ρ and T are order and type of f (z_1,z_2) respectively, Hence

$(\rho T)^{\frac{2}{\rho}} \leq (\rho_1\ T_1)^{\rho_1}.\ (\rho_2\ T_2)^{\rho_2}.$

Corollary

$f_i(z_1,z_2) = \sum_{m,n=0}^{\infty} p_{m,n}^{(i)}\ (z_1,z_2\ )$, i=1,2,….,k be (k) entire polynomials of finite non-zero orders $\rho_1, \rho_2, \ldots\ldots, \rho_k$, and finite non-zero types $T_1,T_2,\ldots\ldots\ldots,T_k$ respectively ,Then the function

$f(z_1,z_2)=\sum_{m,n=0}^{\infty} p_{m,n}\ (z_1,z_2\ )$ with

$c_{m,n} \sim \prod_{i=1}^{k} (a_{m,n}^{(i)})^{\frac{1}{k}}$Is an entire function such that

$(\rho T)^{\frac{k}{\rho}} < \prod_{i=1}^{k} (\rho_i T_i)^{\frac{1}{\rho_i}}$,Where ρ and T are order and type of f (z_1,z_2) respectively and

$\frac{k}{\rho}=\sum_{i=1}^{k} 1/\rho_i$

References.

[1] S.S.Dalal. "On the order and type of integral functions of several complex variables", Indian math soc. 33(1969), 215-220.

[2] D. Kumar and K.N.Arora. "On the (p,q) order and (p,q) type of homogenous polynomials of two complex variables" , math Sci, Res J.S no. 7 (2005).177-189.

[3] H.H .Khan and R.Ali,On the growth of entire functions of several complex variables , Int.Journal of math analysis vol.5,no.4, pp.2141-2146.

[4] P.lelog and L.Gruman, "Entire functions of several complex variables, a series of complex variables studies in mathematics, 282, Springer verlegr Berilir 1986.

[5] R.K.Srivastava andKumar "Onthe order and type of integral functions of several complex variables, Comp .mathcase 2(1968), 161-166.