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On the solutions of nonlinear second order differential equation with finite delay

H. L. Tidke

Department of Mathematics,
North Maharashtra University, Jalgaon-425 001, India
tharibhau@gmail.com

ABSTRACT

The main aim of this paper is to study the approximate solutions, uniqueness and other properties of solutions of nonlinear second order differential equation with given initial values. A variant of the well known Gronwall-Bellman integral inequality with explicit estimate is used to establish the results.

Keywords:

Approximate solutions, second order; differential equation; integral inequality; explicit estimate; closeness of solutions; uniqueness and continuous dependence.

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1. INTRODUCTION

Let $\mathbb{R}^n$ be the real n −dimensional Euclidean space with the corresponding norm $|\cdot|$. Let $\mathbb{R}_+ = [0,\infty)$ be a subset of real numbers. Motivated by the work of ([3,4,5]), in this paper we consider the following differential equation of the form:

$$x''(t) = f(t, x(t), x'(t), x(t-1)) \quad (1.1)$$

for $t \in \mathbb{R}_+$ with the conditions

$$x(t-1) = \phi(t), \quad (0 \le t < 1) \quad (1.2)$$

$$x(0) = x_0, \ \ x'(0) = \overline{x_0}\,, \quad (1.3)$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $\phi(t)$ is a continuous function for which $\lim_{t\to 1-0} \phi(t)$ exists.

When dealing with the equation (1.1) with (1.2)-(1.3), the basic questions to be answered are: (i) under what conditions the systems under considerations have solutions? (ii) how can we find the solutions or closely approximate them? (iii) what are their nature?. The study of such questions gives rise to new results and need a fresh outlook for handling such problems for (1.1)-(1.3). The objective of the present paperis to investigate new estimates on the difference between two approximate solutions of equation (1.1) and convergence properties on solutions of approximate solutions.Subsequently some authors have been studied the problems of existence, uniqueness and other properties of solutions of special forms or the equation (1.1) by using different techniques, see [5,7-10, 13] and the references cited therein. We also refer some papers and monographs [1,2,6,11,12] and the references given therein. Our general formulation of (1.1)-{(1.3) is anattempt to generalize the results of [5].

The paper is organized as follows. In section 2, we present the preliminaries and main result of existence of approximate solutions and uniqueness of the solutions. Finally, in Section 3 deals with closeness and convergence of solutions and also we discuss results on continuous dependence of solutions on initial data, functions involved therein and parameters.

2. MAIN RESULT

Before proceeding to the statement of our main results, we shall set forth some preliminaries that will be used in our subsequent discussion.

Definition 2.1. Let $x_i(t) \in C\,(\mathbb{R}_+, \mathbb{R}^n\,)\,\mathbb{R}(i\, =\, 1, 2)$ be functions such that $x_i''(t)$ exist for $t \in \mathbb{R}_+$ and satisfy the inequalities

$$\left|x_i''(t) - f\left(t, x_i(t), x_i'(t), x_i(t-1)\right)\right| \le \epsilon_i, \quad (2.1)$$

for given constants $\epsilon_i \ge 0$ $(i = 1,2)$, where it is assumed that the initial conditions

$$x_i(t-1) = \phi_i(t), \quad (0 \le t < 1) \quad (2.2)$$

$$x_i(0) = x_i^*, \quad x_i'(0) = \overline{x_i^*} \quad (2.3)$$

for $i = 1,2$ are fulfilled and $\phi_i(t)$ are continuous functions for which $\lim_{t\to 1-0} \phi_i(t)$ exist. Then we call $x_i(t)$ $(i = 1,2)$ theϵ_i −approximate solutions with respect to the equation (1.1) ith initial conditions (2.2)-(2.3).

We require the following Lemma known as the Gronwall-Bellman inequality in our further discussion.

Lemma 2.2 **([2],p. 12)** *Let $u(t), n(t), e(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $n(t)$ be nondecreasing on $\mathbb{R}_+$. If*

$$u(t) \le n(t) + \int_0^t e(s)u(s)ds, \quad (2.4)$$

for$t \in \mathbb{R}_+$, then

$$u(t) \le n(t)\exp(\int_0^t e(s)ds), \quad (2.5)$$

for$t \in \mathbb{R}_+$.

The following theorem estimates the difference between the two approximate solutions of equation (1.1).

Theorem 2.3 *Suppose that the function f in equation* (1.1) *satisfies the condition*

$$|f(t,x,y,z) - f(t,\bar{x},\bar{y},\bar{z})| \le p(t)[|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}|], \quad (2.6)$$

where $p(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. Let $x_i(t)$ $(i = 1,2)$ be respectively ϵ_i −approximate solutions of equation (1.1) *on* $\mathbb{R}_+$ *with* (2.2)-(2.3) *such that*

$$|x_1^* - x_2^*| \le \delta, \quad \left|\overline{x_1^*} - \overline{x_2^*}\right| \le \bar{\delta}, \quad (2.7)$$

where $\delta \ge 0$ and $\bar{\delta} \ge 0$ are constants. Then

$$|x_1(t)-x_2(t)|+|x_1'(t)-x_2'(t)|\le n(t)exp\left(\int_0^t (t-s+1)p(s)\,ds\right),\ (2.8)$$

for $0\le t<1$ and

$$|x_1(t)-x_2(t)|+|x_1'(t)-x_2'(t)|\le n(t)exp\left(\int_0^t [2(t-s)+1][p(s+1)+p(s)]\,ds\right),\ (2.9)$$

for $1\le t<\infty$, where $n(t)=(\epsilon_1+\epsilon_2)\left[t+\frac{t^2}{2}\right]+\delta+(t+1)\bar{\delta}+\int_0^1(2-s)p(s)[|\phi_1(s)-\phi_2(s)|]ds.$

Proof. Since $x_i(t)$ $(i = 1,2)$ for $t\in\mathbb{R}_+$ are respectively ϵ_i-approximate solutions of equation (1.1) with (2.2)-(2.3), we have (2.1). By taking $t = \tau$ in (2.1) and integrating both sides with respect to τ from 0 to t, we have

$$\begin{aligned}\epsilon_i t &\ge \int_0^t \left|x_i''(\tau)-f(\tau,x_i(\tau),x_i'(\tau),x_i(\tau-1))\right|d\tau\\ &\ge \left|\int_0^t\{x_i''(\tau)-f(\tau,x_i(\tau),x_i'(\tau),x_i(\tau-1))\}d\tau\right|\\ &= \left|\left\{x_i'(t)-\overline{x_i^*}-\int_0^t f(\tau,x_i(\tau),x_i'(\tau),x_i(\tau-1))d\tau\right\}\right|,\end{aligned} \quad (2.10)$$

for $i=1,2$.

By taking t = s in (2.10) and integrating both sides with respect to s from 0 to t, we have

$$\begin{aligned}\epsilon_i\frac{t^2}{2} &\ge \int_0^t\left|\left\{x_i'(s)-\overline{x_i^*}-\int_0^s f(\tau,x_i(\tau),x_i'(\tau),x_i(\tau-1))d\tau\right\}\right|ds\\ &\ge \left|\int_0^t\left\{x_i'(s)-\overline{x_i^*}-\int_0^s f(\tau,x_i(\tau),x_i'(\tau),x_i(\tau-1))d\tau\right\}ds\right|\\ &= \left|\left\{x_i(t)-[x_i^*+\overline{x_i^*}t]-\int_0^t(t-s)f(s,x_i(s),x_i'(s),x_i(s-1))ds\right\}\right|,\end{aligned} \quad (2.11)$$

for $i=1,2$. From(2.10), (2.11) and using the elementary inequalities

$$|v-z|\le|v|+|z|,\ |v|-|z|\le|v-z|, \quad (2.12)$$

we observe that

$$\begin{aligned}(\epsilon_1+\epsilon_2)t &\ge \left|\left\{x_1'(t)-\overline{x_1^*}-\int_0^t f(s,x_1(s),x_1'(s),x_1(s-1))ds\right\}\right|\\ &+\left|\left\{x_2'(t)-\overline{x_2^*}-\int_0^t f(s,x_2(s),x_2'(s),x_2(s-1))ds\right\}\right|\\ &\ge\left|\left\{x_1'(t)-\overline{x_1^*}-\int_0^t f(s,x_1(s),x_1'(s),x_1(s-1))ds\right\}\right.\\ &\left.-\left\{x_2'(t)-\overline{x_2^*}-\int_0^t f(s,x_2(s),x_2'(s),x_2(s-1))ds\right\}\right|\\ &\ge|x_1'(t)-x_2'(t)|-\left|\overline{x_1^*}-\overline{x_2^*}\right|\\ &-\int_0^t\left|f(s,x_1(s),x_1'(s),x_1(s-1))-f(s,x_2(s),x_2'(s),x_2(s-1))\right|ds\end{aligned} \quad (2.13)$$

and

$$\begin{aligned}(\epsilon_1+\epsilon_2)\frac{t^2}{2} &\ge \left|\left\{x_1(t)-[x_1^*+\overline{x_1^*}t]-\int_0^t f(s,x_1(s),x_1'(s),x_1(s-1))ds\right\}\right|\\ &+\left|\left\{x_2(t)-[x_2^*+\overline{x_2^*}t]-\int_0^t f(s,x_2(s),x_2'(s),x_2(s-1))ds\right\}\right|\\ &\ge\left|\left\{x_1(t)-[x_1^*+\overline{x_1^*}t]-\int_0^t f(s,x_1(s),x_1'(s),x_1(s-1))ds\right\}\right.\\ &\left.-\left\{x_2(t)-[x_2^*+\overline{x_2^*}t]-\int_0^t f(s,x_2(s),x_2'(s),x_2(s-1))ds\right\}\right|\\ &\ge|x_1(t)-x_2(t)|-|[x_1^*+\overline{\overline{x_1^*}}t]-[x_2^*+\overline{x_2^*}t]|\\ &-\int_0^t(t-s)\left|f(s,x_1(s),x_1'(s),x_1(s-1))-f(s,x_2(s),x_2'(s),x_2(s-1))\right|ds.\end{aligned} \quad (2.14)$$

Let $u(t)=|x_1(t)-x_2(t)|+|x_1'(t)-x_2'(t)|,\quad t\in\mathbb{R}_+$. From (2.13) and (2.14) and using the hypotheses, we get

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$$u(t) \le (\epsilon_1+\epsilon_2)\frac{t^2}{2} + |x_1^* - x_2^*| + |\overline{x_1^*} - \overline{x_2^*}|t + (\epsilon_1+\epsilon_2)t + |\overline{x_1^*} - \overline{x_2^*}|$$

$$+ \int_0^t (t-s+1)\left|f(s,x_1(s),x_1'(s),x_1(s-1)) - f(s,x_2(s),x_2'(s),x_2(s-1))\right| ds$$

$$\le (\epsilon_1+\epsilon_2)\left[\frac{t^2}{2} + t\right] + \delta + (t+1)\bar{\delta}$$

$$+ \int_0^t (t-s+1)p(s)[|x_1(s),-x_2(s)| + |x_1'(s) - x_2'(s)| + |x_1(s-1) - x_2(s-1)|]\, ds$$

$$\le (\epsilon_1+\epsilon_2)\left[t + \frac{t^2}{2}\right] + \delta + (t+1)\bar{\delta}$$

$$+ \int_0^t (t-s+1)p(s)[u(s) + |x_1(s-1) - x_2(s-1)|]\, ds. \tag{2.15}$$

We consider the following two cases

Case I: $0 \le t < 1$. From (2.15) and hypotheses, we observe that

$$u(t) \le (\epsilon_1+\epsilon_2)\left[t + \frac{t^2}{2}\right] + \delta + (t+1)\bar{\delta} + \int_0^t (t-s+1)p(s)[u(s) + |\phi_1(s) - \phi_2(s)|]\, ds$$

$$\le (\epsilon_1+\epsilon_2)\left[t + \frac{t^2}{2}\right] + \delta + (t+1)\bar{\delta} + \int_0^1 (2-s)p(s)[|\phi_1(s) - \phi_2(s)|] ds$$

$$+ \int_0^t (t-s+1)p(s)u(s)\, ds$$

$$\le n(t) + \int_0^t (t-s+1)p(s)u(s)\, ds. \tag{2.16}$$

Clearly $n(t)$ is nondecreasing in t. Now an application of Lemma 2.2 to (2.16) yields (2.8).

Case II: $1 \le t < \infty$. From (2.15) and hypotheses, we observe that

$$u(t) \le (\epsilon_1+\epsilon_2)\left[t + \frac{t^2}{2}\right] + \delta + (t+1)\bar{\delta} + \int_0^t (t-s+1)p(s)[u(s) + |x_1(s-1) - x_2(s-1)|]\, ds$$

$$\le (\epsilon_1+\epsilon_2)\left[t + \frac{t^2}{2}\right] + \delta + (t+1)\bar{\delta} + \int_0^1 (2-s)p(s)[|\phi_1(s) - \phi_2(s)|] ds$$

$$+ \int_0^t (t-s+1)p(s)u(s)\, ds + \int_1^t (t-s+1)p(s)[u(s) + |x_1(s-1) - x_2(s-1)|]\, ds$$

$$\le n(t) + \int_0^t (t-s+1)p(s)u(s)\, ds + I_1, \tag{2.17}$$

where

$$I_1 = \int_1^t (t-s+1)p(s)[|x_1(s-1) - x_2(s-1)|]\, ds. \tag{2.18}$$

By making the change of variable $s-1=\tau$, then from (2.18), we obtain

$$I_1 = \int_0^{t-1} (t-\tau)p(\tau+1)[|x_1(\tau) - x_2(\tau)|]\, d\tau$$

$\le \int_0^t (t-\tau)\ p(\tau+1)u(\tau)d\tau.$

Therefore, we have

$$I_1 \le \int_0^t (t-s)\ p(s+1)u(s)ds. \tag{2.19}$$

Using (2.19) in (2.17), we get

$$u(t) \le n(t) + \int_0^t [2(t-s)+1][p(s) + p(s+1)]u(s)ds. \tag{2.20}$$

Now an application of Lemma 2.2 to (2.20) yields (2.9). This completes the proof of the theorem.

Remark 2.4:

(i) We note that the estimates obtained in (2.8) and (2.9) yield not only the bound on the difference between the two approximate solutions of equation (1.1) but also the bound on the difference between their derivatives.

(ii) If $x_1(t)$ is a solution of equation (1.1) with $x_1(0) = x_1^*,\ x_1'(0) = \overline{x_1^*}$, then we have $\epsilon_1 = 0$ and from (2.8) and (2.9), we see that $x_2(t) \to x_1(t)$ as $\epsilon_2 \to 0$ and $\delta \to 0,\ \bar{\delta} \to 0$ and $\phi_2(t) \to \phi_1(t)$ for $0 \le t < 1$.

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(iii) Moreover, if we put (a) $\epsilon_1 = \epsilon_2 = 0$ and $x_1^* = x_2^*$, $\overline{x_1^*} = \overline{x_2^*}$, $\phi_1(t) = \phi_2(t)$ $(0 \le t < 1)$ in (2.8) and (2.9), then the uniqueness of solutions of equation (1.1) is established and (b) $\epsilon_1 = \epsilon_2 = 0$ in (2.8) and (2.9), then we get the bound which shows the dependency of solutions of equation (1.1) on given initial values.

2. CLOSENESS OF SOLUTIONS

In this section we study the continuous dependence of solutions to (1.1)–(1.3) on the function f and the closeness of the solutions of following equations (3.1)–(3.3).

Consider the initial value problem

$$y''(t) = \overline{f}(t, y(t), y'(t), y(t-1)) \quad (3.1)$$

for $t \in \mathbb{R}_+$ with the conditions

$$y(t-1) = \psi(t), \quad (0 \le t < 1) \quad (3.2)$$

$$y(0) = y_0, \quad y'(0) = \overline{y_0}, \quad (3.3)$$

where $\overline{f} \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $\psi(t)$ is a continuous function for which $\lim_{t \to 1-0} \psi(t)$ exists.

Next theorem deals with the closeness of the solutions of initial value prblems (1.1)–(1.3) and (3.1)–(3.3).

Theorem 3.1 *Suppose that the function f in equation (1.1) satisfies the condition (2.6) and there exist constants $\overline{\epsilon_0} \ge 0, \delta_1 \ge 0$, $\delta_2 \ge 0$ such that*

$$\left|f(t,x,y,z) - \overline{f}(t,x,y,z)\right| \le \overline{\epsilon}, \quad (3.4)$$

$$|x_0 - y_0| \le \delta_1, \quad |\overline{x_0} - \overline{y_0}| \le \delta_2, \quad (3.5)$$

Let $x(t)$ and $y(t)$ be respectively, solutions of the prblems (1.1)–(1.3) and (3.1)–(3.3) on $\mathbb{R}_+$. Then

$$|x(t) - x(t)| + |x'(t) - y'(t)| \le d(t)\, exp\left(\int_0^t (t-s+1)p(s)\, ds\right), \quad (3.6)$$

for $0 \le t < 1$ and

$$|x(t) - x(t)| + |x'(t) - y'(t)| \le d(t)\, exp\left(\int_0^t [2(t-s)+1][p(s+1) + p(s)]\, ds\right), \quad (3.7)$$

for $1 \le t < \infty$, where

$$d(t) = \delta_1 + \delta_2(1+t) + \left(\frac{t^2}{2} + t\right)\overline{\epsilon} + \int_0^1 (2-s)p(s)|\phi(s) - \psi(s)|\, ds.$$

Proof. Let $v(t) = |x(t) - y(t)| + |x'(t) - y'(t)|$, $t \in \mathbb{R}_+$. Using the facts that $x(t)$ and $y(t)$ are the solutions of IVP (1.1)-(1.3) and IVP (3.1)-(3.3) and hypotheses, we observe that

$$\begin{aligned}
v(t) \le\ & |x_0 - y_0| + |\overline{x_0} - \overline{y_0}|t \\
& + \int_0^t (t-s)\left|f(s, x(s), x'(s), x(s-1)) - \overline{f}(s, y(s), y'(s), y(s-1))\right| ds \\
& + |\overline{x_0} - \overline{y_0}| + \int_0^t \left|f(s, x(s), x'(s), x(s-1)) - \overline{f}(s, y(s), y'(s), y(s-1))\right| ds \\
\le\ & \delta_1 + \delta_2(1+t) \\
& + \int_0^t (t-s)\left|f(s, x(s), x'(s), x(s-1)) - f(s, y(s), y'(s), y(s-1))\right| ds \\
& + \int_0^t (t-s)\left|f(s, y(s), y'(s), y(s-1)) - \overline{f}(s, y(s), y'(s), y(s-1))\right| ds \\
& + \int_0^t \left|f(s, x(s), x'(s), x(s-1)) - f(s, y(s), y'(s), y(s-1))\right| ds \\
& + \int_0^t \left|f(s, y(s), y'(s), y(s-1)) - \overline{f}(s, y(s), y'(s), y(s-1))\right| ds \\
\le\ & \delta_1 + \delta_2(1+t) + \int_0^t (t-s)\overline{\epsilon}\, ds\ ) + \int_0^t \overline{\epsilon}\, ds \\
& + \int_0^t (t-s)p(s)[v(s) + |x(s-1) - y(s-1)|]\, ds + \int_0^t p(s)[v(s) + |x(s-1) - y(s-1)|]\, ds \\
\le\ & \delta_1 + \delta_2(1+t) + \left(\frac{t^2}{2} + t\right)\overline{\epsilon} + \int_0^t (t-s+1)p(s)[v(s) + |x(s-1) - y(s-1)|]\, ds. \quad (3.8)
\end{aligned}$$

We consider the following two cases

Case I: $0 \le t < 1$. From (3.8) and hypotheses, we observe that

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$$v(t) \le \delta_1 + \delta_2(1+t) + \left(\frac{t^2}{2}+t\right)\overline{\epsilon} + \int_0^1 (2-s)p(s)|\phi(s)-\psi(s)|\,ds$$

$$+\int_0^t (t-s+1)p(s)v(s)ds$$

$$\le d(t) + \int_0^t (t-s+1)p(s)v(s)ds. \quad (3.9)$$

Clearly $d(t)$ is nondecreasing in t. Now an application of Lemma 2.2 to (3.9) yields (3.6).

Case II: $1 \le t < \infty$. From (3.8) and hypotheses, we observe that

$$v(t) \le \delta_1 + \delta_2(1+t) + (\frac{t^2}{2}+t)\overline{\epsilon} + \int_0^t (t-s+1)p(s)[v(s) + |x(s-1) - y(s-1)|]\,ds$$

$$\le d(t) + \int_0^t (t-s+1)p(s)v(s)\,ds + \int_1^t (t-s+1)p(s)|x(s-1)-y(s-1)|\,ds$$

$$\le d(t) + \int_0^t (t-s+1)p(s)v(s)\,ds + I_2, \quad (3.10)$$

where

$$I_2 = \int_1^t (t-s+1)p(s)[|x(s-1)-y(s-1)|]\,ds. \quad (3.11)$$

By making the change of variable $s-1=\tau$, then from (3.11), we obtain

$$I_2 = \int_0^{t-1} (t-\tau)p(\tau+1)[|x(\tau)-y(\tau)|]\,d\tau$$

$$\le \int_0^t (t-\tau)\ p(\tau+1)v(\tau)d\tau.$$

Therefore, we have

$$I_2 \le \int_0^t (t-s)\ p(s+1)v(s)ds. \quad (3.12)$$

Using (3.12) in (3.10), it is easy to observe that

$$v(t) \le d(t) + \int_0^t [2(t-s)+1][p(s) + \ p(s+1)]v(s)ds. \quad (3.13)$$

Now an application of Lemma 2.2 to (3.13) yields (3.7). This completes the proof of the theorem.

Remark 3.2: The result given in Theorem 3.1 relates the solutions of IVP (1.1)-(1.3) and of IVP (3.1)-(3.3) in the sense that if f is close to $\overline{f}$, x_0 is close to y_0 and $\overline{x_0}$is close to $\overline{y_0}$, then not only the solutions of IVP (1.1)-(1.3) and of IVP (3.1)-(3.3) are close to each other, but also depend continuously on the functions involved therein .

Consider the initial value problem

$$y''(t) = f_k(t, y(t), y'(t), y(t-1)) \quad (3.14)$$

for $t \in \mathbb{R}_+$ with the conditions

$$y_k(t-1) = \psi_k(t), \quad (0 \le t < 1) \quad (3.15)$$

$$y_k(0) = c_k, \ \ y_k'(0) = \overline{c_k}\ , \quad (3.16)$$

for $k = 1,2,\cdots$, where $f_k \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $\psi_k(t)$ are continuous functions for which $\lim_{t\to 1-0} \psi_k(t)$ exist.

As an immediate consequence of Theorem 3.1, we have the following corollary.

Corollary 3.3 *Suppose that the function f in equation* (1.1) *satisfies the condition* (2.6) *and there exist constants* $\epsilon_k \ge 0, \delta_k \ge 0,\ \overline{\delta_k} \ge 0\ \ (k = 1,2,\cdots)$ *such that*

$$|f(t,x,y,z) - f_k(t,x,y,z)| \le \epsilon_k, \quad (3.17)$$

$$|x_0 - c_k| \le \delta_k, \quad |\overline{x_0} - \overline{c_k}| \le \overline{\delta_k}, \quad (3.18)$$

with $\epsilon_k \to 0$ *and* $\delta_k \to 0,\ \overline{\delta_k} \to 0$ *as* $k \to \infty$*. If* $y_k(t)\ (k = 1,2,\cdots)$ *and* $x(t)$ *are respectively the solutions of the prblems*(3.14)-(3.16) *and*(1.1)–(1.3), *then*$y_k(t) \to x(t)$*as* $k \to \infty$ *on* $\mathbb{R}_+$.

Proof. For$k = 1,2,\cdots$, the conditions of Theorem 3.1 hold. As an application of of Theorem 3.1 and Lemma 2.2 yields

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$$|y_k(t) - x(t)| + |y_k'(t) - x'(t)| \leq d_k(t)\, exp\left(\int_0^t (t-s+1)p(s)\, ds\right), \quad (3.19)$$

for $0 \leq t < 1$ and

$$|y_k(t) - x(t)| + |y_k'(t) - x'(t)| \leq d_k(t)\, exp\left(\int_0^t [2(t-s)+1][p(s+1)+p(s)]\, ds\right), \quad (3.20)$$

for $1 \leq t < \infty$, where

$$d_k(t) = \delta_k + \overline{\delta_k}(1+t) + \left(\frac{t^2}{2} + t\right)\epsilon_k + \int_0^1 (2-s)p(s)|\phi(s) - \psi_k(s)|\, ds.$$

The required results follows from (3.19) and (3.20). It follows that the problem (1.1)–(1.3) depends continuously on the functions involved therein. This completes the proof.

Remark 3.4: The result obtained in Corollary 3.3 provide sufficient conditions that ensures solutions of IVPs (4.14)–(4.16) will converge to the solutions of IVP (1.1)–(1.3).

We consider the initial value problem

$$x''(t) = F(t, x(t), x'(t), x(t-1), \mu_1) \quad (3.21)$$

$$x''(t) = F(t, x(t), x'(t), x(t-1), \mu_2) \quad (3.22)$$

for $t \in \mathbb{R}_+$ with the conditions (1.2)-(1.3), where $F \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and μ_1, μ_2 are real parameters

The following theorem states the continuous dependence of solutions to (3.21) and (3.22) with the initial conditions given by (1.2)–(1.3) on parameters.

Theorem 3.5 *Suppose that the function F satisfies the condition*

$$|F(t,x,y,z,\mu_1) - F(t,\overline{x},\overline{y},\overline{z},\mu_2)| \leq h(t)[|x-x| + |y-y| + |z-z| + |\mu_1 - \mu_2|], \quad (3.23)$$

where $h \in C(\mathbb{R}_+, \mathbb{R}_+)$. Let $x(t)$ and $y(t)$ be respectively, solutions of the prblems (3.21) *with* (1.2)–(1.3) *and* (3.22) *with* (1.2)–(1.3) *on* $\mathbb{R}_+$. Then

$$|x(t) - y(t)| + |x'(t) - y'(t)| \leq e(t)\, exp\left(\int_0^t (t-s+1)h(s)\, ds\right), \quad (3.24)$$

for $0 \leq t < 1$ and

$$|x(t) - y(t)| + |x'(t) - y'(t)| \leq e(t)\, exp\left(\int_0^t [2(t-s)+1][h(s+1)+h(s)]\, ds\right), \quad (3.25)$$

for $1 \leq t < \infty$, where

$$e(t) = |\mu_1 - \mu_2| \int_0^1 (2-s)h(s)\, ds.$$

Proof. Let $v(t) = |x(t) - y(t)| + |x'(t) - y'(t)|$, $t \in \mathbb{R}_+$. Using the facts that $x(t)$ and $y(t)$ are respectively, solutions of the prblems (3.21) with (1.2) –(1.3) and (3.22) with (1.2) –(1.3) on $\mathbb{R}_+$ and hypotheses, we observe that

$$v(t) \leq \int_0^t (t-s)|F(s,x(s),x'(s),x(s-1),\mu_1) - F(s,y(s),y'(s),y(s-1),\mu_2)|\, ds$$

$$+ \int_0^t |F(s,x(s),x'(s),x(s-1),\mu_1) - F(s,y(s),y'(s),y(s-1),\mu_2)|\, ds$$

$$\leq \int_0^t (t-s+1)|F(s,x(s),x'(s),x(s-1),\mu_1) - F(s,y(s),y'(s),y(s-1),\mu_2)|\, ds$$

$$\leq \int_0^t (t-s+1)h(s)[v(s) + |x(s-1) - y(s-1)| + |\mu_1 - \mu_2|]\, ds. \quad (3.26)$$

We consider the following two cases

Case I: $0 \leq t < 1$. From (3.26) and hypotheses, we observe that

$$v(t) \leq |\mu_1 - \mu_2| \int_0^1 (2-s)h(s)\, ds + \int_0^1 (t-s+1)h(s)[|\phi(s) - \phi(s)|]\, ds + \int_0^t (t-s+1)h(s)v(s)\, ds$$

$$\leq |\mu_1 - \mu_2| \int_0^1 (2-s)h(s)\, ds + \int_0^t (t-s+1)h(s)v(s)\, ds$$

$$\leq e(t) + \int_0^t (t-s+1)h(s)v(s)\, ds. \quad (3.27)$$

Clearly $e(t)$ is nondecreasing in t. Now an application of Lemma 2.2 to (3.27) yields (3.24).

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Case II: $1 \le t < \infty$. From (3.26) and hypotheses, we observe that

$$v(t) \le |\mu_1 - \mu_2| \int_0^1 (2-s)h(s)\,ds + \int_0^1 (t-s+1)h(s)[|\phi(s) - \phi(s)|]\,ds$$

$$+ \int_1^t (t-s+1)h(s)|x(s-1) - y(s-1)|\,ds + \int_0^t (t-s+1)h(s)v(s)\,ds$$

$$\le |\mu_1 - \mu_2| \int_0^1 (2-s)h(s)\,ds + \int_0^t (t-s+1)h(s)v(s)\,ds + I_3 \quad (3.28)$$

where

$$I_3 = \int_1^t (t-s+1)h(s)[|x(s-1) - y(s-1)|]\,ds \quad (3.29)$$

By making the change of variable $s - 1 = \tau$, then from (3.29), we obtain

$$I_2 = \int_0^{t-1} (t-\tau)h(\tau+1)[|x(\tau) - y(\tau)|]\,d\tau$$

$\le \int_0^t (t-\tau)\ p(\tau+1)v(\tau)d\tau.$

Therefore, we have

$$I_3 \le \int_0^t (t-s)\ h(s+1)v(s)ds. \quad (3.30)$$

Using (3.30) in (3.28), it is easy to observe that

$$v(t) \le e(t) + \int_0^t [2(t-s)+1][h(s) + \ h(s+1)]v(s)ds. \quad (3.31)$$

Now an application of Lemma 2.2 to (3.31) yields (3.25). This completes the proof of the theorem.

Remark 3.6: The result dealing with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars. Notice that the initial conditions do not involve parameters. The dependence on parameters are an important aspect in various physical problems.

REFERENCES

[1] Bellman, R. and Cooke, K. L. , Differential-Difference Equations, *Academic Press, New York,* 1963..

[2] Pachapatte, B. G., Inequalities for Differential and Integral Equations, Academic Press, New York, 1998.

[3] Pachapatte, B. G., Some basic theorems on difference-differential equations, Electronic Journal of DifferentialEquations, Vol. 2008(2008), No. 75, 1-11.

[4] Pachapatte, B. G., Existexne and uniqueness theorems on certain difference-differential equations, Electronic Journal of Differential Equations, Vol. 2009(2009), No. 49, 1-12.

[5] Pachapatte, B. G., Approximation of solutions of a difference-differential equation, ArchivumMathematicum, Tomus 46(2010), 145-155.

[6] Pachapatte, B. G., Approximate solutions of a certain second order integrodifferential equation, Fasciculi Mathematici, (2010), Nr 45, 99-109.

[7] Sugiyama, S., On the existence and uniqueness theorems of difference-differential equations, K dai Math. Sem.Rep.,12(1960), 179-190.

[8] Sugiyama, S., On the boundednes of solutions of difference-differential equations, Proc. Japan Acad., 36(1960), 456-460.

[9] Sugiyama, S., Existence theorems on difference-differential equations, Proc. Japan Acad., 38(1962), 145-149.

[10] Sugiyama, S., Dependence properties of solutions of the retardation and initial values in the theory of difference-differential equations, K dai Math. Sem. Rep.,15(1963), 67-78.

[11] Tidke, H. L. and Dhakne, M. B., Approximate solutions to nonlinear mixed type integrodifferential equation with nonlocal condition, Communications on Applied Nonlinear Analysis, Vol. 17(2010), No. 2, 35-44.

[12] Tidke, H. L. and Dhakne, M. B., Approximate solutions of abstract nonlinear integrodifferential equation of second order with nonlocal condition, Far East Journal of Applied Mathematics, Vol. 41(2010), No. 2, 121-135.

[13] Tidke, H. L. Approximate solutions to fractional semilinear evolution equation with nonlocal condition, International Journal of Mathematical Archive, Vol. 2(11), 2011, 2403-2412.