ISSN 2347-1921

ON GOLDBACH'S CONJECTURE

J.V. Leyendekkers
Faculty of Science, The University of Sydney, NSW 2006, Australia
A.G. Shannon
Emeritus Professor, University of Technology Sydney, NSW 2007,
&
Campion College, PO Box 3052, Toongabbie East, NSW 2146, Australia
t.shannon@campion.edu.au, tshannon38@gmail.com

ABSTRACT

This paper considers some aspects of Goldbach's conjecture as a conjecture and estimates the number of prime pairs in some intervals in order to portray a compelling picture of some of the computational issues generated by the conjecture.

Indexing terms/Keywords

Conjecture; twin primes; sieves; semiprimes; experimental mathematics

Academic Discipline And Sub-Disciplines

Number Theory, History; Education;

SUBJECT CLASSIFICATION

AMS Classification Numbers: 11A41, 11-01.

TYPE (METHOD/APPROACH)

This paper considers conjectures in general and their mathematical context with particular applications to Goldbach's conjecture and the Twin Primes Conjecture and the role of experimental mathematics.

Council for Innovative Research
Peer Review Research Publishing System
Journal: JOURNAL OF ADVANCES IN MATHEMATICS
Vol .11, No. 9
www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

INTRODUCTION

Two seemingly not unrelated famous open problems in prime number theory are the Goldbach conjecture and the twin prime conjecture. Christian Goldbach (1690-1764), a German mathematician and lawyer, formulated a number of conjectures written in a 1742 letter to Leonard Euler (1707-1783), a Swiss mathematician [5]. (In reading them note that Goldbach considered the number 1 to be a prime, a convention that is no longer followed.) Alphonse de Polignac (1826–1863), a French mathematician, in 1849, the year he was admitted to the École Polytechnique in Paris, came up with what is more or less the 'twin prime conjecture'.

The two conjectures can be expressed in quite similar forms in that if p_1 and p_2 are prime numbers then:

- $p_1 + p_2 = N$ has at least one solution for any given even integer $N \geq 4$: a Goldbach conjecture;
- $p_1 - p_2 = 2$ has infinitely many solutions: the twin prime conjecture.

In view of this similarity, it is not surprising that the partial developments of progress on these two conjectures have paralleled each other [18]. Not surprisingly, many published attempts from the elementary [3], including combinatorial [4] and computational bounds [15], to those which introduce new techniques have appeared [6,11] including Farey sieves [11]. While Hardy [13] dismissed, the former, the latter have themselves enriched the literature; for instance, the twin primes constant, Π_2, of Halberstam and Richert [12] resulted in the extended Goldbach conjecture that

$$R(n) \sim 2\Pi_2 \prod_{\substack{k=2 \\ p_k|n}} \frac{p_k - 1}{p_k - 2} \int_2^n (\ln x)^{-2}\, dx$$

in which $R(n)$ is the number of representation of an even number, n, as the sum of two prime numbers. Sieve methods have also been tried; for instance, the Farey sieve associates counts with fractions in Farey sequences [2] and uses these counts to characterize prime numbers [7]. Eminent high-profile mathematicians, such as Erdös and his problem solvers [1], Green, Tao and their colleagues [10], have made other fascinating progress on these conjectures.

CHECKING GOLDBACH

The Goldbach conjecture can be expressed as

$$2N = p_1 + 2t + p_2 - 2t \tag{2.1}$$

in which $p_1 = N - 2t,\ p_2 = N + 2t.$

Then either $p_1 = p_2 = N$ or one of the primes must be greater than N (half the even number) [14]. This is illustrated in Table 1.

Table 1: Examples of Equation (2.1)

Even Number	Sum of 2 prime numbers
$104 = 2 \times 52$	59 + 45 = 61-2 + 43+2 = 61 + 43 71 + 33 = 67+4 +37-4 = 67 +37 91 + 13 = 87+4 + 17-4 = 87 + 17
$2862 = 2 \times 1431$	1437 + 1425 = 1433+4 + 1429-4 1437 + 1425 = 1437+2 + 1425-2 = 1439 + 1423
$54908 = 2 \times 27454$	27451 + 27457 = 27451-24 + 27457+24 =27427+27481 = 27451+30 + 27457-30 = 27481 + 27427
$99714 = 2 \times 49857$	49857 + 49857 = 49857-34 + 49857+34 = 49823 + 49891
$535670 = 2 \times 267835$	267835 + 267835 = 267835+72 + 267835-72 = 267907 + 267763

Some of the philosophical implications of this approach are outlined in [17]. If there is no restriction on t, then since there is an infinity of primes numbers a t should exist such that Equation (2.1) should be satisfied. An example of the relatively large values which t can take is shown by

389965026819938 = 5569 + 389965026814369

or

$$38996502681 9938 = 5569 + 19498251340 4400 + 38996502681 4369 - 19498251340 4400$$
$$= 2 \times 19498251340 9969;$$

that is, it is clear that in this case the value of t ($N \pm 2t$) is greater than the gaps between the primes.

While the gaps between the primes become very large, the numbers themselves are relatively very large so that even half of these numbers would be greater than the gaps believed to exist between primes. Jens Kruse Andersen and his colleagues have investigated these issues computationally, and in the process improved some numerical techniques. For instance, the large prime number ($2^{57885161}$-1) has 174251170 digits and the next largest prime has 129781889 digits so that half of the adjacent even integer would be larger than the gap between the primes so that a t could be found.

The value of t will depend on the number of primes in the range [N,2N]. The smaller prime in (2.1) has the associated range [3,N-1], so the question is about the probability of find a matching t given that there are so many choices of primes in this range. For example, if the expected value, $E(t)$, is the weighted average value of all the possible values of t in a given range, then

$$E(t) = kP(t - k)$$

and, as noted above, there are n possible values of t in a given range where n is the number of prime numbers in the region.

GOLDBACH INDICATIVE RATIOS

The number of primes to (M-3) yields a smooth curve with ln$M$ which appears to reach a constant value around 0.02. Table 2 lists a rough estimate of the number of prime pairs for a given even number M as n^2/M. The ratio of the predicted number over the actual number is approximately 1.5 when $M \in \bar{0}_4 \subset Z_4$ [15], but when $M \in \bar{2}_4, M = 4r_2 + 2$, the ratio is approximately 0.8 and the ratio of the number of prime pairs to the number of available primes averages around 0.1 for the digit range in Table 2.

Table 2: Proportion of prime pairs

No. of digits	M	Class	No. n of primes	n^2/M-3 A	No. of prime pairs, B	A/B	n/M
2	20	$\bar{0}_4$	7	2.9	2	1.95	0.350
	40	$\bar{0}_4$	11	3.3	3	1.10	0.275
	58	$\bar{2}_4$	15	4.1	4	1.03	0.276
	80	$\bar{0}_4$	22	6.3	6	1.05	0.275
3	258	$\bar{2}_4$	55	12	16	0.75	0.213
	440	$\bar{0}_4$	85	16	15	1.07	0.193
	728	$\bar{0}_4$	129	23	15	1.53	0.177
	920	$\bar{0}_4$	157	27	23	1.17	0.171
4	1170	$\bar{2}_4$	193	32	50	0.64	0.165
	3206	$\bar{2}_4$	454	64	72	0.89	0.143
	4916	$\bar{0}_4$	658	88	51	1.73	0.134
	6738	$\bar{2}_4$	870	112	140	0.80	0.129
	8686	$\bar{2}_4$	1083	135	107	1.3	0.125
5	10290	$\bar{2}_4$	1263	155	295	0.53	0.123
	34298	$\bar{2}_4$	3668	392	300	1.31	0.107
	59322	$\bar{2}_4$	6005	608	566	1.07	0.101
	92764	$\bar{0}_4$	9129	898	514	1.75	0.098
6	651064	$\bar{0}_4$	45625	3197	3084	1.04	0.070
7	1162858	$\bar{2}_4$	75586	4913	4530	1.08	0.065

An exploratory relation between *M* and *n*/*M* is explored in Figure 1. This suggests a refinement in Table 3 which is then represented by a line of "best fit" in Figure 2. While the coefficient of determination, r^2, is relatively high, near enough is not good enough, unless one is satisfied with asymptotic proofs!

Figure 1: *X* (ln *M* – 5) vs *Y* ((*n*/*M*)x10^2)

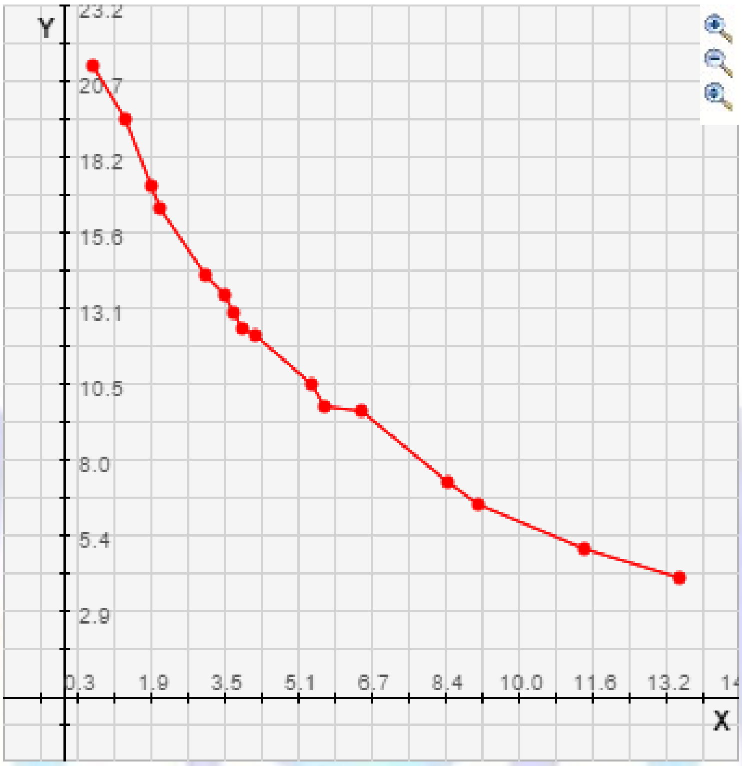
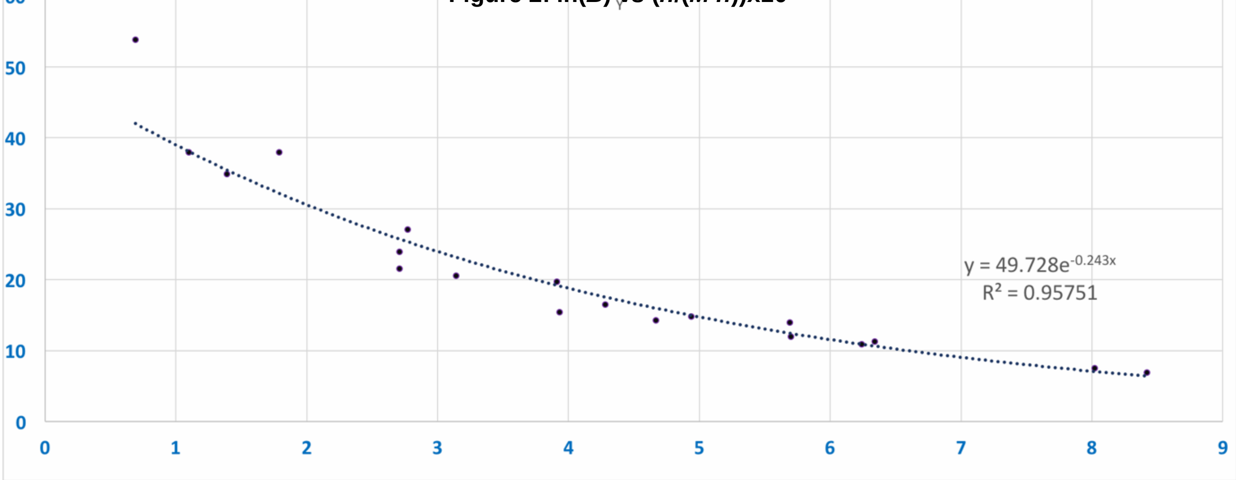


Table 3: Proportion of prime pairs

No. of digits	*M*	No. *n* of primes	No. of prime pairs, *B*	ln *B*	$\left(\frac{n}{M-n}\right)\times 10^2$
2	20	7	2	0.69	53.84
	40	11	3	1.10	37.93
	58	15	4	1.39	34.88
	80	22	6	1.79	37.93
3	258	55	16	2.77	27.09
	440	85	15	2.71	23.94
	728	129	15	2.71	21.54
	920	157	23	3.14	20.58
4	1170	193	50	3.91	19.75
	3206	454	72	4.28	16.50
	4916	658	51	3.93	15.45
	6738	870	140	4.94	14.83
	8686	1083	107	4.67	14.24
5	10290	1263	295	5.69	13.99
	34298	3668	300	5.70	11.98
	59322	6005	566	6.34	11.26
	92764	9129	514	6.24	10.92
6	651064	45625	3084	8.03	7.54
7	1162858	75586	4530	8.42	6.95

ISSN 2347-1921

Figure 2: ln(*B*) vs (*n*/(*M*-*n*))x10^2



Since the smaller prime could be 3, the number of primes available for certain *M* values could be very large. The results in Figure 1 for *n*/*M* vs ln*M* suggest that the ratio *n*/*M* reaches a constant positive value. Since n^2/M approximates to the number of prime pairs, and with *K* the limiting value for *n*/*M*, the number of prime pairs is approximately *Kn*, that is, greater than zero which is necessary for the Goldbach conjecture to be established. Although the range in Table 2 is relatively small, the stability of the integer structure shown in [15] should ensure that the ln*M* function is valid up to very large values of *M*.

CONCLUDING COMMENTS

Some of these ideas which flow from Equation (2.1) could also be tested on semi-primes which are asymptotically denser than primes. A “semiprime” (or biprime) is an integer which is the product of two (not necessarily distinct) prime numbers. In this context Lemoine's conjecture, named after the French mathematician, Émile Lemoine (1840-1912), states that all odd integers greater than 5 can be represented as the sum of an odd prime number and an even semiprime [5]. The material also lends itself to undergraduate projects in heuristic mathematics [19] particularly where iPads and similar devices are part of the teaching and learning process [9]. While Hardy typifies a certain approach to conjectures, Polya [16] and Franklin [8] characterise a broader view of inductive heuristics which can engage the attention of the ‘amateurs’, such as Pascal and Fermat.

ACKNOWLEDGMENTS

Our thanks are due to Steve Sam, a postgraduate student at Warrane College, the University of New South Wales, for some invaluable computing assistance.

REFERENCES

[1] Alladi, Krishnaswami, Elliott, P.DT.A, Granville, A, and Tenenbaum, G. 1998. Analytic and Elementary Number Theory: A Tribute to Mathematical legend Paul Erdös. Dordrecht: Kluwer.

[2] Alladi, Krishnaswami, and Shannon, A.G. 1977. On a property of Farey-Fibonacci fractions. The Fibonacci Quarterly. 15 (2): 153-155.

[3] Ball, W. W. R. and Coxeter, H.S.M.. 1987. Mathematical Recreations and Essays, 13th Edition. New York: Dover.

[4] Clarke, J.H. and Shannon, A.G. 1983. A combinatorial approach to Goldbach’s conjecture. Mathematical Gazette. 67 (439): 44–46.

[5] Dickson, Leonard E. 1971. History of the Theory of Numbers. Volume 1. New York: Chelsea. p. 424.

[6] Erdös, Paul, and Jabotinsky, Eri. 1958. On sequences of integers generated by a sieving process. Nederlandse Akademie van Wetenschappen. Series A. 6: Part I: 115-123; Part II: 124-128.

[7] Franel, Jérôme. 1924. Les suites de Farey et les problèmes des nombres premiers. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse. 198–206.

[8] Franklin, James. 2015. The Science of Conjecture: Evidence and Probability before Pascal. With a New Preface. Baltimore, MD: Johns Hopkins University Press.

[9] Furnell, Steven. 2015. Using iPads to Enable Cultural Change in Technology-enhanced Learning. In Nicos Souleles and Claire Pillar (eds). iPads in Higher Education. Newcastle-upon-Tyne: Cambridge Scholars Publishing, pp.227-238.

[10] Green, Ben, and Tao, Terence. 2008. The primes contain arbitrarily long arithmetic progressions. Annals of Mathematics. 167 (2): 481–547.

[11] Guthery, Scott B. 2011. The Farey Sieve. arXiv:0909.4006v5 [math.NT] 23 March.

[12] Halberstam, H. and Richert, H-E. 1974. Sieve Methods. New York: Academic Press,

[13] Hardy, G. H. 1999. Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work. 3rd Edition. New York: Chelsea.

[14] Heath-Brown, D.R. and Puchta, J.C. 2002. Integers represented as a sum of primes and powers of two. Asian Journal of Mathematics. 6 (3): 535–565.

[15] Leyendekkers, J.V. and Shannon, A.G. 2000. The Goldbach-Conjecture Primes within a Modular Ring, Notes on Number Theory and Discrete Mathematics, 6 (4): 101-112.

[16] Polya, G. 1954. Mathematics and Plausible Reasoning. Princeton, NJ: Princeton University Press.

[17] Posey, Carl. 2005. Intuitionism and Philosophy. In Stewart Shapiro (ed.) The Oxford Handbook of Philosophy of Mathematics and Logic. New York: Oxford University Press, p.328.

[18] Sándor, József. 2014. Remark on twin primes. Notes on Number Theory and Discrete Mathematics. 20 (3): 29-30.

[19] Stein, M. L. and Stein, P.R. 1965. New Experimental Results on the Goldbach Conjecture. Mathematics Magazine. 38 (1): 72-80.

Authors' biographies with Photos

Dr Jean Leyendekkers was awarded a Doctor of Science (D.Sc) degree on Solution Theory by the University of Sydney. Since retiring from the Faculty of Science there Jean has written papers on Number Theory and now has eighty seven published. A few of the earlier papers were written with Janet Rybak but most have been co-authored by Professor Tony Shannon. The emphasis has been on Integer Structure influence in Number Theory. Jean has been active for 30 years in community work on urban planning and in particular on the regeneration of bushland. Jean enjoys classical music, mysteries and loves cats, dogs, possums and all other animals and birds.

Professor A. G. (Tony) Shannon AM is an Adjunct Professor of Central Queensland University and the University of Notre Dame Australia, and an Emeritus Professor of the University of Technology, Sydney, where he was Foundation Dean of the Graduate Research School. He holds the degrees of PhD, EdD and DSc. He is co-author of numerous books and articles in medicine, mathematics and education. His research interests are in the philosophy of education, number theory, and epidemiology, particularly through the application of generalized nets and intuitionistic fuzzy logic. Professor Shannon is a Fellow of several professional societies. He is presently Registrar of Campion College, a liberal arts degree granting institution in Sydney. In 1987 he was appointed a Member of the Order of Australia (AM) for services to education.