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**Exponentially Varying Load on Rayleigh Beam Resting on Pasternak Foundation**

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**Abstract:**

This paper investigates the dynamic behaviour of uniform Rayleigh beam resting on Pasternak foundation and subjected to exponentially varying magnitude moving the load. The solution techniques are based on finite Fourier sine transformed Laplace transformation and convolution theorem. The results show that for a fixed value of axial force, damping coefficient and rotatory inertia, increases in shear modulus and foundation modulus reduces the response amplitude of the dynamical system. It was also found that increases in axial force, rotary inertia, and damping coefficient for fixed values of shear modulus and foundation modulus lead to decreases in the deflection profile of the Rayleigh beam resting on Pasternak foundation. Finally, it was found that the effect of shear modulus is more noticeable that that of the foundation modulus.

**Keywords:** Pasternak Foundation, Shear Modulus, Uniform Beam, Exponentially Varying Moving Load, Damping Coefficient, Foundation Modulus, Axial Force.

## 1 INTRODUCTION

The response of structural and flexible members to moving loads has been the subject of numerous researches owing to its relevance in many diverse areas. The structure has commonly been modelled either as a beam or a plate in most of the analytical studies in engineering, applied mathematics, and mathematical physics. The dynamic response analysis is very complex in such structural members under the passage of moving loads due to the interaction between the passing load and the structure. The response of railroads to moving trains, the response of bridges to moving vehicles, belt drives to the conveyor and computer tape drive to floppy disks are some of the application of such moving load problem Adams [1] and Shabnam et al. [2].

Moreover, modern means of transport are ever faster and heavier, while the structure over which they move is ever more slender and lighter Fryba [3]. The dynamic stresses produce are larger by far than the static ones. Theism has continued to motivate a lot of research activities in this area.

The dynamic response of a simply supported beam, traversed by a constant force moving at a uniform speed was first studied by Kyrylo [4]. He used the method of expansion of eigenfunction to obtain his results. Later Timoshenko [5] used energy methods to obtain solutions in series form for the simply supported finite beam on an elastic foundation subjected to the time dependent points loads moving with uniform velocities across the beam. Kenny [6] similarly investigated the dynamic response of infinite elastic beams on elastic foundation under the influence of load moving at constant speeds. He included the effects of viscous damping in the governing differential equation.

In a recent development, some of the researchers that made tremendous feat in the dynamic study of structures under moving loads includes Oni and Awodola [7], Liu and Chang [8], Oni and Omolefe [9], Misra [10], Oni and Omolefe [11], Achawakorn and Jearsiri Pongkul [12]. However, all the researchers aforementioned considered only one parameter model which has various shortcoming because it predicts discontinuities in the deflection of the surface of the foundation at the ends of a finite beam and this is in contradiction to an observation made in practice.

Several researchers in recent time that considered two parameters models in their studies so as to overcome the shortcoming of the one parameter model include Jimoh and Ajoge [13], Jimoh and Ajoge [14], Oni and Jimoh [15], Oni and Jimoh [16], Oni and Jimoh [17], Oni and Ogunbamike [18].

In all those researches, exponentially varying magnitude moving load were not considered. The present paper is concerned with the response of uniform Rayleigh beam resting on Pasternak foundation and traversed by exponentially varying magnitude moving load.

## 2. THE INITIAL BOUNDARY-VALUE PROBLEM

The governing partial differential equation for a uniform Rayleigh beam length L on Pasternak foundation and traversed by a moving load Q(x,t) of mass, m moving with velocity c and damping term included is given by Frybal [3]

$$EI\frac{\partial^4 Y(x,t)}{\partial x^4}+\mu\frac{\partial^2 Y(x,t)}{\partial t^2}-N\frac{\partial^2 Y(x,t)}{\partial x^2}-\mu R_0^2\frac{\partial^4 Y(x,t)}{\partial x^2\partial t^2}+\frac{\varepsilon\partial Y(x,t)}{\partial t}+KY(x,t)+G\frac{\partial^2 Y(x,t)}{\partial x^2}=Q(x.t) \tag{1}$$

Where

$EI$ = flexural rigidity of the structures,

$\mu$ = mass per unit length of the beam

$K$ = foundation modulus,

G = Shear modulus

$N$ = axial force,

$\varepsilon$ = damping coefficient,

$R_o$ =rotatory inertia

$E$= young's modulus,

$I$ = moment of inertia,

$Y(x,t)$ = transverse displacement,

$x$ = spatial coordinate and

$t$ = time coordinate

The boundary condition of the structure under consideration is simply supported and is given as

$$Y(0,L)=0=\ Y(L,t)$$

$$\frac{\partial^2 Y(0,L)}{\partial x^2} = 0 = \frac{\partial^2 Y(L,t)}{\partial x^2} = 0 \qquad (2)$$

And the initial conditions are taken as

$$Y(x,0) = 0 = \frac{\partial Y(x,0)}{\partial t} = 0 \qquad (3)$$

The load moving on the elastic beam is assumed to be an exponentially varying magnitude moving load of the form

$$Q(x,t) = Qe^{-dt} f(x - ct) \qquad (4)$$

Where

$Q$ = $Mg$ = moving force of constant magnitude

$g$ = acceleration due to gravity

$c$ = constant velocity of the load motion

$d$ = arbitrary constant

*f(x-ct)* = Dirac Delta function which is defined as the unit impulse function of point $x = ct$

By putting (4) into (1), we obtained

$$EI\frac{\partial^4 Y(x,t)}{\partial x^4} + \mu\frac{\partial^2 Y(x,t)}{\partial t^2} - N\frac{\partial^2 Y(x,t)}{\partial x^2} - \mu R_0^2\frac{\partial^4 Y(x,t)}{\partial x^2 \partial t^2} + \frac{\varepsilon \partial Y(x,t)}{\partial t} + KY(x,t) + G\frac{\partial^2 Y(x,t)}{\partial x^2} = Qe^{-dt} f(x - ct) \qquad (5)$$

**3. SOLUTION OF THE MATHEMATICAL PROBLEM**

In this section, we proceed to solve the initial boundary value problem described by 2, 3, and 5. We remark that the integral transform techniques have proved suitable and effectively applicable to solving moving load problems such as the one under investigation [3]. Therefore, this method is adopted in the solution of the IBVP. Specifically, the Fourier transformation for the length coordinate and the Laplace transformation for the time coordinate with boundary and initial conditions are used in this work.

**3.1 FINITE FOURIER TRANSFORMED GOVERNING EQUATION**

Here, we proceed to take the Fourier transform of the governing partial differential equation (5). We find, however, that the boundary conditions in equation (2) may be accommodated only by using a finite Fourier sine transform, so we shall apply the finite Fourier sine integral transformation for the length coordinate, and this is defined as

$$Y(n,t) = \int_0^L V(x,t) \sin\frac{n\pi x}{L}\ dx, n = 1,2,3 \qquad (6)$$

With the inverse transform defined as

$$V(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} Y(n,t)\sin\frac{n\pi x}{L} \qquad (7)$$

Thus, by invoking equation (6) on equation (5), we have

$$EI\left(\frac{n\pi}{L}\right)^4 Y(n,t) + \mu Y_{tt}(n,t) - \left(-N\frac{n^2\pi^2}{L^2}Y(n,t)\right) - (-\mu R_0^2\frac{n^2\pi^2}{L^2}Y_{tt}(n,t)) + \varepsilon Y_t(n,t) + KY(n,t)$$
$$+ (-G\frac{n^2\pi^2}{L^2}Y(n,t)) = Qe^{-dt}\sin\frac{m\pi t}{L} \quad (8)$$

Equation (8) can be conveniently written as

$$Y_{tt}(n,t) + a_{11}Y_t(n,t) + a_{22}Y(n,t) = a_{33}\sin d_0 t\, e^{-dt} \qquad (9)$$

where

$$d_0 = \frac{n\pi c}{L}, \quad d_1 = 1 + R_0^2\frac{n^2\pi^2}{L^2}, \quad d_2 = \frac{\varepsilon}{\mu}, \quad d_3 = \frac{EIn^4\pi^4}{\mu L^4}, \quad d_4 = \frac{n^2\pi^2}{\mu L^2}(N-G)$$

$$d_5 = \frac{K}{\mu}, \quad d_6 = \frac{Q}{\mu}, \quad Q = Mg, \quad a_{11} = \frac{d_2}{d_1}, \quad a_{22} = \frac{d_3+d_4+d_5}{d_1},$$

$$a_{33} = \frac{d_6}{d_1} \qquad (10)$$

Equation (9) represents the finite Fourier transform governing equations of the uniform Rayleigh beam resting on Pasternak foundation and subjecting to exponentially varying magnitude moving load moving with constant velocity

### 3.2 LAPLACE TRANSFROMED SOLUTIONS

We apply the method of the Laplace integral transformation for the time coordinate between *0* and ∞ to solve equation (9). The operator of the Laplace transform is here indicated by

$$L[f(t)] = \int_0^{\infty} f(t)\, e^{-st}dt \qquad (11)$$

Where

*L*= Laplace transform operator

*S* = Laplace transform variable

That is, in particular, we use

$$L[Y(n,t)] = Y(s,t) = \int_0^{\infty} Y(n,t)\, e^{-st}dt \qquad (12)$$

Using the information (12) on (9) we have

$$e^{-st}dt + a_{11}\int_0^{\infty} Y_t(n,t)\, e^{-st}dt + a_{22}\int_0^{\infty} Y(n,t)\, e^{-st}dt = a_{33}\int_0^{\infty} \sin d_0 t\, e^{-st}dt \;\; e^{-dt} \;\; dt \qquad (13)$$

On evaluating each term of equation (13) by the method of integration by parts and using the set of initial conditions (3), we obtain

$$S^2Y(n,s) + a_{11}SY(n,s) + a_{22}Y(n,s) = a_{33}\left(\frac{d_0}{(s+d)^2 + d_0^2}\right) \qquad (14)$$

Equation (14) can be re-written as

$$Y(n,s) = \frac{a_{33}}{(S-b_1)(S-b_2)}\left(\frac{d_0}{(s+d)^2 + d_0^2}\right) \qquad (15)$$

Where

$$b_1 = \frac{-a_{11}}{2} + \frac{\sqrt{a_{11}^2 - 4a_{22}}}{2} \qquad (16)$$

$$b_2 = \frac{-a_{11}}{2} + \frac{\sqrt{a_{11}^2 - 4a_{22}}}{2} \qquad (17)$$

Equation (15) can be simplified to obtain

$$Y(n,s) = \frac{a_{33}}{(b_1 - b_2)}\left[\left(\frac{1}{S-b_1}\right)\left(\frac{d_0}{(s+d)^2 + d_0^2}\right) - \left(\frac{1}{S-b_2}\right)\left(\frac{d_0}{(s+d)^2 + d_0^2}\right)\right] \qquad (18)$$

### 3.3 THE INVERSE INTEGRAL TRANSFORMATION

We proceed in this section to obtain the inverse transformation of equation (18). To obtain the inverse transformation of (18), we shall adopt the following representation.

$$F_1(S) = \frac{1}{S-b_1}, \quad F_2(S) = \frac{1}{S-b_2}, \quad g(S) = \frac{d_0}{(s+d)^2 + d_0^2} \qquad (19)$$

So that the Laplace inversion of (18) is the convolution of $f_i$ and $g$ defined as

$$f_i * g = \int_0^t F_1(t-u)g(u)du, i = 1,2 \ldots \qquad (20)$$

Thus, the Laplace inversion of equation (18) is given by

$$Y(n,t) = 1 + e^{b_1 t}J_1 - e^{b_2 t}J_2 \qquad (21)$$

Where

$$J_1 = \int_0^t e^{-(b_1-d)u} \sin d_0 u \ dt \qquad (22)$$

$$J_2 = \int_0^t e^{-(b_2-d)u} \sin d_0 u \ dt \qquad (23)$$

$$H = \frac{a_{33}}{(b_1 - b_2)} \qquad (24)$$

By evaluating the integrals in (22) and (23), we obtain

$$J_1 = \frac{d_0}{d_0^2 + (b_1 + d)^2}\left[1 - e^{-(b_1+d)t}\cos d_0 t - \left(\frac{b_1 + d}{d_0}\right)e^{-(b_1+d)t}\sin d_0 t\right] \quad (25)$$

$$J_2 = \frac{d_0}{d_0^2 + (b_2 + d)^2}\left[1 - e^{-(b_2+d)t}\cos d_0 t - \left(\frac{b_2 + d}{d_0}\right)e^{-(b_2+d)t}\sin d_0 t\right] \quad (26)$$

Using equations (25) and (26) in equation (21) and upon evaluation, we have

$$\begin{aligned} Y(n,t) = {} & \frac{Hd_0}{d_0^2 + (b_1 + d)^2}\left(e^{b_1 t}e^{-dt}\cos d_0 t - \left(\frac{b_1 + d}{d_0}\right)e^{-dt}\sin d_0 t\right) \\ & - \frac{Hd_0}{d_0^2 + (b_2 + d)^2}\left(e^{b_2 t}e^{-dt}\cos d_0 t - \left(\frac{b_2 + d}{d_0}\right)e^{-dt}\sin d_0 t\right) \quad (27) \end{aligned}$$

Using equation (27) in equation (7), we have

$$\begin{aligned} V(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} & \frac{Hd_0}{d_0^2 + (b_1 + d)^2}\left(e^{b_1 t}e^{-dt}\cos d_0 t - \left(\frac{b_1 + d}{d_0}\right)e^{-dt}\sin d_0 t\right) \\ & - \frac{Hd_0}{d_0^2 + (b_2 + d)^2}\left(e^{b_2 t}e^{-dt}\cos d_0 t \right. \\ & \left. - \left(\frac{b_2 + d}{d_0}\right)e^{-dt}\sin d_0 t\right)\sin\frac{n\pi x}{L} \quad (28) \end{aligned}$$

Equation (28) represents the transverse displacement of the uniform Rayleigh beam resting on Pasternak foundation subjected to exponentially varying moving load.

## 4.0 NUMERICAL ANALYSIS AND DISCUSSION OF RESULTS

To illustrate the theory described in this paper numerically, the values of the physical constants and parameters considered are

$$\pi = \frac{22}{7}, \quad EI = 15000Nm^2, \quad P = 15N, \quad \mu = 0.065, \quad L = 12.9m \ and\ I = 2.87698 \times 10^{-3}\, m$$

The responses of the uniform Rayleigh beam to exponentially varying magnitude moving load for various values of foundation modulus (K), shear modulus (G), axial force *(N)*, Rotatory inertia (Ro) and damping coefficient are shown in the following figures (Figures 1-5)

The response of the beam to exponentially varying magnitude moving load for various values of the shear modulus (K) at fixed value of rotatory inertia (Ro), shear modulus (G), damping coefficient ($\varepsilon$) and axial force (*N*) is shown in Figure 1 while figure 2 shows the dynamic response of the system. Figures 1 and 2 shows that the response amplitude of the beam decreases with increases values of shear modulus and foundation modulus, respectively.

Figures 3, 4, and 5 show the deflection profiles of the beam for various values of the axial force, damping coefficient, and rotatory inertia, respectively. From the figures, it is observed that increasing the values of each of the parameters decreases the response amplitudes of the beam.

## 5. CONCLUSION

The dynamic response of uniform Rayleigh beam resting on Pasternak foundation and subjected to exponentially varying magnitude moving load has been studied in this paper. The governing partial differential equations of the problem have been solved by using the finite Fourier integral sine transformed, Laplace transformation and convolution theorem and the prescribed initial and boundary conditions. It is assumed that the beam is of uniform cross-section and of constant mass. The effects of shear modulus, foundation modulus, axial force, rotatory inertia, and damping coefficient on the deflections of the beams are highlighted. It is found that the amplitudes of the deflections profiles of the beam decrease with increasing values of each of the structural parameters. Furthermore, it is also found that the effect of shear modulus is more noticeable than that of the foundation modulus.

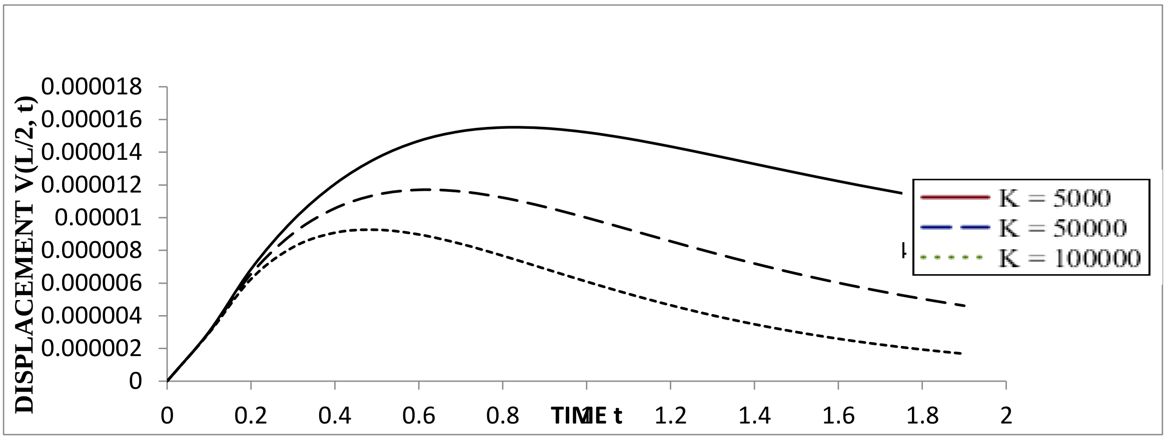
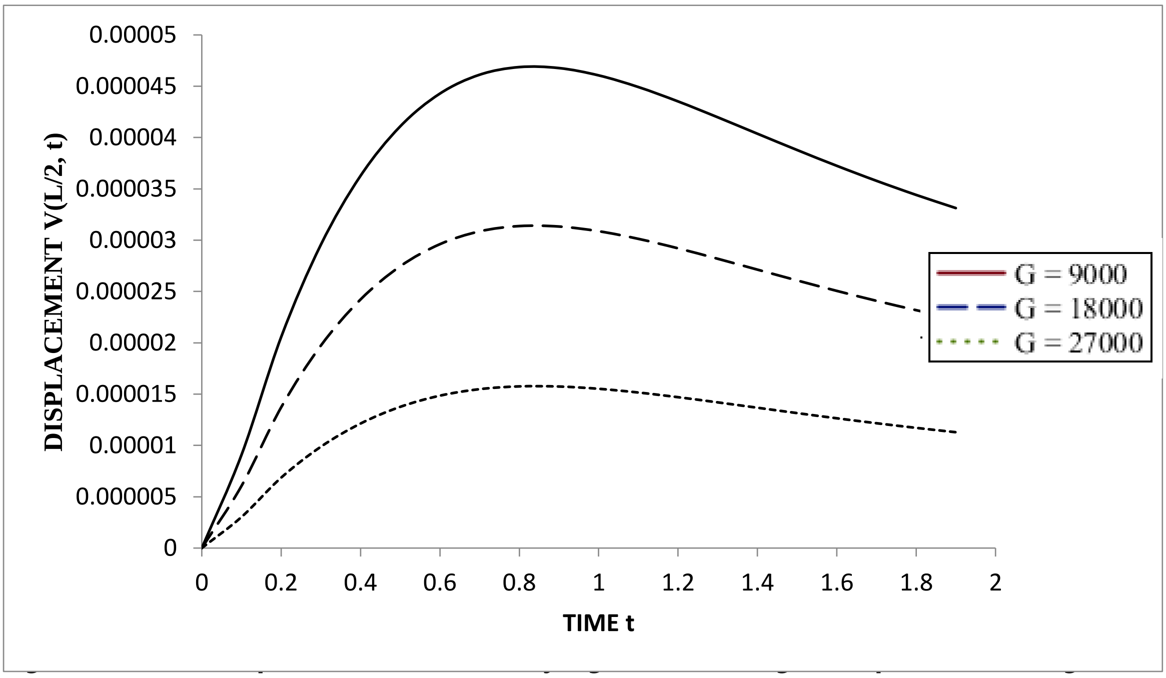
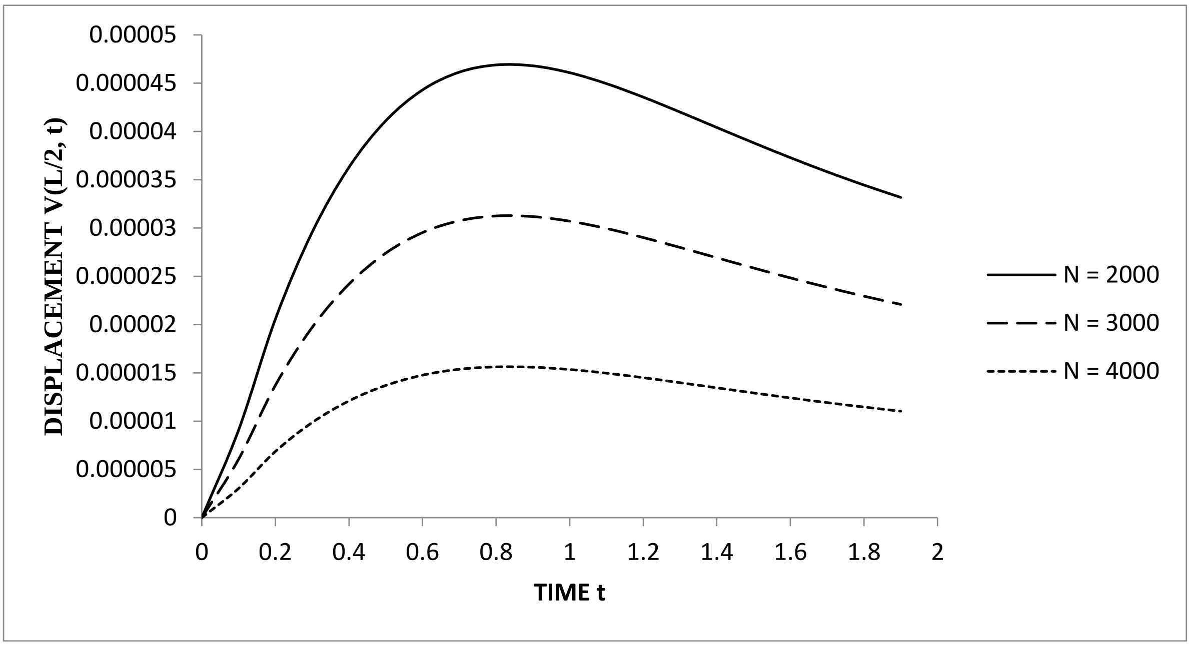



**Figure 1: Deflection profile of a uniform Rayleigh beam resting on bi-parametric subgrades under exponentially varying magnitude moving Load for various values of foundation modulus (K).**



**under exponentially varying magnitude moving Load for various values of shear modulus (G).**




**under exponentially varying magnitude moving Load for various values of the axial force (N).**

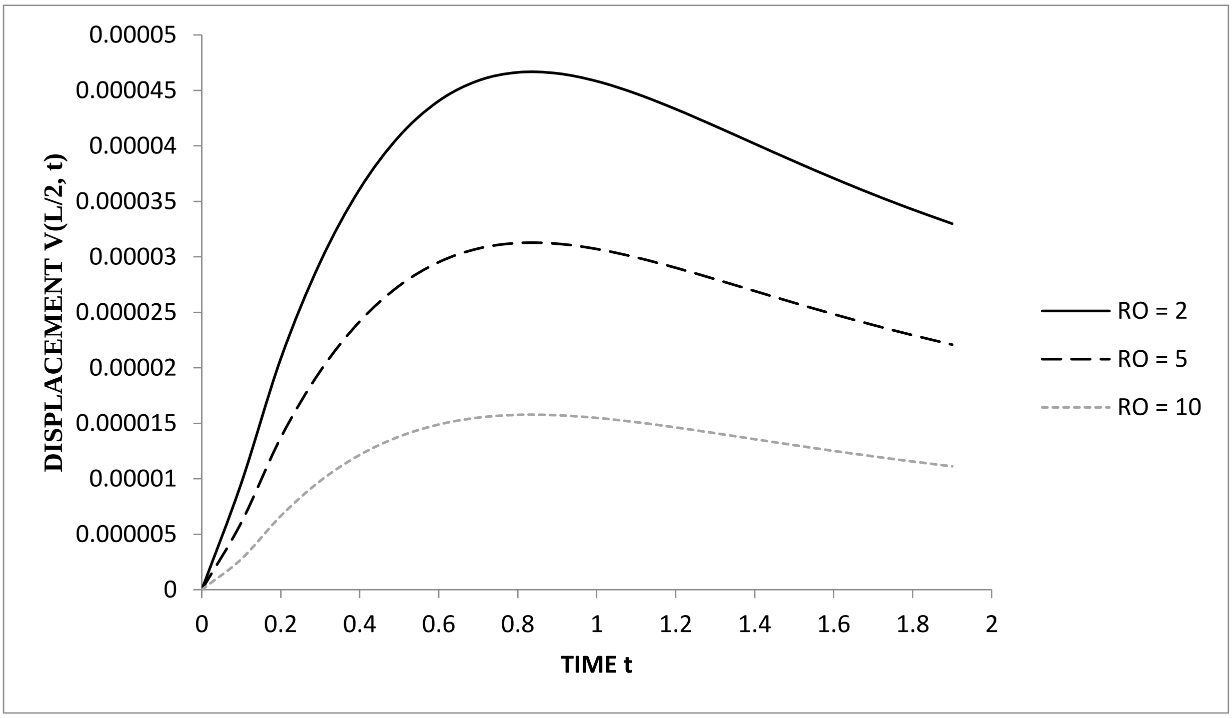


**Figure 4: Deflection profile of a uniform Rayleigh beam resting on bi-parametric subgrades under exponentially varying magnitude moving Load for various values of rotatory inertia (RO).**

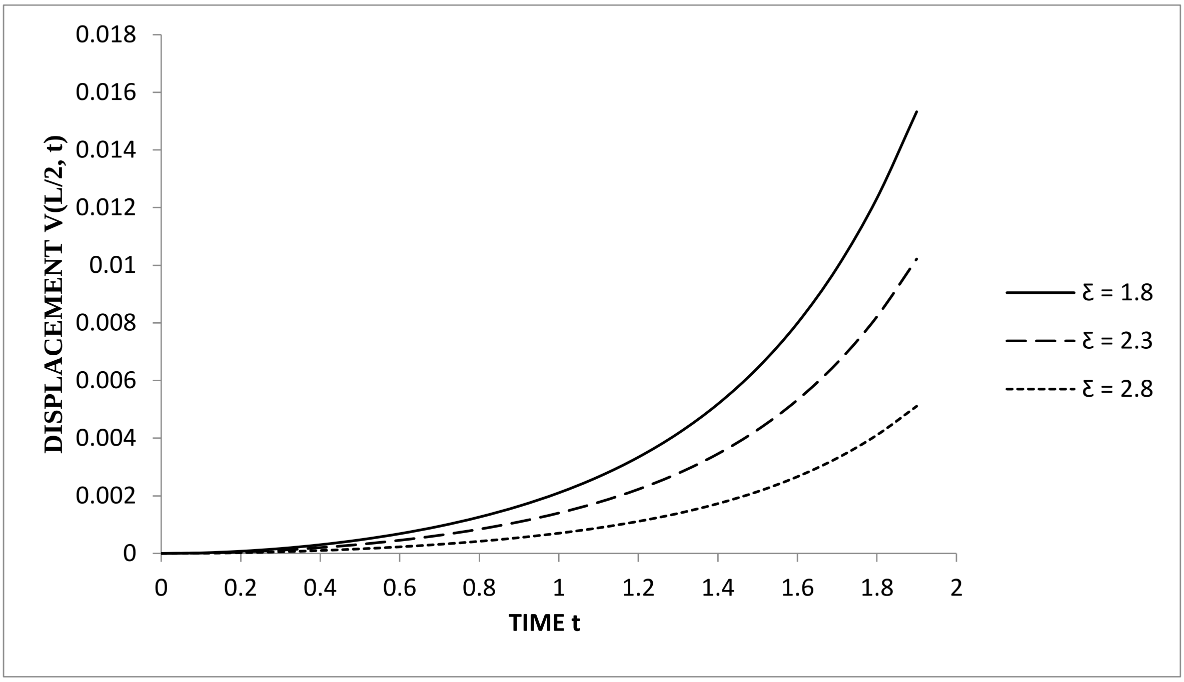



**Figure 5: Deflection profile of a uniform Rayleigh beam under exponentially varying magnitude moving Load for various values of damping coefficient (Ɛ).**

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