ISSN 2347-1921

ON SYMMETRIC BI-DERIVATIONS OF KU-ALGEBRAS

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Abstract. The notion of left-right (resp. right-left) symmetric bi-derivation of KU-algebras is introduced and some related properties are investigated.

keywords : KU-algebras; symmetric bi-derivation; Kernel; Fixed; trace.

1 INTRODUCTION

BCK and BCI algebras are two important classes of algebras of logic introduced by Imai and Iseki and also have been deeply studied by many researchers in [6, 7, 8]. C. Prabpayak and U. Leerawat introduced a nwe algebraic structure that is called KU-algebra. Y. B. Jun and X. L. Xin applied the notion of derivation in ring and near ring theory to BCI-algebras [4]. And H. A. S. Abujabal and N. O. Al-Shehri investigated some fundamental properties and proved some results on derivations of BCI-algebras in [5]. S. M. Mostafa, R.A.K. Omar and A. Abd-eldayem defined the derivation on a KU-algebra and they studied some related properties in [3]. The concept of symmetric bi-derivation was introduced by Gy. Maksa in [9] (see also [10]). J. Vukman proved some results concerning symmetric bi-derivation on prime and semi prime rings [11, 12]. Y.Çeven introduced symmetric bi-derivation in lattices and investigated some related properties [13]. S. Ilbira and A. Firat [14] introduced the notion of left-right (resp. right-left) symmetric bi-derivation of BCI-algebras. In this paper the notion of left-right (resp. right-left) symmetric bi-derivation of KU-algebras is introduced and some of its properties are investigated.

2 Preliminaries

Definition 2.1 [1]**A KU-algebra is an algebra**

where is a binary operation and is a constant

satisfying the following axioms for all $x, y, z \in X$ **:**

(KU_1) $(x*y)*[(y*z)*(x*z)] = 0$.

(KU_2) $x*0 = 0$.

(KU_3) $0*x = x$.

(KU_4) If $x*y = y*x = 0$ implies $x = y$.

Define a binary relation $\le$ by : $x \le y \Leftrightarrow y*x = 0$, we can prove that $(X,*)$ is a partially ordered set. By the binary relation $\le$,we can write the previous axioms in another form as follows:

($KU_1^{'}$) $(y*z)*(x*z) \le (x*y)$.

($KU_2^{'}$) $0 \le x$.

($KU_3^{'}$) $x \le y \Leftrightarrow y*x = 0$.

($KU_4^{'}$) If $x \le y$ and $y \le x \Rightarrow x = y$.

Corollary 2.2 [2] In a KU-algebra X the following identities are true for all $x, y, z \in X$:

(i) $z*z = 0$

(ii) $z*(x*z) = 0$

(iii) If $x \le y$ then $y*z \le x*z$

(iv) $z*(y*x) = y*(z*x)$

(v) $y*[(y*x)*x] = 0$

Definition 2.3 [1] A nonempty subset S of a KU-algebra X is called a sub-algebra of X if $x*y \in S$, whenever $x, y \in S$.

Definition 2.4 [1, 2] A nonempty subset A of a KU-algebra X is called ideal of X if it satisfies the following conditions:

(i) $0 \in A$

(ii) $y*z \in A$ iplies $z \in A$ for all $y, z \in X$.

For a KU-algebra X we will denote $x \wedge y = (x*y)*y$.

Proposition 2.5 [3] Let (X, *, 0) be a KU-algebra then the following identities are true for all $x, y, z \in X$:

(i) $(x*y)*(x*z) \le y*z$

(ii) If $x \le y$ then $z*x \le z*y$

(iii) $z*(x*y) \le (z*x)*(z*y)$

(iv) $x \wedge y \le x$ and $x \wedge y \le y$.

Definition 2.6 Let X be a KU-algebra. A mapping $D(.,.): X \times X \to X$ is called symmetric if $D(x, y) = D(y, x)$ for all $x, y, z \in X$.

Definition 2.7 Let X be a KU-algebra. A mapping $d : X \to X$ defined by $d(x) = D(x, x)$ is called the trace of $D(.,.)$, where $D(.,.): X \times X \to X$ is a symmetric mapping.

3 The Symmetric Bi-Derivations on KU-algebras

The following definition introduces the notion of symmetric bi-derivation for Ku-algebras.

Definition 3.1 Let X be a KU-algebra and $D(.,.): X \times X \to X$ be a symmetric mapping. If D satisfies the identity $D(x*y, z) = D(x, z)*y \wedge x*D(y, z)$ for all $x, y, z \in X$, then D is called $left-right \ \ symmetric \ \ bi-derivation$ (briefly $(l,r)-symmetric \ \ bi-derivation$). If D satisfies the identity $D(x*y, z) = x*D(y, z) \wedge D(x, z)*y$ for all $x, \ y, z \in X$, then we say that D is $right-left \ symmetric \ \ bi-derivation$ (briefly $(r, \ l)-symmetric \ \ bi-derivation$). Moreover if D is both an $(r, \ l)-$ and a $(l, \ r)-symmetric \ \ bi-derivation$, it is said that D is $symmetric \ \ bi-derivation$.

Example 3.1 Let $X := \{0,1,2,3,4\}$ be a set in which the operation $*$ is defined in as follows with the Cayley table[3];

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	2	4
2	0	0	0	1	4
3	0	0	0	0	4
4	0	1	1	1	0

The mapping $D(.,.): X \times X \to X$ will be defined by

$$D(x, y) = \begin{cases} 4, & \text{if} x = y = 4, \\ 0, & \text{otherwise} \end{cases}$$

Then it can be checked that D is both $(l,r)-symmetric \ \ bi-derivation$ and $(r,l)-symmetric \ \ bi-derivation$ on X .

Example 3.2 Let $X := \{0,1,2,3,4\}$ be a set in which the operation $*$ is defined in as follows with the Cayley table[3];

ISSN 2347-1921

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	1	0	0	0

The mapping $D(.,.): X \times X \to X$ will be defined by

$$D(x,y) = \begin{cases} 3, & \text{if} x = y = 4, \\ 0, & \text{otherwise} \end{cases}$$

It is easy to check that D is $(l,r)-symmetric\ \ bi-derivation$ on X . But since

$$D(1*4,4) = D(4,4) = 3$$

and

$$1*D(4,4) \wedge D(1,4)*4 = 1*3 \wedge 3*4 = 3 \wedge 2 = (3*2)*2 = 2*2 = 0$$

D is not $(r,l)-symmetric\ \ bi-derivation$.

Proposition 3.2 Let D be a $symmetric\ \ bi-derivation$ on X . Let $x \in X$ and by using the definition of $(l,r)-symmetric\ \ bi-derivation$ on X we have

$$\begin{aligned} D(0,x) &= D(x*0,x) = (D(x,x)*0) \wedge (x*D(0,x)) \\ &= 0 \wedge (x*D(0,x)) \\ &= (0*(x*D(0,x))*(x*D(0,x)) \\ &= (x*D(0,x))*(x*D(0,x)) \\ &= 0 \end{aligned}$$

Similarly, by using the definition of $(r,l)-symmetric\ \ bi-derivation$ on X we can find $D(0,x) = 0$.

Let X be a KU-algebra and $D(.,.): X \times X \to X$ be a symmetric bi-derivation on X . Then $D(0,x) = 0$ for all $x \in X$

Proof.

Corollary 3.3 Every symmetric bi-derivation on a KU-algebra is regular.

Proof. It is clear from Proposition 3.2.

Proposition 3.4 Let X be a KU-algebra and $D(.,.): X \times X \to X$ be a symmetric mapping. Then

- If D is a $(l,r)-symmetric\ \ bi-derivation$, then $D(x,z) = x \wedge\ D(x,z)$ for all $x, z \in X$
- If D is a $(r,l)-symmetric\ \ bi-derivation$, then $D(x,z) = D(x,z) \wedge\ x$ for all $x, z \in X$.

Proof. i) Let $x, z \in X$ and D be a $(l,r)-symmetric\ \ bi-derivation$ on X . Then we have

$$\begin{aligned} D(x,z) &= D(0*x,z) \\ &= D(0,z)*x) \wedge (0*D(x,z)) \\ &= (0*x) \wedge D(x,z) \\ &= x \wedge D(x,z). \end{aligned}$$

ii)Let $x \in L$ and D be a $(r,l)-symmetric\ \ bi-derivation$ on X. Then we have

$$\begin{aligned} D(x,z) &= D(0*x,z) \\ &= (0*D(x,z)) \wedge (D(0,z)*x) \\ &= D(x,z) \wedge (0*x) \\ &= D(x,z) \wedge x \end{aligned}$$

Proposition 3.5 Let X be a KU-algebra and d be the trace of symmetric bi-derivation D on X. Then

- $D(x,z) \le x$.
- $d(x) \le x$.
- $D(x*y,z) \le D(x,z)*y$.
- $D(x*y,z) \le x*D(y,z)$.
- $d^{-1}(0) = \{x \in X \mid d(x) = 0\}$ is a subalgebra of X.

Proof. Let X be a KU-algebra and d be the trace of symmetric bi-derivation D on X.

[(i)] Let D be a $(r,l)-symmetric\ \ bi-derivation$ on X by using Proposition $3.4(ii)$ and Corollary $2.2(ii)$ we have

$$x*D(x*z) \quad = x*(D(x,z) \wedge x) = 0$$

So $D(x,z) \le x$.

[(ii)] This can be easily obtained from (i).

[(iii)] Let D be a $(l,r)-symmetric\ \ bi-derivation$ on X and by using Corollary $2.2(v)$ we have

$$\begin{aligned} (D(x,z)*y)*(D(x*y,z)) &= (D(x,z)*y)*[(D(x,z)*y) \wedge (x*D(y,z))] \\ &= (D(x,z)*y)*[(D(x,z)*y)*(x*D(y,z)))*(x*D(y,z))] \\ &= 0 \end{aligned}$$

So $D(x*y,z) \le D(x,z)*y$.

[(iv)] Let D be a $(r,l)-symmetric\ \ bi-derivation$ on X and by using Corollary $2.2(v)$ we have

$$\begin{aligned} (x*D(y*z))*[D(x*y,z)] &= (x*D(y*z))*[(x*D(y,z)) \wedge (D(x,z)*y)] \\ &= (x*D(y*z))*[(x*D(y,z))*((D(x,z)*y))*(D(x,z)*y)] \\ &= 0 \end{aligned}$$

So $D(x*y,z) \le x*D(y,z)$.

[(v)] Since d is regular we have $d^{-1}(0) \neq \varnothing$. Let $x, y \in d^{-1}(0)$ then we have $d(x) = d(y) = 0$. By using the definition of symmetric bi-derivation and KU_1, KU_2 and Corollary $2.2(i)$ we have

$$\begin{aligned} d(x*y) = D(x*y, x*y) &= (x*D(y,x*y)) \wedge (D(x,x*y)*y) \\ &= (x*[(x*D(y,y)) \wedge (D(y,x)*y)]) \wedge ([(x*D(x,y)) \wedge (D(x,x)*y)]*y) \\ &= (x*[(x*0) \wedge (D(y,x)*y)]) \wedge ([(x*D(x,y)) \wedge (0*y)]*y) \\ &= (x*[0 \wedge (D(y,x)*y)]) \wedge ([(x*D(x,y)) \wedge y]*y) \\ &= (x*0) \wedge ([(x*D(x,y)) \wedge y]*y) \\ &= 0 \wedge ([(x*D(x,y)) \wedge y]*y) \\ &= 0 \end{aligned}$$

We have $x*y \in d^{-1}(0)$ Hence $d^{-1}(0)$ is KU-subalgebra of X.

Definition 3.6 Let X be a KU-algebra. A nonempty subset A of X is said to be D-invariant if $D(A,A) \subseteq A$ where $D(A,A) = \{D(x,x) \mid x \in A\}$.

Proposition 3.7 Let D be a symmetric bi-derivation of the KU-algebra X. Then every ideal A is D-invariant.

Proof.

Let $y \in D(A,A)$ then $y = D(x,z)$ for some $x, z \in A$. We have $D(x,z) \le x$ so $x*D(x,z) = 0$ and $x \in A$ and since A is an ideal then we have $D(x,z) = y \in A$. Therefore, $D(A,A) \subseteq A$.

Proposition 3.8 Let X be a KU-algebra and D be the symmetric bi-derivation on X. Then

i) If $x \le y$ then $D(x,z) \le y$.

ii) If $y \le x$ then $D((y*z)*(x*z),t) = 0$.

Proof. i) Let $x \le y$. then by Corollary 2.2 (iii) we have $y*D(x,z) \le x*D(x,z)$. Since $0 \le y*D(x,z)$ and $x*D(x,z) = 0$ we have $y*D(x,z) = 0$.Hence $D(x,z) \le y$.

ii) Let $y \le x$ then we have $(y*z)*(x*z) \le x*y$. So $D((y*z)*(x*z),t) \le x*y$. Hence $D((y*z)*(x*z),t) \le 0$ and $0 \le D((y*z)*(x*z),t)$. So, $D((y*z)*(x*z),t) = 0$.

Proposition 3.9 If D is a (r,l) symmetric bi-derivation defined on the KU-algebra X then we have $D(x*y,z) \le D(x,z)*D(y,z)$ for all $x, y, z \in X$.

Proof. Let $x, y, z \in L$. Then by using the definition of (r,l) symmetric bi-derivation, Corollary 2.2 (iv) we have

$$\begin{aligned} (D(x,z)*D(y,z))*D(x*y,z) &= (D(x,z)*D(y,z))*[(x*D(y,z)) \wedge (D(x,z)*y)] \\ &= (D(x,z)*D(y,z))*[((x*D(y,z))*(D(x,z)*y))*(D(x,z)*y)] \\ &= ((x*D(y,z))*(D(x,z)*y))*[(D(x,z)*D(y,z))*(D(x,z)*y)] \\ &\le (D(x,z)*D(y,z))*(x*D(y,z)) \\ &\le x*D(x,z) = 0 \end{aligned}$$

But $0 \le (D(x,z)*D(y,z))*D(x*y,z)$.

So $(D(x,z)*D(y,z))*D(x*y,z) = 0$. Hence $D(x*y,z) \le D(x,z)*D(y,z)$.

Definition 3.10 Let D be a symmetric bi-derivation of the KU-algebra X, and let d be the trace of D. We can define KerD;

$$Ker_D := \{x \in X \mid D(x,x) = d(x) = 0\}$$

Theorem 3.11 Let D be a symmetric bi-derivation of the KU-algebra X. If $y \in Ker_D$ and $x \in X$ then $x \wedge y \in Ker_D$.

Proof. Let D be a symmetric bi-derivation of the KU-algebra X and $y \in Ker_D$ and $x \in X$. By using the definition of (l,r)-$symmetric\ bi-derivation$ on X and the property (KU_2) of a KU-algebra we have;

$$\begin{aligned} d(x \wedge y) &= D(x \wedge y, x \wedge y) \\ &= D((x * y) * y, x \wedge y) \\ &= D(x * y, x \wedge y) * y \wedge (x * y) * D(y, x \wedge y) \\ &= D(x * y, x \wedge y) * y \wedge (x * y) * D(y, (x * y) * y) \\ &= D(x * y, x \wedge y) * y \wedge ((x * y) * [D(y,y) * (x * y) \wedge (x * y) * D(y,y)]) \\ &= D(x * y, x \wedge y) * y \wedge ((x * y) * [0 * (x * y) \wedge (x * y) * 0] \\ &= 0 \end{aligned}$$

Therefore, $x \wedge y \in Ker_D$.

Definition 3.12 Let D be a symmetric bi-derivation on a KU-algebra X. Then for a fixed element $a \in X$ we can define a set $Fix_D(\text{L})$ by

$$Fix_D(X) := \{x \in X \mid D(x,a) = x\}$$

Proposition 3.13 Let D be a symmetric bi-derivation on a KU-algebra X. Then $Fix_D(X)$ is a subalgebra of X.

Proof. Let $x, y \in Fix_D(X)$ we have $D(x,a) = x$ and $D(y,a) = y$ and so by using the definition of (l,r) symmetric bi-derivation we get

$$\begin{aligned} D(x * y, a) &= D(x,a) * y \wedge x * D(y,a) \\ &= x * y \wedge x * y \\ &= x * y \end{aligned}$$

Hence $x * y \in Fix_D(X)$.

Proposition 3.14 Let D be a symmetric bi-derivation on a KU-algebra X. If $x, y \in Fix_D(X)$ then $x \wedge y \in Fix_D(X)$.

Proof. Let $x, y \in Fix_D(X)$. Then we have $D(x,a) = x$ and $D(y,a) = y$. By using the definition of (l,r) symmetric bi-derivation and Proposition 3.13 we have

$$\begin{aligned} D(x \wedge y, a) &= D((x * y) * y, a) \\ &= D(x * y, a) * y \wedge (x * y) * D(y,a) \\ &= ((x * y) * y) \wedge ((x * y) * y) \\ &= (x * y) * y \\ &= x \wedge y \end{aligned}$$

Therefore, $x \wedge y \in Fix_D(X)$.

References

[1] C. Prabpayak and U. Leerawat, On Ideals and Congruence in KU-Algebras, Scientia Magna International Book Series, Vol. 5 (2009), No.1, 54-57.

[2] C. Prabpayak and U. Leerawat, On Isomorphisms of KU-Algebras, Scientia Magna International Book Series, Vol. 5 (2009), No.3, 25-31.

[3] S.M. Mostafa, R.A.K. Omar, A. Abd-eldayem Propeties of Derivations on KU-Algebras, Journal Of Advances in Mathematics, Vol.9, No 10,3085-3097.

[4] Jun, Y. B. and Xin, L. , On Derivations of BCI-Algebras, Information Sciences 159(2004), 167-176.

[5] Abujabal, H. A. S. and Al-Shehri, N. O., Some Results on Derivations of BCI-Algebras., J. Nat. sci. Math., 46:1 & 2(April & October 2006), 13-19.

[6] Y. Imai and Iseki K.,On Axiom Systems of Propositional Calculi, XIV, Proc. Japan Acad. Ser A, MAth Sci., 42(1966),19-22.

[7] Iseki K.,An Algebra related with a Propositional Calculi , Proc. Japan Acad. Ser A, MAth Sci., 42(1966),26-29.

[8] Iseki K. and Tanaka S.,An Introduction to Theory of BCK-Algebras, Math. Japo., 23(19789,1-26.

[9] Maksa Gy., A Remark on Symmetric Biadditive Functions Having Nonnegative Diagonalization, Glasnik Math., 15:35(1980), 279-282.

[10] Maksa Gy.,On the Trace of Symmetric Bi-Derivation, C. R. Math. Rep. Acad. Sci. Canada, 9(1987), 303-307.

[11] Vukman J., Symmetric Bi-Derivations on Prime and Semi-Prime Rings, Aequations Math., 38(1989), 245-254.

[12] Vukman J.,Two result Concerning Symmetric Bi-Derivations on Prime Rings, Aequations Math., 40(1990), 181-189.

[13] Çeven, Y., Symmetric bi-derivations of lattices, Quaestiones Mathematicae 32(2009), 241-245.

[14] S. Ilbira and A. Firat, ON Symmetric bi-derivations of BCI-Algebras, Applied Mathematical Sciences, Vol. 5, 2011, no. 60, 2957-2966.