Numerical Solution of Eikonal Equation Using Finite Difference Method

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Abstract

In this paper, a method to calculate tsunami wave front is introduced using the finite difference method to solve the ill-posed problem and to calculate perturbed velocity of the wave front. Comparison between the actual and approximate solution will be proposed in a table form and a graphic form.

Keywords

Wave Propagation, Eikonal Equation, Wave Front, Partial Differential Equation, Analytical Solutions, Tsunami Wave.

1.Introduction

Recently, many applications of eikonal equation have been introduced. In1987, Hitashi Ishii published a paper in the proceeding of the American mathematical Society regarding the Hamilton-Jacobi equations of eikonal type: $H(x,u,Du)=0,\ \ in\,\Omega\ \ where\,\Omega$ is an open subset of R^N,

$H:\Omega\times R\times R^N\to R, u\in C(\Omega)$ and denote the gradient of u by $|Du|$.**[1]**

Also, E.D. Moskalensky who used two-dimensional eikonal equation. Describing the front of a disturbance propagating. **[2],[3]**

$$\boxed{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\frac{1}{v^2(x,y)}}$$

Analytical solutions for the tsunami wave-front behavior above an uneven bottom can be described by the 2-D eikonal equation

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\frac{1}{v^2(x,y)} \qquad (1)$$

where $v(x,y)$ is the wave propagation velocity distribution in the environment. The position of a wave front at the time is given by the equation

$$f(x,y)=c$$

As a matter of fact, solutions of equation (1) are known only for rather a limited number of functions v, therefore usually the wave front position is obtained by numerical methods**.[4]** we will use the finite difference method to obtain the position of wave front.

2. Finite Difference Method

The finite Difference Method for solving differential equations is simple to understand and implement. However, it has one significant drawback: it can only be applied to meshes in which the cell faces are lined up with the coordinate axes. As such it becomes difficult, if not outright impossible to resolve curved boundaries- like those encountered when dealing with any realistic geometry. **[5],[6]**

The two- dimensional Eikonal equation is:

$$f_x{}^2 + f_y{}^2 = \emptyset^2$$

Where:

$\emptyset = \frac{1}{v}$ *and* $v(x, y)$ *the wave propagation velocity*

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{v^2(x, y)}$$

The Eikonal equation using finite difference method:

$$\left(\frac{f_{i+1,j} - f_{i,j}}{h}\right)^2 + \left(\frac{f_{i,j+1} - f_{i,j}}{k}\right)^2 = \emptyset^2(x_i, y_j)$$

$$\left(\frac{f_{i,j+1} - f_{i,j}}{k}\right)^2 = \emptyset^2(x_i, y_j) - \left(\frac{f_{i+1,j} - f_{i,j}}{h}\right)^2$$

$$\frac{f_{i,j+1}^2 - 2 * f_{i,j} * f_{i,j+1} + f_{i,j}^2}{k^2} = \emptyset^2(x_i, y_j) - \frac{(f_{i+1,j}^2 - 2 * f_{i,j} * f_{i+1,j} + f_{i,j}^2)}{h^2}$$

$$f_{i.j+1} = \sqrt{k^2 * \emptyset^2(x_i, y_j) + 2 * f_{i,j} * f_{i,j+1} - f_{i,j}^2 + \frac{k^2}{h^2} * (-f_{i+1,j}^2 + 2 * f_{i,j} * f_{i+1,j} - f_{i,j}^2)}$$

$$f_{i.j+1} = \sqrt{k^2 * \emptyset^2(x_i, y_j) + 2 * f_{i,j} * f_{i,j+1} - \frac{k^2}{h^2} * f_{i+1,j}^2 + 2 * \frac{k^2}{h^2} * f_{i,j} * f_{i+1,j} - \left(1 + \frac{k^2}{h^2}\right) * f_{i,j}^2}$$

$$\boldsymbol{when}\ \emptyset(x, y) = \frac{\sqrt{x^2 + y^2}}{\sqrt{(x+1)^2 + y^2}},$$

$$\boldsymbol{exact\ solution} = x - \frac{1}{2} * \ln((x+1)^2 + y^2)$$

$$\boldsymbol{when}\ \emptyset(x, y) = \frac{1}{|x|}$$

$$\boldsymbol{exact\ solution} = -\log\left(\frac{\sqrt{x^2 + y^2} + y}{x}\right), \quad for\ \boldsymbol{x > 0}$$

3.Numerical Results of Finite Difference Method

The numerical results of the finite difference method illustrated in the following tables corresponding to different selected values of i, j and hWhere $i = 1,2,3,4$; $j = 0,1,2,..,30$; with a corresponding step size ($h = 0.2$) and where $i = 1,2,3, \dots ,9$; $j = 0,1,2, \dots ,1$; with a corresponding step size ($h = 0.1$) illustrated in Table.(1) and Table.(2), respectively.

3.1 The Results when $\emptyset(x, y) = \frac{\sqrt{x^2+y^2}}{\sqrt{(x+1)^2+y^2}}$ [2]

Table. (1): Numerical results of the 2-D Eikonal equation using F.D.M when $\emptyset(x,y)=\frac{\sqrt{x^2+y^2}}{\sqrt{(x+1)^2+y^2}}$

i	j	$X(i)$	$Y(j)$	W	U	E
1	0	1.2	1	0.3176346015	0.3176346015	0.0000000000
	1	1.2	1.03	0.3019791618	0.3118651250	0.0098859632
	2	1.2	1.06	0.2869939907	0.3059755550	0.0189815643
	3	1.2	1.10	0.2726192591	0.2999708640	0.0273516049
	4	1.2	1.13	0.2588118878	0.2938559755	0.0350440877
	5	1.2	1.16	0.2455422899	0.2876357560	0.0420934661
	6	1.2	1.20	0.2327917355	0.2813150100	0.0485232745
	7	1.2	1.23	0.2205502103	0.2748984675	0.0543482572
	8	1.2	1.26	0.2088146611	0.2683907835	0.0595761224
	9	1.2	1.30	0.1975875570	0.2617965285	0.0642089715
	10	1.2	1.33	0.1868757077	0.2551201850	0.0682444773
	11	1.2	1.36	0.1766892972	0.2483661420	0.0716768448
	12	1.2	1.40	0.1670411039	0.2415386940	0.0744975901
	13	1.2	1.43	0.1579458869	0.2346420330	0.0766961461
	14	1.2	1.46	0.1494199205	0.2276802495	0.0782603290
	15	1.2	1.50	0.1414806722	0.2206573295	0.0791766573
	16	1.2	1.53	0.1341466160	0.2135771525	0.0794305365
	17	1.2	1.56	0.1274371827	0.2064434885	0.0790063058
	18	1.2	1.60	0.1213728482	0.1992600000	0.0778871518
	19	1.2	1.63	0.1159753668	0.1920302400	0.0760548732
	20	1.2	1.66	0.1112681614	0.1847576520	0.0734894906
	21	1.2	1.70	0.1072768845	0.1774455680	0.0701686835
	22	1.2	1.73	0.1040301734	0.1700972160	0.0660670426
	23	1.2	1.76	0.1015606301	0.1627157120	0.0611550819
	24	1.2	1.80	0.0999060656	0.1553040640	0.0553979984
	25	1.2	1.83	0.0991110647	0.1478651770	0.0487541123
	26	1.2	1.86	0.0992289422	0.1404018500	0.0411729078
	27	1.2	1.90	0.1003241881	0.1329167800	0.0325925919
	28	1.2	1.93	0.1024755256	0.1254125600	0.0229370344
	29	1.2	1.96	0.1057797498	0.1178916880	0.0121119382
	30	1.2	2	0.1103565620	0.1103565620	0.0000000000

2	0	1.4	1	0.4444885550	0.4444885550	0.0000000000
	1	1.4	1.03	0.4253273535	0.4395003705	0.0141730170
	2	1.4	1.06	0.4069293542	0.4344003505	0.0274709963
	3	1.4	1.10	0.3892903849	0.4291923875	0.0399020026
	4	1.4	1.13	0.3724102382	0.4238803580	0.0514701198
	5	1.4	1.16	0.3562929077	0.4184681160	0.0621752083
	6	1.4	1.20	0.3409465299	0.4129594870	0.0720129571
	7	1.4	1.23	0.3263831215	0.4073582620	0.0809751405
	8	1.4	1.26	0.3126181761	0.4016681925	0.0890500164
	9	1.4	1.30	0.2996701807	0.3958929840	0.0962228033
	10	1.4	1.33	0.2875600912	0.3900362930	0.1024762018
	11	1.4	1.36	0.2763108058	0.3841017220	0.1077909162
	12	1.4	1.40	0.2659466616	0.3780928180	0.1121461564
	13	1.4	1.43	0.2564929791	0.3720130640	0.1155200849
	14	1.4	1.46	0.2479756699	0.3658658800	0.1178902101
	15	1.4	1.50	0.2404209214	0.3596546200	0.1192336986
	16	1.4	1.53	0.2338549672	0.3533825680	0.1195276008
	17	1.4	1.56	0.2283039486	0.3470529390	0.1187489904
	18	1.4	1.60	0.2237938693	0.3406688720	0.1168750027
	19	1.4	1.63	0.2203506407	0.3342334360	0.1138827953
	20	1.4	1.66	0.2180002132	0.3277496200	0.1097494068
	21	1.4	1.70	0.2167687842	0.3212203400	0.1044515558
	22	1.4	1.73	0.2166830682	0.3146484340	0.0979653658
	23	1.4	1.76	0.2177706108	0.3080366630	0.0902660522
	24	1.4	1.80	0.2200601215	0.3013877120	0.0813275905
	25	1.4	1.83	0.2235817927	0.2947041860	0.0711223933
	26	1.4	1.86	0.2283675609	0.2879886140	0.0596210531
	27	1.4	1.90	0.2344512564	0.2812434520	0.0467921956
	28	1.4	1.93	0.2418685678	0.2744710740	0.0326025062
	29	1.4	1.96	0.2506567285	0.2676737800	0.0170170515
	30	1.4	2	0.2608538000	0.2608538000	0.0000000000
3	0	1.6	1	0.5755088330	0.5755088330	0.0000000000
	1	1.6	1.03	0.5567670515	0.5711606700	0.0143936185

	2	1.6	1.06	0.5388356746	0.5667092870	0.0278736124
	3	1.6	1.10	0.5217044149	0.5621577540	0.0404533391
	4	1.6	1.13	0.5053674542	0.5575091390	0.0521416848
	5	1.6	1.16	0.4898237435	0.5527665100	0.0629427665
	6	1.6	1.20	0.4750769592	0.5479329230	0.0728559638
	7	1.6	1.23	0.4611352224	0.5430114240	0.0818762016
	8	1.6	1.26	0.4480106623	0.5380050410	0.0899943787
	9	1.6	1.30	0.4357188857	0.5329167800	0.0971978943
	10	1.6	1.33	0.4242784065	0.5277496200	0.1034712135
	11	1.6	1.36	0.4137100666	0.5225065140	0.1087964474
	12	1.6	1.40	0.4040364855	0.5171903810	0.1131538955
	13	1.6	1.43	0.3952815585	0.5118041050	0.1165225465
	14	1.6	1.46	0.3874700195	0.5063505310	0.1188805115
	15	1.6	1.50	0.3806270864	0.5008324640	0.1202053776
	16	1.6	1.53	0.3747781945	0.4952526650	0.1204744705
	17	1.6	1.56	0.3699488227	0.4896138500	0.1196650273
	18	1.6	1.60	0.3661644177	0.4839186860	0.1177542683
	19	1.6	1.63	0.3634504105	0.4781697930	0.1147193825
	20	1.6	1.66	0.3618323211	0.4723697400	0.1105374189
	21	1.6	1.70	0.3613359428	0.4665210420	0.1051850992
	22	1.6	1.73	0.3619875911	0.4606261650	0.0986385739
	23	1.6	1.76	0.3638143995	0.4546875170	0.0908731175
	24	1.6	1.80	0.3668446358	0.4487074540	0.0818628182
	25	1.6	1.83	0.3711080074	0.4426882740	0.0715802666
	26	1.6	1.86	0.3766359127	0.4366322230	0.0599963103
	27	1.6	1.90	0.3834615843	0.4305414890	0.0470799047
	28	1.6	1.93	0.3916200518	0.4244182020	0.0327981502
	29	1.6	1.96	0.4011478315	0.4182644400	0.0171166085
	30	1.6	2	0.4120822220	0.4120822220	0.0000000000
4	0	1.8	1	0.7103565620	0.7103565620	0.0000000000
	1	1.8	1.03	0.6961652382	0.7065375980	0.0103723598
	2	1.8	1.06	0.6827772816	0.7026238060	0.0198465244
	3	1.8	1.10	0.6701093050	0.6986176210	0.0285083160

	4	1.8	1.13	0.6580988462	0.6945214920	0.0364226458
	5	1.8	1.16	0.6467003138	0.6903378750	0.0436375612
	6	1.8	1.20	0.6358817230	0.6860692260	0.0501875030
	7	1.8	1.23	0.6256220504	0.6817180040	0.0560959536
	8	1.8	1.26	0.6159090772	0.6772866600	0.0613775828
	9	1.8	1.30	0.6067376257	0.6727776410	0.0660400153
	10	1.8	1.33	0.5981081154	0.6681933810	0.0700852656
	11	1.8	1.36	0.5900253820	0.6635363020	0.0735109200
	12	1.8	1.40	0.5824977220	0.6588088070	0.0763110850
	13	1.8	1.43	0.5755361314	0.6540132840	0.0784771526
	14	1.8	1.46	0.5691537192	0.6491520960	0.0799983768
	15	1.8	1.50	0.5633652809	0.6442275830	0.0808623021
	16	1.8	1.53	0.5581870226	0.6392420600	0.0810550374
	17	1.8	1.56	0.5536364335	0.6341978120	0.0805613785
	18	1.8	1.60	0.5497323077	0.6290970970	0.0793647893
	19	1.8	1.63	0.5464949157	0.6239421380	0.0774472223
	20	1.8	1.66	0.5439463405	0.6187351280	0.0747887875
	21	1.8	1.70	0.5421109886	0.6134782220	0.0713672334
	22	1.8	1.73	0.5410163008	0.6081735420	0.0671572412
	23	1.8	1.76	0.5406936874	0.6028231720	0.0621294846
	24	1.8	1.80	0.5411797335	0.5974291600	0.0562494265
	25	1.8	1.83	0.5425177236	0.5919935100	0.0494757864
	26	1.8	1.86	0.5447595595	0.5865181920	0.0417586325
	27	1.8	1.90	0.5479681650	0.5810051350	0.0330369700
	28	1.8	1.93	0.5522205045	0.5754562240	0.0232357195
	29	1.8	1.96	0.5576113741	0.5698733060	0.0122619319
	30	1.8	2	0.5642581860	0.5642581860	0.0000000000

3.2 The Results when $\emptyset = 1/|x|$

Table. (2): Numerical results of the 2-D Eikonal equation using F.D.M when $\emptyset = \mathbf{1}/|\boldsymbol{x}|$

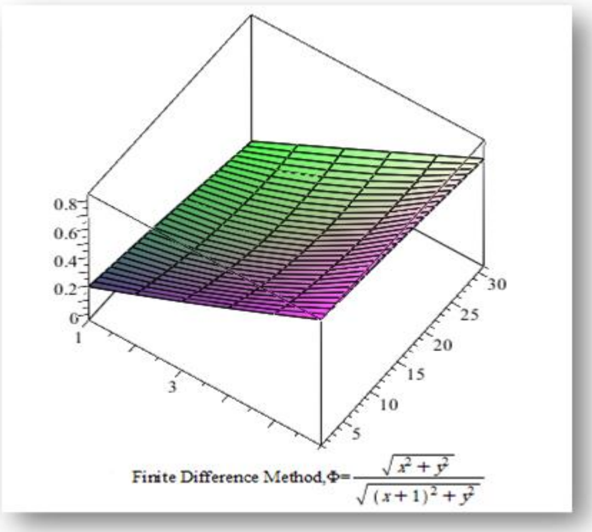
i	j	$X(i)$	$Y(j)$	W	U	E
1	0	1.1	1	-0.815608901	-0.81560890	0.000000000
	1	1.1	1.1	-0.903569533	-0.88137359	0.022195946
	2	1.1	1.2	-0.978699255	-0.94421786	0.034481392
	3	1.1	1.3	-1.045743821	-1.00428118	0.041462646
	4	1.1	1.4	-1.106819220	-1.06171232	0.045106898
	5	1.1	1.5	-1.162945528	-1.11666280	0.046282733
	6	1.1	1.6	-1.214532568	-1.16928220	0.045250371
	7	1.1	1.7	-1.261481324	-1.21971524	0.041766083
	8	1.1	1.8	-1.303036613	-1.26809990	0.034936713
	9	1.1	1.9	-1.337235593	-1.31456638	0.022669211
	10	1.1	2	-1.359236682	-1.35923668	0.000000000
2	0	1.2	1	-0.758486137	-0.75848614	0.000000000
	1	1.2	1.1	-0.853257751	-0.82120395	0.032053802
	2	1.2	1.2	-0.933399301	-0.88137359	0.052025714
	3	1.2	1.3	-1.003131280	-0.93908906	0.064042216
	4	1.2	1.4	-1.064923635	-0.99445749	0.070466141
	5	1.2	1.5	-1.120108734	-1.04759301	0.072515721
	6	1.2	1.6	-1.169195745	-1.09861229	0.070583456
	7	1.2	1.7	-1.211942911	-1.14763132	0.064311594
	8	1.2	1.8	-1.247221574	-1.19476322	0.052458357
	9	1.2	1.9	-1.272630303	-1.24011680	0.032513508
	10	1.2	2	-1.283795663	-1.28379566	0.000000000
3	0	1.3	1	-0.708460844	-0.70846084	0.000000000

	1	1.3	1.1	-0.803901347	-0.76830412	0.035597231
	2	1.3	1.2	-0.885212052	-0.82592176	0.059290293
	3	1.3	1.3	-0.955516663	-0.88137359	0.074143076
	4	1.3	1.4	-1.016986216	-0.93473416	0.082252055
	5	1.3	1.5	-1.070913486	-0.98608752	0.084825969
	6	1.3	1.6	-1.117803845	-1.03552306	0.082280786
	7	1.3	1.7	-1.157388484	-1.08313244	0.074256045
	8	1.3	1.8	-1.188544698	-1.12900727	0.059537429
	9	1.3	1.9	-1.209141199	-1.17323748	0.035903718
	10	1.3	2	-1.215910213	-1.21591021	0.000000000
4	0	1.4	1	-0.664330605	-0.66433060	0.000000000
	1	1.4	1.1	-0.757079921	-0.72147408	0.035605841
	2	1.4	1.2	-0.836889765	-0.77667027	0.060219493
	3	1.4	1.3	-0.906024346	-0.82995362	0.076070730
	4	1.4	1.4	-0.966233446	-0.88137359	0.084859859
	5	1.4	1.5	-1.018634168	-0.93099029	0.087643876
	6	1.4	1.6	-1.063677398	-0.97887086	0.084806536
	7	1.4	1.7	-1.101122046	-1.02508660	0.076035449
	8	1.4	1.8	-1.129997587	-1.06971074	0.060286846
	9	1.4	1.9	-1.148581408	-1.11281682	0.035764592
	10	1.4	2	-1.154477394	-1.15447740	0.000000000
5	0	1.5	1	-0.625145117	-0.62514512	0.000000000
	1	1.5	1.1	-0.713233398	-0.67976049	0.033472913
	2	1.5	1.2	-0.789757172	-0.73266826	0.057088917
	3	1.5	1.3	-0.856425627	-0.78388429	0.072541336
	4	1.5	1.4	-0.914634944	-0.83343897	0.081195975

	5	1.5	1.5	-0.965312194	-0.88137359	0.083938606
	6	1.5	1.6	-1.008839666	-0.92773729	0.081102372
	7	1.5	1.7	-1.045014685	-0.97258456	0.072430124
	8	1.5	1.8	-1.073032079	-1.01597313	0.057058945
	9	1.5	1.9	-1.091509952	-1.05796243	0.033547523
	10	1.5	2	-1.098612289	-1.09861229	0.000000000
6	0	1.6	1	-0.590143686	-0.59014369	0.000000000
	1	1.6	1.1	-0.672279141	-0.64239610	0.029883039
	2	1.6	1.2	-0.744263814	-0.69314718	0.051116633
	3	1.6	1.3	-0.807503219	-0.74239938	0.065103835
	4	1.6	1.4	-0.863147872	-0.79016873	0.072979146
	5	1.6	1.5	-0.911964719	-0.83648187	0.075482850
	6	1.6	1.6	-0.954269434	-0.88137359	0.072895847
	7	1.6	1.7	-0.989889401	-0.92488458	0.065004826
	8	1.6	1.8	-1.018149550	-0.96705963	0.051089919
	9	1.6	1.9	-1.037895696	-1.00794617	0.029949530
	10	1.6	2	-1.047593013	-1.04759301	0.000000000
7	0	1.7	1	-0.558710602	-0.55871060	0.000000000
	1	1.7	1.1	-0.633838444	-0.60875665	0.025081797
	2	1.7	1.2	-0.700270424	-0.65747892	0.042791505
	3	1.7	1.3	-0.759303670	-0.70487039	0.054433283
	4	1.7	1.4	-0.811930203	-0.75093638	0.060993820
	5	1.7	1.5	-0.858785126	-0.79569230	0.063092822
	6	1.7	1.6	-0.900121422	-0.83916153	0.060959890
	7	1.7	1.7	-0.935788122	-0.88137359	0.054414535
	8	1.7	1.8	-0.965208324	-0.92236254	0.042845785

	9	1.7	1.9	-0.987364843	-0.96216567	0.025199171
	10	1.7	2	-1.000822379	-1.00082238	0.000000000
8	0	1.8	1	-0.530342598	-0.53034260	0.000000000
	1	1.8	1.1	-0.597294307	-0.57832922	0.018965090
	2	1.8	1.2	-0.657172861	-0.62514512	0.032027744
	3	1.8	1.3	-0.711291526	-0.67077669	0.040514840
	4	1.8	1.4	-0.760504334	-0.71522142	0.045282911
	5	1.8	1.5	-0.805309808	-0.75848614	0.046823671
	6	1.8	1.6	-0.845894790	-0.80058527	0.045309524
	7	1.8	1.7	-0.882134484	-0.84153933	0.040595154
	8	1.8	1.8	-0.913557556	-0.88137359	0.032183969
	9	1.8	1.9	-0.939267014	-0.92011684	0.019150174
	10	1.8	2	-0.957800449	-0.95780045	0.000000000
9	0	1.9	1	-0.504624665	-0.50462467	0.000000000
	1	1.9	1.1	-0.561711042	-0.55068879	0.011022247
	2	1.9	1.2	-0.613904580	-0.59571437	0.018190210
	3	1.9	1.3	-0.662429908	-0.63968336	0.022746550
	4	1.9	1.4	-0.707880160	-0.68258761	0.025292551
	5	1.9	1.5	-0.750554946	-0.72442747	0.026127480
	6	1.9	1.6	-0.790559118	-0.76521040	0.025348721
	7	1.9	1.7	-0.827817822	-0.80494974	0.022868083
	8	1.9	1.8	-0.862039585	-0.84366355	0.018376036
	9	1.9	1.9	-0.892590064	-0.88137359	0.011216476
	10	1.9	2	-0.918104422	-0.91810442	0.000000000

4.Graph of Eikonal Equation Using F.D.M



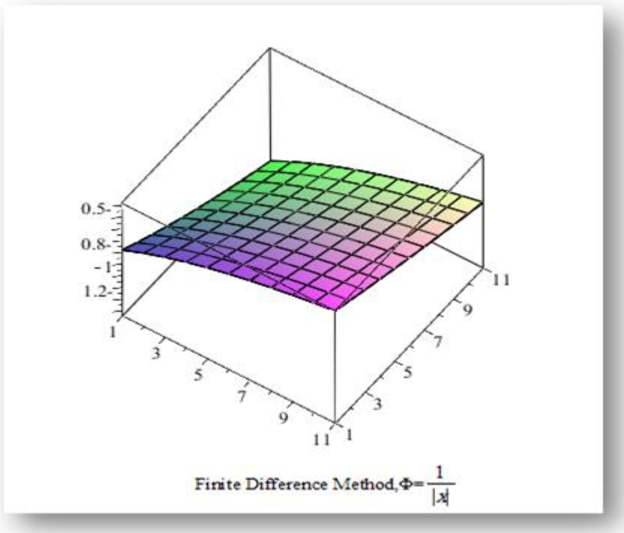


Figure 1. Plot of the 2-D Eikonal equation using F.D.M when $\emptyset(x,y)=\frac{\sqrt{x^2+y^2}}{\sqrt{(x+1)^2+y^2}}$

Figure 2. Plot of the 2-D Eikonal equation using F.D.M when $\emptyset = \mathbf{1}/|\boldsymbol{x}|$

5.Graphs of the Comparison Between Actual Solution and Approximate Solution

6. Conclusion

The purpose of this paper is to represent the solution of two-dimensional Eikonal equation using finite difference method. The results obtained by this method can by described as good results and that was achieved by applying the comparison between the results of the actual solution and the approximate solution.

Also, by adding more information about a particular sea, we can save human beings from the distractive effects due to tsunami. Finally, earthquakes can be fore cared using this dynamical case.

ACKNOWLEDGEMENT

The author wishes to gratefully acknowledge the referees of this paper who helped to improve the presentation.

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