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**CONGRUENCES ON *-SIMPLE TYPE A I-SEMIGROUPS**

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**Abstract**

This paper obtains a characterisation of the congruences on *-simple type A I-semigroups. The *-locally idempotent-separating congruences, strictly *-locally idempotent-separating congruences and minimum cancellative monoid congruences, are characterised.

**Keywords:** Type A I-Semigroup, *-Locally Idempotent-Separating, Cancellative Monoid Congruence, Generalized Bruck-Reilly *-Extensions.

## 1. Introduction

For a semigroup $S$, $E(S)$ will denote the set of idempotents of $S$. If $S$ is a semigroup with non-empty set of idempotents $E(S)$, we define a partial order "≤" on E (S) such that $e \leq f$ if and only if $ef = fe = e$. Let $I$ denote the set of all integers and let $\mathbb{N}^0$ denote the set of non-negative integers. A semigroup $S$ is said to be an $I$-semigroup if and only if $E(S)$ is order isomorphic to $I$ under the reverse of the partial order.

The structure theorem for *-simple type A I-semigroups was established in [8], as an extension of the structure theorem for simple I-inverse semigroups and *-simple type A $\omega$-semigroups due to Warne [10] and Asibong-Ibe [1]. This paper is a follow up of the study of congruences on *-bisimple type A I-semigroups studied by Ndubuisi and Asibong-Ibe [7], where the congruences were identified as idempotent-separating congruence and minimum cancellative monoid congruence.

Earlier investigations in [6] and [10] studied congruences on *-simple type A $\omega$-semigroups and congruences on simple I-inverse semigroups respectively. Determination of congruences throughout this paper is based on their description in [6].

This work is divided as follows. Section 2 contains a minimum of results concerning *-simple type A $I$-semigroups. The content of section 3 is a determination of *-locally idempotent-separating congruences, strictly *-locally idempotent-separating congruences and minimum cancellative monoid congruences of a *-simple type A I-semigroup.

Let us recall some definitions which will be useful in the study.

Let $S$ be a semigroup and let $a, b \in S$. Then the elements $a$ and $b$ are said to to be $\mathcal{R}^*$-related written $a\,\mathcal{R}^*\,b$ if and only if for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$. The relation $\mathcal{L}^*$ is defined

dually. The join of the equivalence relations $\mathcal{R}^*$ and $\mathcal{L}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. Thus $a\,\mathcal{H}^*b$ if and only if $a\,\mathcal{R}^*b$ and $a\,\mathcal{L}^*b$. In general $\mathcal{R}^* \circ \mathcal{L}^* \neq \mathcal{L}^* \circ \mathcal{R}^*$ as shown in [3].

Following Fountain [4] a semigroup is an abundant semigroup if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class in $S$ contain idempotents. An abundant semigroup $S$ is adequate [3] if $E(S)$forms a semilattice. In an adequate semigroup every $\mathcal{L}^*$-class $\mathcal{R}^*$-class contains a unique idempotent.

Let $a$ be an element of an adequate semigroup $S$, and $a^*$ $(a^\dagger)$ denotes the unique idempotent in the $\mathcal{L}^*$-class $L_a^*$ ( $\mathcal{R}^*$-class $R_a^*$ ) containing $a$.

We remark that a type A (in particular, right type A) semigroup realized in Fountain [2] as a special type of right PP monoid with $e$-cancellable element where $e \in E(S)$, the set of idempotents in $S$. An adequate semigroup $S$ is said to be a type A semigroup if $ea = a(ea)^*$ and $ae = (ae)^\dagger a$ for all $a \in S$ and $e \in E(S)$.

We conclude this section by defining the relation $\mathcal{J}^*$. Let $S$ be a semigroup and $I^*$ be an ideal of $S$. Then $I^*$ is said to be a *-ideal if $L_a^* \subseteq I^*$ and $R_a^* \subseteq I^*$ for all $a \in I^*$. The smallest *-ideal containing 'a' is the principal *-ideal generated by 'a' and is denoted by $J^*(a)$. For $a, b \in S, a\, \mathcal{J}^* b$ if and only if $J^*(a) = J^*(b)$. The relations $\mathcal{J}^*$ contains $\mathcal{D}^*$.

A semigroup $S$ is said to be *-simple if the only *-ideal of $S$ is itself. Clearly a semigroup is *-simple if all its elements are $\mathcal{J}^*$-related. To have a clear picture of $\mathcal{J}^*$-related elements we recall the following Lemma.

**Lemma 1.1 [3].** Let $S$ be a semigroup and $a, b \in S$. Then $b \in J^*(a)$ if and only if there are elements $a_0, a_1, \ldots, a_n \in S$, $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $a_i \mathcal{D}^* x_i a_{i-1} y_i$, for $i = 1,2, \ldots, n$.

Other basic results discussed in [3] will be assumed. The notation adopted in this paper is similar to that in Fountain [3], Howie [5], Asibong-Ibe [1] and Makanjuola [6].

Recently type A semigroups have been shown to be special type of restriction semigroups. In this case type A $\omega$-semigroup will essentially be an $\omega$-restriction semigroups. The idea developed here will prove useful in the study of restriction semigroups.

However, we will in this work retain the term type A semigroups generally.

## 2. *-Simple Type A I-Semigroups

Following [9], let $T = \bigcup_{i=0}^{d-1} M_i$ be a chain of cancellative monoids. Each element $x_i \in T$ is necessarily in $M_i$ for $0 \le i \le d-1$. An identity $e_i \in M_i$ is an idempotent in $T$. Clearly $e_i \in T$ form a chain of idempotents $e_0 > e_i > \cdots > e_{d-1}$.

Let $\theta : T \to M_0$ be a monoid morphism and let $S = T \times I \times I$ (where $I$ is the set of all integers) be the set of all ordered triples $(x_i, m, n)$ where $m \in \mathbb{N}^0, n \in I$, $0 \le i \le d-1$ and $x_i \in T$.

Define multiplication on S by the rule

$$(x_i, m, n)(y_j, p, q) = \begin{cases} \left(x_i . f_{n-p,p}^{-1} . y_j \theta^{n-p} . f_{n-p,q}, m, n+q-p\right) & \text{if } n \ge p \\ \left(f_{p-n,m}^{-1} . x_i \theta^{p-n} . f_{p-n,n} . y_j, m+p-n, q\right) & \text{if } n \le p \end{cases}$$

where $\theta^0$ is the identity automorphism of $T$, and for $m \in \mathbb{N}^0$, $n \in I$, $f_{0,n} = e_i$, the identity of $M_i$, while for $m > 0$, $f_{m,n} = u_{n+1}\theta^{m-1} . u_{n+2}\theta^{m-2} \ldots u_{n+(m-1)}\theta . u_{n+m}$, and

$f_{m,n}^{-1} = u_{n+m}^{-1} . u_{n+(m-1)}^{-1}\theta \ldots u_{n+2}^{-1}\theta^{m-2} . u_{n+1}^{-1}\theta^{m-1}$, where $\{u_n : n \in I\}$ is a collection of T with $u_n = e_i$ for $n > 0$. Denote a semigroup formed by $S = GBR^*(T, \theta)$ where $T = \bigcup_{i=0}^{d-1} M_i$.

If for each $i$ we now let $M_i = \{e_i\}$, a monoid with one element, we obtain the set $I \times I$ under the multiplication

$$(md+i, nd+i)(pd+j, qd+j) = \begin{cases} (md+i, (n+q-p)d+i) & \text{if } n \ge p \\ ((m+p-n)d+j, qd+j) & \text{if } n \le p \end{cases}$$

We denote $I \times I$ under the above multiplication by $B_d^*$ and call it the extended bicyclic semigroup.

If we let $(x_i, m, n)$ be an idempotent in $S$. Then

$$(x_i, m, n) = (x_i, m, n)(x_i, m, n) = \begin{cases} \left(x_i . f_{n-m,m}^{-1} . x_i\theta^{n-m} . f_{n-m,n}, m, n-m+n\right) & \text{if } n \geq m \\ \left(f_{m-n,m}^{-1} . x_i\theta^{m-n} . f_{m-n,n} . x_i, m-n+m, n\right) & \text{if } n \leq m \end{cases}$$

in which case $m = n$, $x_i^2 = x_i$ .

Conversely, suppose $x_i^2 = x_i$ then we have that $(x_i, m, n)(x_i, m, n) = (x_i, m, n)$. Thus $(x_i, m, n)$ is an idempotent if and only if $m = n$ and $x_i$ is an idempotent in $S$.

The following results were proved in [8].

**Lemma 2.1.** Let $S = GBR^*(T, \theta)$ be a generalized Bruck-Reilly *-extension of a monoid $T$, where $T = \bigcup_{i=0}^{d-1} M_i$ is a finite chain of cancellative monoids $M_i$. Let $(x_i, m, n), (y_j, p, q) \epsilon\, S$ . Then

i) $(x_i, m, n)\, \mathcal{R}^* (y_j, p, q)$ if and only if $m = p$ and $i = j$.

ii) $(x_i, m, n)\, \mathcal{L}^* (y_j, p, q)$ if and only if $n = q$ and $i = j$.

iii) $(x_i, m, n)\, \mathcal{J}^* (y_j, p, q)$. That is $S$ is *-simple.

**Lemma 2.2.** $S = GBR^*(T, \theta)$ is a type A semigroup if and only $T$ is a type A semigroup

**Theorem 2.3.** Let $S = GBR^*(T, \theta)$ be the generalized Bruck-Reilly *-extension of the monoid $T$ where $T = \bigcup_{i=0}^{d-1} M_i$ . Then $S$ is a *-simple type A I-semigroup with $d$ $\mathcal{D}^*$-classes.

We conclude this section, with the structure theorem of *-simple type A I-semigroups.

**Theorem 2.4 [8].** Let $S$ be a *-simple type A I-semigroup with $d$ $\mathcal{D}^*$-classses. Then $S$ is isomorphic to a generalized Bruck-Reilly *-extension $S = GBR^*(T, \theta)$ of a monoid $T$ , where $T = \bigcup_{i=0}^{d-1} M_i$ is a finite chain of cancellative monoids $M_i$ and $\theta$ is an endomorphism of $T$ with image in $M_0$ .

**3. The Congruences**

In this section, we will determine the congruence relations on a *-simple type A $I$-semigroup $S = GBR^*(T, \theta)$. We first present the properties of the congruences and then show that every congruence relation $\rho$ on $S$ is either a *-locally idempotent-separating congruence (if no two distinct $\mathcal{D}^*$-related idempotents are $\rho$-related) or all the idempotents are in one $\rho$-class. We also provide a method for constructing the strictly *-locally idempotent separating congruences. Lastly, we show that there is a minimum cancellative monoid congruence on $S$ .

**Lemma 3.1.** Let $S = GBR^*(T, \theta)$ be a *-simple type A I-semigroup where $T = \bigcup_{i=0}^{d-1} M_i$ is a semilattice of cancellative monoids. Then $\mathcal{H}^*$ is a congruence on $S$, and $S/\mathcal{H}^* \cong B_d^*$.

**Proof.** The mapping $\theta : S \to B_d^*$ by

$$(x_i, m, n)\theta = (md + i, nd + i)$$

is onto. It is a homomorphism since

$$\big((x_i,m,n)(y_j,p,q)\big)\theta = \begin{cases} \big(x_i.f^{-1}_{n-p,p}.y_j\theta^{n-p}.f_{n-p,q},m,n+q-p\big) & \text{if } n \geq p \\ \big(f^{-1}_{p-n,m}.x_i\theta^{p-n}.f_{p-n,n}.y_j,m+p-n,q\big) & \text{if } n \leq p \end{cases} \times \theta$$

$$= \begin{cases} \big(x_i.f^{-1}_{n-p,p}.y_j\theta^{n-p}.f_{n-p,q},m,n+q-p\big)\theta & \text{if } n \geq p \\ \big(f^{-1}_{p-n,m}.x_i\theta^{p-n}.f_{p-n,n}.y_j,m+p-n,q\big)\theta & \text{if } n \leq p \end{cases}$$

$$= \begin{cases} (md+i,(n+q-p)d+i) & \text{if } n \geq p \\ \big((m+p-n)d+j,qd+j\big) & \text{if } n \leq p \end{cases}$$

$$= (md+i,nd+i)(pd+j,qd+j)$$

$$= (x_i,m,n)\theta\,\big(y_j,p,q\big)\theta.$$

Thus $\theta$ is a homomorphism.

Furthermore, $\big((x_i,m,n)(y_j,p,q)\big) \epsilon\ \mathcal{H}^*$ if and only if $(md+i,\ nd+i) = (pd+j,qd+j)$; hence $\theta \circ \theta^{-1} = \mathcal{H}^*$ and the result follows.

**Lemma 3.2.** Let $\rho$ be a congruence on a *-simple type A $I$-semigroup $S = GBR^*(T,\theta)$ where $T = \cup_{i=0}^{d-1} M_i$ . Suppose that

(i) $(e_i,m,m)\ \rho\ \big(e_j,m,m\big)$ then for any $n \,\epsilon\, I$, $(e_i,n,n)\ \rho\ \big(e_j,n,n\big)$

(ii) $(e_i,m,m)\ \rho\ \big(e_j,m+1,m+1\big)$ then for any $n \,\epsilon\, I$, $(e_i,n,n)\ \rho\ \big(e_j,n+1,n+1\big)$

**Proof.** i) Let $(e_0,n,m),(e_0,m,n) \,\epsilon\, S$, then

$$(e_0,n,m)(e_i,m,m)(e_0,m,n) = (e_i,n,n).$$

$$(e_0,n,m)\big(e_j,m,m\big)(e_0,m,n) = \big(e_j,n,n\big),$$

and

$$(e_0,m,n)(e_i,n,n)(e_0,n,m) = (e_i,m,m).$$

$$(e_0,m,n)\big(e_j,n,n\big)(e_0,n,m) = \big(e_j,m,m\big).$$

ii) Let $(e_0,n,m),(e_0,m,n) \,\epsilon\, S$, then we have

$$(e_0,n,m)(e_i,m,n) = (e_i,n,n).$$

$$(e_0,n,m)\big(e_j,m+1,m+1\big)(e_0,m,n) = \big(e_j,n+1,n+1\big),$$

and

$$(e_0,m,n)(e_i,n,n)(e_0,n,m) = (e_i,m,m).$$

$$(e_0,m,n)\big(e_j,n+1,n+1\big)(e_0,n,m) = \big(e_j,m+1,m+1\big).$$

Hence the proof.

We now establish an important property of congruences on *-simple type A $I$-semigroups.

**Theorem 3.3.** A congruence $\rho$ on a *-simple type A *I*-semigroup is either a *-locally idempotent-separating congruence or all the idempotents are in one $\rho$-class.

**Proof.** Suppose that the idempotent elements of $S$ are not in one $\rho$-class and $e_{md+i}\,\rho\, e_{(m+k)d+i}$ for some $m,k \epsilon I, k>0$ and $0 \le i \le d-1$. We are to show that no two distinct $\mathcal{D}^*$-related idempotents are $\rho$-related. Let $k=1$ which implies that $e_{md+i}\,\rho\, e_{(m+1)d+i}$ . Using Lemma 3.2 and the fact that $e_m\,\rho\, e_n$ implies $e_m\,\rho\, e_k$ for every $n \le k \le m$ together with the transitive property of the congruence, we see that the idempotents are in one $\rho$-class which is contrary to our assumption. Thus, no two distinct $\mathcal{D}^*$-related idempotents are $\rho$-related. Therefore $\rho$ is a *-locally idempotent-separating congruence. This completes the proof.

A typical *-idempotent-separating congruence of a *-simple type A *I*-semigroup is characterized in the theorem.

**Theorem 3.4.** Let $S = GBR^*(T,\theta)$ where $T = \cup_{i=0}^{d-1} M_i$ . The relation $\rho$ on $S = GBR^*(T,\theta)$ defined by the rule:

$$(x_i, m, n)\;\rho\;(y_j, p, q) \text{ if and only if } m=p,\; n=q, i=j \text{ and } (x_i, y_j) \epsilon \ker\theta$$

is a *-locally idempotent separating congruence

**Proof.** It can be easily shown that $\rho$ is reflexive and symmetric. To show transitivity, we let $(x_i,m,n)\,\rho\,(y_j,p,q)$, $(y_j,p,q)\,\rho\,(z_k,u,v)$ for all $(x_i,m,n),(y_j,p,q),(z_k,u,v) \epsilon S$. Then $m=p,\; n=q, i=j, (x_i,y_j) \epsilon \ker\theta$ and $p=u,\; q=v,\; j=k, (y_j,z_k) \epsilon \ker\theta$.

Consequently, $m=u,\; n=v, i=k$. Hence $(x_i,z_k) \epsilon \ker\theta$, which means that $\rho$ is transitive.

Next is to show that $\rho$ is a congruence. Now let $a=(x_i,m,n),\; b=(y_j,p,q)$. That $\rho$ is a congruence entails showing that

$$a\,\rho\, b \text{ implies } ag\,\rho\, bg \quad \text{(for right congruence)}$$

$$a\,\rho\, b \text{ implies } ga\,\rho\, gb \quad \text{(for left congruence)}$$

$\forall\; g=(z_k,w,l) \epsilon S = GBR^*(T,\theta)$.

Consequently,

$$ag = (x_i,m,n)(z_k,w,l)$$

$$= \begin{cases} (x_i.\, f^{-1}_{n-w,w}.\, z_k\theta^{n-w}.\, f_{n-w,l}, m, n+l-w) & \text{if } n \ge w \\ (f^{-1}_{w-n,m}.\, x_i\theta^{w-n}.\, f_{w-n,n}.\, z_k, m+w-n, l) & \text{if } n \le w \end{cases}$$

$$bg = (y_j,p,q)(z_k,w,l)$$

$$= \begin{cases} (y_j.\, f^{-1}_{q-w,w}.\, z_k\theta^{q-w}.\, f_{q-w,l}, p, q+l-w) & \text{if } q \ge w \\ (f^{-1}_{w-q,p}.\, y_j\theta^{w-n}.\, f_{w-q,q}.\, z_k, p+w-q, l) & \text{if } q \le w \end{cases}$$

So, if $(x_i,m,n)\,\rho\,(y_j,p,q)$, then

$$(x_i,m,n)(z_k,w,l)\;\rho\;(y_j,p,q)(z_k,w,l) =$$

$$\begin{cases}(x_i.f_{n-w,w}^{-1}.z_k\theta^{n-w}.f_{n-w,l},m,n+l-w) & \text{if } n\geq w\\ (f_{w-n,m}^{-1}.x_i\theta^{w-n}.f_{w-n,n}.z_k,m+w-n,l) & \text{if } n\leq w\end{cases}$$

$$\rho\ \begin{cases}(y_j.f_{q-w,w}^{-1}.z_k\theta^{q-w}.f_{q-w,l},p,q+l-w) & \text{if } q\geq w\\ (f_{w-q,p}^{-1}.y_j\theta^{w-q}.f_{w-q,q}.z_k,p+w-q,l) & \text{if } q\leq w\end{cases}$$

But $(x_i,m,n)\ \rho\ (y_j,p,q)$ if and only if $m=p,\ n=q,\ i=j$ and $(x_i,y_j)\ \epsilon \ker\theta$.

Thus,
$$\begin{cases}(x_i.f_{n-w,w}^{-1}.z_k\theta^{n-w}.f_{n-w,l},m,n+l-w) & \text{if } n\geq w\\ (f_{w-n,m}^{-1}.x_i\theta^{w-n}.f_{w-n,n}.z_k,m+w-n,l) & \text{if } n\leq w\end{cases}$$

$$\rho\ \begin{cases}(y_j.f_{n-w,w}^{-1}.z_k\theta^{n-w}.f_{n-w,l},m,n+l-w) & \text{if } n\geq w\\ (f_{w-n,m}^{-1}.y_j\theta^{w-n}.f_{w-n,n}.z_k,m+w-n,l) & \text{if } n\leq w\end{cases}$$

Hence $\rho$ is a right congruence.

That $\rho$ is a left congruence follows similarly. Thus $\rho$ is a congruence.

Furthermore, $(e_i,m,m)\ \rho\ (e_i,n,n)$ implies $m=n$ which implies $(e_i,m,m)=(e_i,n,n)$. Thus any two distinct idempotent elements which are $\mathcal{D}^*$-related cannot lie in the same $\rho$-class. Hence the proof.

We will now construct the strictly *-locally idempotent-separating congruences on *-simple type A $I$-semigroups.

**3.5. Notation.** Let $k_0,k_1,k_2,k_3,\dots,k_t$ be a sequence of non-empty integers, satisfying $0\leq k_0<k_1\dots<k_t<d-1,\ k_0=-1,\ k_{t+1}=d-1$.

Define a relation $\rho=\rho(k_0,k_1,\dots,k_t)$ on $S=GBR^*(T,\theta)$ by

$$(x_i,m,n)\ \rho\ (y_j,p,q)\ \text{ implies } \begin{cases}m=p,\ n=q \text{ for } k_{v-1}<i,j\leq k_v,\ \ 0\leq v\leq t+1\\ \text{or } \ m=p+1,\ n=q+1 \text{ for } i\leq k_0 \text{ and } j>k_t\\ \text{or } \ m+1=p,\ n+1=q \text{ for } j\leq k_0 \text{ and } i>k_t\end{cases}$$

**Lemma 3.6.** With the notation introduced, $\rho=\rho(k_0,k_1,\dots,k_t)$ is a strictly *-locally idempotent-separating congruence on $S=GBR^*(T,\theta)$.

**Proof.** Suppose $\rho$ is a strictly *-locally idempotent-separating congruence on $S=GBR^*(T,\theta)$. Then we have that

$$(x_i,m,n)\ \rho\ (y_j,p,q) \text{ implies } (y_j,n,m)\ \rho\ (x_i,q,p)$$

since it is evident that the relation $\rho$ defined above is a congruence on a type A semigroup (where $(y_j,n,m)$ is inverse of $(x_i,m,n)$ and $(x_i,q,p)$ is the inverse of $(y_j,p,q)$ ).

Now we have that $(x_i,m,n)^\dagger\ \rho\ (y_j,p,q)^\dagger$ and $(x_i,m,n)^*\rho\ (y_j,p,q)^*$ implies $(e_i,m,m)\ \rho\ (e_i,p,p)$ and $(e_i,n,n)\ \rho\ (e_i,q,q)$.

That is, we have that $e_{md+i}\ \rho\ e_{pd+j}$ and $e_{nd+i}\ \rho\ e_{qd+j}$ .

Suppose $md+i\geq(p+1)d+j$ then $e_{pd+j}\ \rho\ e_{(p+1)d+j}$ then $i<j,\ m\leq p+1.\quad i>j,\ m\leq p.$

Similarly, we have $j < i,\ p \le m+1.\quad j > i,\ p \le m.$

Consequently, we have $i < j,\ m \le p+1 \le m+1$. That is $m = p$ or $p+1$.

$i > j,\ m \le p \le m+1.$ That is $p = m$ or $m+1$.

Interchanging the roles of $m$ and $n$, p and $q$ we have that

$i < j,\ n = q$ or $q+1.\quad i > j,\ q = n$ or $n+1$.

Now using Lemma 3.2 and considering some cases, we have the desired result.

We now consider cancellative monoid congruences. These can be characterized as follows:

**Theorem 3.7.** Let $S = GBR^*(T,\theta)$ be a *-simple type A *I*-semigroup. Define a relation $\sigma$ on $S$ by

$$(x_i, m, n)\ \sigma\ (y_j, p, q)$$

if and only if $m - n = p - q$ and $x_i = y_j$. Then

i) $\sigma$ is the minimum congruence on $S$.

ii) $S/\sigma$ is a cancellative monoid.

**Proof.** i) That $\sigma$ is reflexive and symmetric can be easily checked. To show transitivity, let $(x_i, m, n)\ \sigma\ (y_j, p, q)$ and $(y_j, p, q)\ \sigma\ (z_k, r, c)$ for $(x_i, m, n), (y_j, p, q), (z_k, r, c) \in S$. Then we have $m - n = p - q,\ x_i = y_j$ and $p - q = r - c,\ y_j = z_k$. This implies $m - n = r - c$ and $x_i = z_k$. Thus $\sigma$ is transitive.

Now let $a = (x_i, m, n),\ b = (y_j, p, q)$. That $\sigma$ is a congruence entails showing that

$$a\ \sigma\ b \Longrightarrow au\ \sigma\ bu \qquad \text{(for right congruence)}$$

$$a\ \sigma\ b \Longrightarrow ua\ \sigma\ ub \qquad \text{(for left congruence)}$$

$\forall\ u = (z_k, r, c) \in S$. So, we have that

$$au = (x_i, m, n)(z_k, r, c) = \begin{cases} \left(x_i . f_{n-r,r}^{-1} . z_k\theta^{n-r} . f_{n-r,c}, m, n+c-r\right) & \text{if } n \ge r \\ \left(f_{r-n,n}^{-1} . x_i\theta^{r-n} . f_{r-n,n} . z_k, m+r-n, c\right) & \text{if } n \le r \end{cases}$$

$$bu = (y_j, p, q)(z_k, r, c) = \begin{cases} \left(y_j . f_{q-r,r}^{-1} . z_k\theta^{q-r} . f_{q-r,c}, p, q+c-r\right) & \text{if } q \ge r \\ \left(f_{r-q,q}^{-1} . y_j\theta^{r-q} . f_{r-q,q} . z_k, p+r-q, c\right) & \text{if } q \le r \end{cases}$$

Suppose $(x_i, m, n)\ \sigma\ (y_j, p, q)$, we have

$m - (n+c-r) = (m-n) + (r-c)$ and $p - (q+c-r) = (p-q) + (r-c)$

$m + r - n - c = (m-n) + (r-c)$ and $p + r - q - c = (p-q) + (r-c)$.

Since $m - n = p - q$, we have that $(m-n) + (r-c) = (p-q) + (r-c)$.

Consequently, $\sigma$ is a right congruence. That $\sigma$ is a left congruence follows similarly. Thus $\sigma$ is a congruence.

Suppose $\rho$ is any other congruence. Then we have $(1,m,m)\ \rho\ (1,0,0)$ for all $m \in I$. If $(x_i,m,n)\ \sigma\ (y_j,p,q)$, then $(x_i,m,n)(1,p,p) = (y_j,p,q)(1,p,p)$ for some $p \in I$

Since $(1,m,m)\ \rho\ (1,0,0)$, then $(x_i,m,n)(1,p,p)\ \rho\ (x_i,m,n)$.

Similarly, $(y_j,p,q)(1,p,p)\ \rho\ (y_j,p,q)$ so that $(x_i,m,n)\ \rho\ (y_j,p,q)$. Hence $\sigma \subseteq \rho$.

ii) Obviously the class of $\sigma$ containing the idempotents is the identity element for $S/\sigma$. So we have $(1,m,n)\sigma\,(y_j,p,q)\sigma = (y_j,p,q)\sigma$ . Thus $S/\sigma$ is a monoid.

To show that $S/\sigma$ is cancellative, let $a = (x_i,m,n), b = (y_j,p,q)$. That $S/\sigma$ is cancellative entails showing that

$$a\sigma\ u\sigma = b\sigma\ u\sigma \Longrightarrow a\sigma = b\sigma \qquad \text{(for right cancellative)}$$

$$u\sigma\ a\sigma = u\sigma\ b\sigma \Longrightarrow a\sigma = b\sigma \qquad \text{(for left cancellative)}$$

$\forall\ u = (z_k,r,c) \in S$. So, we have that

$$a\sigma\ u\sigma = (x_i,m,n)\sigma\ (z_k,r,c)\sigma = (y_j,p,q)\sigma\ (z_k,r,c)\sigma$$

$$= b\sigma\ u\sigma\ .$$

The rest of the proof follows from a routine calculation.

For the remainder of this section the group of integers under addition will be denoted by $\mathbb{Z}$.

We now describe the nature of $S/\sigma$ in the case where $\theta$ is the identity mapping.

**Theorem 3.8.** Let $S = GBR^*(T,\theta)$ be a *-simple type A I-semigroup in which $\theta$ is the identity mapping. Define a multiplication on the set $T \times \mathbb{Z}$ by the rule that

$$(x_i, md+i)(y_j, nd+i) = (x_i y_j, (md+i)+(nd+i))$$

for $x_i, y_j \in T$, $m,n \in \mathbb{Z}$ . Then $S/\sigma \cong T \times \mathbb{Z}$ .

**Proof.** Define a map $\varphi : S \to T \times \mathbb{Z}$ by the rule that $(x_i,m,n)\varphi = (x_i y_j, (md+i)-(nd+i))$.

Evidently, $\varphi$ is well defined. It is known that $T \times \mathbb{Z}$ is a cancellative monoid with identity $(1,0)$.

Now let $(x_i,m,n)$ and $(y_j,p,q)$ be any two elements of $S$. Then

$$\big((x_i,m,n)(y_j,p,q)\big)\,\varphi = \begin{cases} \left(x_i.\, f_{n-p,p}^{-1}.\, y_j\theta^{n-p}.\, f_{n-p,q}, m, n+q-p\right) & \text{if } n \geq p \\ \left(f_{p-n,n}^{-1}.\, x_i\theta^{p-n}.\, f_{p-n,n}.\, y_j, m+p-n, q\right) & \text{if } n \leq p \end{cases} \times \varphi$$

$$= \begin{cases} \left(x_i.\, f_{n-p,p}^{-1}.\, y_j\theta^{n-p}.\, f_{n-p,q}, m, n+q-p\right)\varphi & \text{if } n \geq p \\ \left(f_{p-n,n}^{-1}.\, x_i\theta^{p-n}.\, f_{p-n,n}.\, y_j, m+p-n, q\right)\varphi & \text{if } n \leq p \end{cases}$$

$$= \begin{cases} \left(x_i\, y_j\,, md+i-(n+q-p)d+i\right) & \text{if } n \geq p \\ \left(x_i\, y_j\,, (m+p-n)d+j-qd+j\right) & \text{if } n \leq p \end{cases}$$

$$= \begin{cases} \left(x_i\, y_j\,, (m-n)d + i \ + (p-q)d + i\right) & \text{if } n \geq p \\ \left(x_i\, y_j\,, (m-n)d + j \ + (p-q)d + j\right) & \text{if } n \leq p \end{cases}$$

$$= \left(x_i\, y_j\,, (m-n)d + i \ + (p-q)d + j\right)$$

$$= \left(x_i, (m-n)d + i\right)\left(y_j, (p-q)d + j\right)$$

$$= \left(x_i\,, (md+i) - (nd+i)\right)\left(y_j\,, (pd+j) - (qd+j)\right)$$

$$= (x_i, m, n)\varphi\,\left(y_j, p, q\right)\varphi\,.$$

Thus $\varphi$ is a homomorphism.

Furthermore,

$$(x_i, m, n)\varphi = \left(y_j, p, q\right)\varphi$$

if and only if $\left(x_i\,, (md+i) - (nd+i)\right) = \left(y_j\,, (pd+j) - (qd+j)\right)$

if and only if $(md+i) - (nd+i) = (pd+j) - (qd+j)$ and $x_i = y_j$

if and only if $(x_i, m, n)\sigma = \left(y_j, p, q\right)\sigma$ .

That is $\varphi \circ \varphi^{-1} = \sigma$ .

**References**


1. Asibong-Ibe, U.I. *-Simple type $A$ $\omega$-semigroups, Semigroup Forum 47 (1993), 135-149.
2. Fountain, J.B. A class of right PP monoids, Quart. J. Math. Oxford 2, 28 (1974), 28-44.
3. Fountain, J.B. Adequate semigroups. Proc. Edinburgh Math. Soc 22 (1979), 113-125.
4. Fountain, J.B. Abundant semigroups, Proc. London. Math. Soc., (3) 44 (1982), 103-129.
5. Howie, J.M. Fundamentals of Semigroup Theory, Oxford University Press, Inc. USA, 1995.
6. Makanjuola, S. O. Congruences on type A $\omega$-semigroups. D.Phil Thesis, Ahmadu Bello Univer.,1988.
7. Ndubuisi, R.U and Asibong-Ibe, U.I. Congruences on *-bisimple type $A$ $I$-semigroup. Journal of Semigroup Theory and Applications, Vol. 2018, Article ID 4 (2018), 1-14.
8. Ndubuisi, R.U, Asibong-Ibe, U.I and UdoAkpan, I.U. A class of *-simple type A I-semigroups. Int'l J. Mathematics and its applications, 6(1-E) (2018), 1227-1234.
9. Shang, Y. and Wang, L. *-Bisimple type $A$ $I$-semigroups, Southeast Asian Bull. Math. 36 (2012), 535-545.
10. Warne, R.J. Some properties of simple $I$-regular semigroup. Compositio Math. Vol 22 (1970), 181-195.