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For Some Boundary Value Problems in Distributions

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Abstract

In this paper we give a new proof for the result concerning convergent sequences of functions that give convergent sequence of distributions in D' and find the analytic representation of the distribution obtained by their boundary values. Also, we present two examples.

1. Introduction

It is well known that every function $f \in L^1$ is a regular distribution and its analytic representation is, in fact, the Cauchy representation

$$\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} > .$$

This function is analytic in the complex plane except on the support of f and it holds

$$\hat{f}(x+iy) - \hat{f}(x-iy) \to f(x)$$

as $y \to 0^+$ in D' sense

2. Main results

We give a new proof to the following theorem.

Theorem 1. Let (f_n) be a sequence of functions of L^1 space which converges to the function f in L^1. Let (P_n) be a sequence of analytic functions which converges uniformly to the function P on every compact subset of the real line. Then the sequence of distributions $\left(P_n f_n\right)$ converges to the distribution $P\,f$ in D' sense as $n \to \infty$ and $P\,\hat{f}$ is analytic representation of the distribution $P\,f$.

Proof. Let $\varphi \in D$ and let the support of $\varphi \in D$ lies in $[-a,a]$, for $a > 0$. Then

$$\left|\int_{-\infty}^{\infty} P_n(t) f_n(t)\varphi(t)dt - \int_{-\infty}^{\infty} P(t) f(t)\varphi(t)dt\right| =$$

$$\left|\int_{-\infty}^{\infty} P_n(t)[f_n(t) - f(t)]\varphi(t)dt + \int_{-\infty}^{\infty} [P_n(t) - P(t)] f(t)\varphi(t)dt\right|.$$

Since the sequence $(P_n(t))$ converges uniformly to $P(t)$ on $[-a,a]$, there exists $M > 0$ such that $|P_n(t)|$ and $|P(t)|$ are less than M. So, we have the following expression

$$\left|\int_{-\infty}^{\infty} P_n(t) f_n(t)\varphi(t)dt - \int_{-\infty}^{\infty} P(t) f(t)\varphi(t)dt\right| \le$$

$$\int_{-\infty}^{\infty} |P_n(t)||f_n(t) - f(t)||\varphi(t)|dt + \int_{-\infty}^{\infty} |P_n(t) - P(t)||f(t)||\varphi(t)|dt.$$

For arbitrary $\varepsilon > 0$ there exists n_0 such that for $n \ge n_0$ it holds

$$\|f_n - f\|_1 < \frac{\varepsilon}{2M} \text{ and } |P_n(t) - P(t)| < \frac{\varepsilon}{2\|f\|_1} \text{ for all } t \in [-a,a].$$

This, together with the above expression, proves that the sequence of distributions $\left(P_n f_n\right)$ converges to the distribution Pf in D' sense as $n \to \infty$.

In the following we will prove that $P\,\hat{f}$ is analytic representation of the distribution Pf. Since

$$\int_{-\infty}^{\infty} [P(x+iy)\,\hat{f}(x+iy) - P(x-iy)\,\hat{f}(x-iy)]\varphi(x)dx =$$

$$\int_{-\infty}^{\infty} P(x+iy)\,\hat{f}(x+iy)\varphi(x)dx - \int_{-\infty}^{\infty} P(x-iy)\,\hat{f}(x-iy)\varphi(x)dx,$$

in the following, we will consider the boundary values of $P(x+iy)\,\hat{f}(x+iy)$ and $P(x-iy)\,\hat{f}(x-iy)$ as $y \to 0^+$.

Let $\varphi \in D$ be arbitrary chosen and its support be in $[-a,a]$.

a) The first integral

$$\int_{-\infty}^{\infty} P(x+iy)\,\hat{f}(x+iy)\varphi(x)dx$$

may be written in the form

$$\int_{-\infty}^{\infty} P(x+iy)\varphi(x)\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)dt}{t-x-iy}dx.$$

By Fubini's theorem we get that

$$\int_{-\infty}^{\infty} P(x+iy)\,\hat{f}(x+iy)\varphi(x)dx = \int_{-\infty}^{\infty} f(t)dt\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{P(x+iy)\varphi(x)}{t-x-iy}dx.$$

For the second integral, we have

$$I_1 = \int_{-\infty}^{\infty}\frac{P(x+iy)\varphi(x)}{t-x-iy}dx = \int_{-\infty}^{\infty}\frac{P(x+iy)[\varphi(x)-\varphi(t)]}{t-x-iy}dx + \varphi(t)\int_{-\infty}^{\infty}\frac{P(x+iy)}{t-x-iy}dx = I_2 + \varphi(t)I_3.$$

Then

$$I_3 = \int_{-\infty}^{\infty}\frac{P(x+iy)}{t-x-iy}dx = \int_{-\infty}^{\infty}\frac{P(x+iy)-P(t)}{t-x-iy}dx + \int_{-\infty}^{\infty}\frac{P(t)}{t-x-iy}dx =$$

$$-\int_{-\infty}^{\infty}\frac{P(x+iy)-P(t)}{x-t+iy}dx - P(t)\int_{-\infty}^{\infty}\frac{1}{x-t+iy}dx.$$

i) Having in mind that $x \in (-a,a)$, for the last integral we have

$$\int_{-\infty}^{\infty}\frac{dx}{x-t+iy} = \log(a-t+iy) - \log(-a-t+iy) =$$

$$\ln\sqrt{(a-t)^2+y^2} + i\arg(a-t+iy) - \ln\sqrt{(a+t)^2+y^2} - i\arg(-a-t+iy).$$

ii) Now we consider

$$\int_{-\infty}^{\infty}\frac{P(x+iy)-P(t)}{x-t+iy}dx.$$

We use Taylor's series of $P(z)$ at the point t and get that

$$P(x+iy)-P(t) = \frac{P^{(1)}(t)}{1!}(x-t+iy) + \frac{P^{(2)}(t)}{2!}(x-t+iy)^2 + \dots$$

By Cauchy's formula, we have

$$P^{(n)}(t)=\frac{n!}{2\pi i}\int_{\gamma_R}\frac{P(\zeta)}{(\zeta-t)^{n+1}}d\zeta,$$

where $\gamma_R:|\xi-t|=R.$

Let $K=\max_{\xi\in\gamma_R}|P(\xi)|$. Then we obtain that

$$\left|P^{(n)}(t)\right|\le\frac{n!K}{R^n}.$$

Now

$$\int_{-\infty}^{\infty}\frac{P(x+iy)-P(t)}{x-t+iy}dx=\int_{-\infty}^{\infty}\frac{1}{x-t+iy}\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!}(x-t+iy)^j\,dx=\int_{-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!}(x-t+iy)^{j-1}dx.$$

Since

$$\left|\frac{P^{(j)}(t)}{j!}\right|\le\frac{j!K}{j!R^j}=\frac{K}{R^j}$$

and since we may choose R so that $|x-t+iy|<R$ ($x,\ t\in[-a,a]$ and y is small enough), we may integrate term by term and get

$$\int_{-\infty}^{\infty}\frac{P(x+iy)-P(t)}{x-t+iy}dx=\int_{-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!}(x-t+iy)^{j-1}dx=\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!\,j}[(a-t+iy)^j-(-a-t+iy)^j].$$

So, for I_3, and then $I_1=I_2+\varphi(t)I_3$, we get

$$I_3=-\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!\,j}[(a-t+iy)^j-(-a-t+iy)^j]-$$
$$P(t)[\ln\sqrt{(a-t)^2+y^2}+i\arg(a-t+iy)-\ln\sqrt{(a+t)^2+y^2}-i\arg(-a-t+iy)],$$

$$I_1=-\int_{-\infty}^{\infty}\frac{P(x+iy)[\varphi(x)-\varphi(t)]}{x-t+iy}dx-\varphi(t)P(t)[\ln\sqrt{(a-t)^2+y^2}+i\arg(a-t+iy)-\ln\sqrt{(a+t)^2+y^2}$$
$$-i\arg(-a-t+iy)]-\varphi(t)\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!\,j}[(a-t+iy)^j-(-a-t+iy)^j].$$

Finally,

$$\int_{-\infty}^{\infty}P(x+iy)\,\hat{f}(x+iy)\varphi(x)dx=\int_{-\infty}^{\infty}f(t)dt\frac{1}{2\pi i}I_1.$$

b) Now we consider

$$\int_{-\infty}^{\infty} P(x-iy)\,\hat{f}(x-iy)\varphi(x)dx.$$

In the similar way, we obtain that

$$\int_{-\infty}^{\infty} P(x-iy)\,\hat{f}(x-iy)\varphi(x)dx = \int_{-\infty}^{\infty} f(t)dt\,\frac{1}{2\pi i}J_1,$$

where

$$J_1 = -\int_{-\infty}^{\infty} \frac{P(x-iy)[\varphi(x)-\varphi(t)]}{x-t-iy}dx - \varphi(t)P(t)[\ln\sqrt{(a-t)^2+y^2} + i\arg(a-t-iy) - \ln\sqrt{(a+t)^2+y^2} - i\arg(-a-t-iy)] - \varphi(t)\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!\,j}[(a-t-iy)^j - (-a-t-iy)^j]$$

Now we compute the limits of I_1 and J_1 as $y \to 0^+$.

$$\lim_{y\to 0^+} I_1 = -\int_{-\infty}^{\infty} \frac{P(x)[\varphi(x)-\varphi(t)]}{x-t}dx - \varphi(t)P(t)[\ln(a-t) + i\cdot 0 - \ln(a+t) - i\cdot\pi] - \varphi(t)\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!\,j}[(a-t)^j - (-a-t)^j]$$

and

$$\lim_{y\to 0^+} J_1 = -\int_{-\infty}^{\infty} \frac{P(x)[\varphi(x)-\varphi(t)]}{x-t}dx - \varphi(t)P(t)[\ln(a-t) + i\cdot 0 - \ln(a+t) + i\cdot\pi] - \varphi(t)\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!\,j}[(a-t)^j - (-a-t)^j].$$

Finally we have

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty} [P(x+iy)\,\hat{f}(x+iy) - P(x-iy)\,\hat{f}(x-iy)]\varphi(x)dx =$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t)(\lim_{y\to 0^+} I_1 - \lim_{y\to 0^+} J_1)dt = \frac{1}{2\pi i}\int_{-\infty}^{\infty} 2\pi i\, f(t)P(t)\varphi(t)dt = \int_{-\infty}^{\infty} f(t)P(t)\varphi(t)dt.$$

We note that

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty} P(x+iy)\,\hat{f}(x+iy)\varphi(x)dx =$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t)dt(-\int_{-\infty}^{\infty}\frac{P(x)[\varphi(x)-\varphi(t)]}{x-t}dx)-\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t)P(t)[\ln(a-t)-\ln(a+t)]\varphi(t)dt -$$

$$\frac{1}{2}\int_{-\infty}^{\infty} f(t)P(t)\varphi(t)dt-\frac{1}{2\pi i}\int_{-\infty}^{\infty} f(t)\sum_{j=1}^{\infty}\frac{P^{(j)}(t)}{j!\,j}[(a-t)^j+(-1)^{j+1}(a+t)^j]\varphi(t)dt$$

where the number $a>0$ depends of the function $\varphi\in D$, exists. Indeed, the function

$$\Phi(t)=\int_{-\infty}^{\infty}\frac{P(x)[\varphi(x)-\varphi(t)]}{x-t}dx$$

is continuous and

$$|\Phi(t)|\le\|\varphi'\|\int_{-a}^{a}|P(x)|\,dx\,.$$

Hence the double integral exists.

Also,

$$\int_{-\infty}^{\infty} f(t)P(t)[\ln(a-t)-\ln(a+t)]\varphi(t)dt$$

exists because $\varphi(a)=\varphi(-a)=0$,and, therefore,

$$\lim_{t\to a}\ln(a-t)\varphi(t)=\lim_{t\to a}\ln(a-t)[\varphi(t)-\varphi(a)]=0,$$

and

$$\lim_{t\to -a}\ln(a-t)\varphi(t)=\lim_{t\to -a}\ln(a-t)[\varphi(t)-\varphi(a)]=0.$$

Thus $P(t)[\ln(a-t)-\ln(a+t)]\varphi(t)$ is bounded on the interval $[-a,a]$, and consequently the integral exists.

For the existence of the last integral, we use the Cauchy's formula

$$P^{(j)}(t)=\frac{j!}{2\pi i}\int_{\gamma_R}\frac{P(\zeta)}{(\zeta-t)^{j+1}}d\zeta,$$

where $\gamma_R:|\xi-t|=R$. Since $t\in\operatorname{supp}\varphi\subset[-a,a]$, we may choose R so that $d(\xi,[-a,a])=r>1$.Then

$$\left|P^{(j)}(t)\right|\le\frac{j!KR}{r^{j+1}}.$$

For every $t \in [-a, a]$

$$\left|(a-t)^j + (-1)^{j+1}(a+t)^j\right| \le \left|(2a)^j + (2a)^j\right|$$

and we have

$$\sum_{j=1}^{\infty} \frac{\left|P^{(j)}(t)\right|}{j!\,j} \left|(a-t)^j + (-1)^{j+1}(a+t)^j\right| \le \sum_{j=1}^{\infty} \frac{j!KR}{j!\,j} \frac{2(2a)^j}{r^{j+1}}.$$

If we choose $2a < r$, then it follows that the limits exists for every $\varphi \in D$. So,

$$P(x+iy)\hat{f}(x+iy) \text{ and } P(x-iy)\hat{f}(x-iy)$$

converge in D' sense.

Example 1. Let (P_n) be the sequence of functions $P_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$. It is well known that the sequence (P_n) converges uniformly to the function e^x on any compact set. From the above theorem it follows that for any sequence of functions f_n of L^1 that converges to a function f in L^1, the sequence $(f_n P_n)$ converges in D' to $e^x f(x)$ and its analytic representation is the function $e^z \hat{f}(z)$

We finish with a solution of the problem given in ([1], pg.106), which is of this kind.

Example 2. let f be a function of class $C^1(\mathbb{R})$ and $f(t) = O(|t|^\alpha)$ for some $\alpha < 0$. Then

$$\lim_{y \to 0^+} \hat{f}(x+iy) = F(x),$$

where $F(x)$ is a continuous function on $\mathbb{R}$.

Solution.

$$\hat{f}(x+iy) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)dt}{t-x-iy}$$

Since for any fixed x,

$$\frac{f(t)}{t-x-iy} = O(|t|^{\alpha-1})$$

we conclude that the integral exists for every $x \in \mathbb{R}$. Furthermore,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)dt}{t-x-iy}=\frac{1}{2\pi i}\int_{|t-x|<\delta}\frac{f(t)dt}{t-x-iy}+\frac{1}{2\pi i}\int_{|t-x|\geq\delta}\frac{f(t)dt}{t-x-iy}=I_1+I_2.$$

$$I_1=\frac{1}{2\pi i}\int_{|t-x|<\delta}\frac{f(t)dt}{t-x-iy}=\frac{1}{2\pi i}\int_{|t-x|<\delta}\frac{f(t)-f(x)}{t-x-iy}dt+\frac{1}{2\pi i}\int_{|t-x|<\delta}\frac{f(x)}{t-x-iy}dt.$$

Since $x-\delta<t<x+\delta$ we have

$$2\pi i I_1=\int_{x-\delta}^{x+\delta}\frac{f(t)-f(x)}{t-x-iy}dt+f(x)[\log(\delta-iy)-\log(-\delta-iy)].$$

We may apply the Lebesgue dominated convergence theorem as $y\to 0^+$, so we get

$$2\pi i\lim_{y\to 0^+}I_1=\int_{x-\delta}^{x+\delta}\frac{f(t)-f(x)}{t-x}dt+i\pi f(x).$$

Definitely we have

$$F(x)=\frac{1}{2\pi i}\int_{x-\delta}^{x+\delta}\frac{f(t)-f(x)}{t-x}dt+\frac{1}{2}f(x).$$

It is easy to verify that the function $F(x)$ is continuous on □ , since the function

$$h(t)=\frac{f(t)-f(x)}{t-x}$$

is continuous for $x\neq t$ and $h(x)=f'(x)$ on □ .

References

1. Bremermann G.: Raspredelenija, kompleksnije permenenije I preobrazovanija Fourie, Izdat. ''Mir'', Moskva, 1968.

2. Carmichael R, Mitrovic D.: Distributions and analytic functions, John Wiley and Sons Inc., New York, 1989.

3. V. Manova-Eraković, S. Pilipović, V. Reckovski, Generalized Cauchy transformation with applications to boundary values in generalized function spaces, Integral Transforms and Special Function, Vol. 21, Nos. 1-2, January-February 2010, p.p. 75-83.

4. V. Manova-Eraković, S. Pilipović, V. Reckovski, Analytic representations of sequences in L^p spaces 1 £ p< ¥ , Filomat 31:7 (2017), University of Nis, Serbia, 2017, p.p. 1959-1966

5. V. Manova-Eraković, V. Reckovski, A note on the analytic representations of convergent sequences in S', Filomat 29:6 (2015), University of Nis, Serbia, 2015, p.p. 1419-1424

6. Pathak R.S. Distributions theory and applications, Varnasi, India, 2001.

7. Rudin W.: Real and complex analysis, third edition Mc Graw Hill, 1987.