Least Squares Estimator for Vasicek Model Driven by Fractional Lévy Processes

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Abstract

In this paper, we consider parameter estimation problem for Vasicek model driven by fractional lévy processes defined

$$dX_t = (\mu - \theta X_t)dt + dL_t^d, t \geq 0, X_0 = 0.$$

We construct least squares estimator for drift parameters based on time-continuous observations, the consistency and asymptotic distribution of these estimators are studied in the non-ergodic case. In contrast to the fractional Vasicek model, it can be regarded as a Lévy generalization of fractional Vasicek model.

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1.Introduction

Statistical inference for stochastic differential equations is a main research direction in probability theory and its applications. When the noise is a standard Brownian motion, such problems have been extensively studied and some surveys and complete literatures for this direction could be found in Bishwal [2]. Moreover, since the seminal work of Vasicek [15], the Vasicek model

$$dX_t = (\mu - \theta X_t)dt + dB_t, \quad t \geq 0, \quad X_0 = 0, \tag{1.1}$$

driven by standard Brownian motion $(B_t)_{t\geq 0}$ has been extensively applied in various fields, such as economics and finance, biology, physics, chemistry, medicine and environmental studies, where μ, θ are unknown, the first term describes the so-called drift component $(\mu - \theta X_t)dt$. The parameter θ determines the reversion speed of the stochastic component to their long-term mean $\frac{\mu}{\theta}$. The economic interpretation of this mean-reversion component is that stochastic price fluctuations around the mean and price peaks are only temporarily, caused by for example power plant outages or capacity shortages. Indeed, when this model is used to describe some phenomena, it is important to identify the unknown parameters in this model. As a result, the parameter estimation problem for the Vasicek process driven by Brownian motion has played an important role in econometrics and becomes a interesting problem in the literature.

When $\mu = 0$, the process degenerates into the well-known Ornstein-Uhlenbeck process. If the parameter $\theta \in (-\infty, +\infty)$ is unknown and the process $\{X_t, t \geq 0\}$ can be observed continuously, then an important problem is to estimate the parameter θ based on the (single path) observation $\{X_t, t \geq 0\}$. The most popular approaches are either the maximum likelihood estimators (MLE) or the least squares estimators (LSE), and in this case they coincide. For $\theta > 0$ (ergodic case), the MLE of θ is asymptotically normal (Kutoyants [9]). For $\theta < 0$ (non-ergodic case), the MLE of θ is asymptotically Cauchy (Dietz and Kutoyants [6]).

As an extension of Brownian motion, the fractional Brownian motion (fBm) has become an object of intensive study, due to its interesting properties and its applications in various scientific areas such as hydrology, telecommunications, fluid dynamics, turbulence, image processing, economics and finance. Recall that the fBm $\{B_H(t)\}_{t\geq 0}$ with Hurst index $H \in (0,1)$ is the only centred Gaussian self-similar process with stationary increments, satisfies $B_0^H = 0$, $E[B_t^H] = 0$, $t \geq 0$, the covariance function is given by

$$E[B_t^H B_s^H] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right], \qquad t, s \geq 0. \tag{1.2}$$

It has stochastic integral representation in terms of a standard Brownian motion:

$$B_t^H = \frac{\sqrt{2H\Gamma(\frac{3}{2} - H)}}{\Gamma(H + \frac{1}{2})\Gamma(2-2H)} \int_{\mathbb{R}} \left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] dB_s, \tag{1.3}$$

where $u_+ = \max\{u, 0\}$, B_t is standard Brownian motion. For $H = \frac{1}{2}$, B^H coincides with the standard Brownian motion B, but B^H is neither a semi martingale nor a Markov process unless $H = \frac{1}{2}$.

If the Brownian motion in the Vasicek model (1.1) is replaced with fBm, we get the following fractional Vasicek model (fVm)

$$dX_t = (\mu - \theta X_t)dt + dB_t^H, \quad t \geq 0, \quad X_0 = 0. \tag{1.4}$$

Parameter θ determines the persistence in X_t. Depending on the sign of θ, the model can capture the stationary, the explosive, and the null recurrent behavior. The fVm was first used to describe the dynamics in volatility by Comte and Renault [3]. Other applications of fVm can be found in Comte, Coutin and Renault [4], Corlay, Lebovits and Véhel [5] references therein. Despite many applications of fVm in practice, estimation and the asymptotic theory in fVm has received little attention in the literature. Xiao and Yu [16] propose estimators for μ and θ develop the asymptotic theory for the estimators.

When $\mu = 0$, a very important special case of fVm is the fractional Ornstein-Uhlenbeck process. The parameter estimation for θ has been extensively studied using the MLE method (see Prakasa Rao [11]) or using the LSE technique (see Hu and Nualart [8]).

On the basis of sufficient study of fBm, many authors have proposed to use more general stochastic processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about stochastic processes and fields in general. Therefore, some generalizations of the fBm have been introduced such as sub-fractional Brownian motion, bifractional Brownian motion, weighted-fractional Brownian motion, fractional Lévy processes. However, in contrast to the extensive studies on fBm, there has been only a little systematic investigation on the statistical inference of other fractional processes. The main reason for this is the complexity of dependence structures. Recently, Fink and Klüppelberg [7] proved that the fractional Lévy driven Ornstein-Uhlenbeck processes (FLOUP) has unique stationary pathwise solution of the corresponding Langevin equation and the increments of an FLOUP exhibits long-range dependence.

However, there has been no study on parametric inference for Vasicek model with fractional Lévy noises yet. Motivated by the aforementioned works, as a first attempt, in this paper, we consider the generalized Vasicek model driven by fractional Lévy process L_t^d (the precise definition is given below in Definition 2.1), and it is defined by the following stochastic differential equations

$$dX_t = (\mu - \theta X_t)dt + dL_t^d, \quad t \geq 0, \quad X_0 = 0. \tag{1.5}$$

In the present paper, we assume that the parameters $\mu \in R$ and $\theta < 0$ are unknown. We shall use the least square method to construct their estimators under the continuous observations, respectively. Our main results and aims are described as follows.

Firstly, we use the least square method to obtain the estimators of μ and θ. We introduce least squares estimators of μ and θ of the forms

$$\hat{\theta}_T = \theta - \frac{T\int_0^T X_t dL_t^d - L_T^d \int_0^T X_t dt}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}, \tag{1.6}$$

and

$$\hat{\mu}_T = \frac{1}{T}(X_T + \hat{\theta}_T \int_0^T X_t dt) \tag{1.7}$$

for all $T > 0$. The two estimators are motivated by the following heuristic argument. By minimizing the contrast function

$$L(\theta, \mu) = \int_0^T |\dot{X}_t - (\mu - \theta X_t)|^2 dt \tag{1.8}$$

where $\dot{X}_t$ denotes the differentiation of X_t with respect to t.

As a result, we can explicitly get the two least squares estimators $\hat{\theta}_T$ and $\hat{\mu}_T$ as follows

$$\hat{\theta}_T = \frac{X_T \int_0^T X_t dt - T \int_0^T X_t dX_t}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}, \tag{1.9}$$

$$\hat{\mu}_T = \frac{X_T \int_0^T X_t^2 dt - \int_0^T X_t dX_t \int_0^T X_t dt}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}, \tag{1.10}$$

for all $T > 0$, where the integral $\int_0^T X_t dX_t$ is interpreted as the Young integral (see, for example, Young [10]). We shall prove the consistency of $\hat{\theta}_T$ and $\hat{\mu}_T$, that is,

$$\hat{\theta}_T \xrightarrow{a.s.} \theta, \qquad T \to \infty,$$

and

$$\hat{\mu}_T \xrightarrow{a.s.} \mu, \qquad T \to \infty,$$

where the notation $\xrightarrow{a.s.}$ denotes "almost surely convergence".

The rest of this paper is organized as follows. In Section2, we present some preliminaries for lévy process and fractional lévy process. In Section3, we study the consistency of the least square's estimator $\hat{\theta}_T$ and $\hat{\mu}_T$.

2.Preliminaries

2.1. **Lévy processes.** In this subsection, we mainly introduce the elementary properties of Lévy processes that will be used in following. More studies on the Lévy process can be found in Sato [13], Samorodnitsky and Taqqu [12] and the references therein.

Let $L = \{L(t)\}_{t\geq 0}$ be Lévy processes in $\mathbb{R}$ without Brownian component. It is determined by its characteristic function in the Lévy-Khintchine form

$$\mathbb{E}[\exp\{iuL(t)\}] = \exp\{t\psi(u)\}, \ t \geq 0,$$

where

$$\psi(u) = i\gamma u + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx), \ u \in \mathbb{R},$$

where $\gamma \in \mathbb{R}$ and ν is the Lévy measure of L on $\mathbb{R}$ that satisfies

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty, \quad \int_{|x|>1} |x|^2 \nu(dx) < \infty.$$

This is a necessary and sufficient condition for L to have finite mean and variance given by

$$\mathrm{var}(L(t)) = t\mathrm{var}(L(1)) = t\int_{\mathbb{R}} x^2 \nu(dx), \ t \geq 0.$$

Furthermore, we restrict $E[L(1)] = 0$, then

$$\gamma = -\int_{|x|>1} x\nu(dx),$$

and

$$\psi(u) = \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu(dx),\ u \in \mathbb{R}.$$

Throughout this paper we will use a two side Lévy process $L = \{L(t)\}_{t\in R}$ constructed by taking two independent copies $\{L_1(t)\}_{t\geq 0}, \{L_2(t)\}_{t\geq 0}$ of a one-side Lévy process and setting

$$L(t) = \begin{cases} L_1(t), & t \geq 0, \\ -L_2(-t_-), & t < 0. \end{cases} \tag{2.1}$$

2.2 **Fractional Lévy processes.** In this subsection, we briefly recall the definition and properties of fractional Lévy process.

As an extension of fractional Brownian motion, fractional Lévy process is of interest in practical applications because of its stationarity of increments and long-range dependence. However, it is not Gaussian. Actually, the very large utilization of the fractional Brownian motion in practice (hydrology, telecommunications) are due to these properties (long range dependence). One prefers in general fractional Brownian motion before other processes because it is Gaussian and the calculus for it is easier. However, in concrete situations when the Gaussianity is not plausible for the model, one can use for example the fractional Lévy process. There exists a consistent literature that focuses on different theoretical and applications aspects of the fractional Lévy process. For example, Bender, Lindner and Schick'[1] studied the finite variation of fractional Lévy processes, Tikanmäki and Mishura [14] define fractional Lévy processes using the compact interval representation and proved that the fractional Lévy processes presented via different integral transformations have the same finite dimensional distributions if and only if they are fractional Brownian motions.

In this paper, we are interested in fractionally integrated processes. Therefore, we will work with the fractional integration parameter $d = H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$ rather than the Hurst parameter H. Moreover, we restrict ourselves to $d \in (0, \frac{1}{2})$ as we are interested in the long range dependence case. Based on the moving average integral representation of fractional Brownian motion, the class of fractional Lévy processes is introduced by replacing the Brownian motion by a general Lévy process with zero mean, finite variance and no Brownian component.

Definition 2.1. (*Marquardt [10]*) *Let $L = (L_t)_{t\in R}$ be a zero-mean two-sided Lévy process with $E[L(1)^2] < \infty$ and without a Brownian component. For fractional integration parameter $d \in (0, \frac{1}{2})$, a stochastic process*

$$L_t^d := \frac{1}{\Gamma(d+1)} \int_{\infty}^{\infty} [(t-s)_+^d - (-s)_+^d] L(ds), \qquad t \in R$$

is called a fractional Lévy process (fLp).

fLp has the sample path properties as follows.

• Hölder continuity. For every $\beta < d$, there exists a continuous modification of L^d and there exist an almost surely positive random variable H_ϵ and a constant $\delta > 0$ such that

$$P\left[\omega \in \Omega : \sup_{0<h<H_\epsilon(\omega)} \left(\frac{L_{t+h}^d(\omega) - L_t^d(\omega)}{h^\beta}\right) \leq \delta\right] = 1.$$

- Stationary increments. L^d is a process with stationary increments.
- Symmetry. $\{L^d_{-t}\}_{t\in\mathbb{R}} \stackrel{d}{=} \{-L^d_t\}_{t\in\mathbb{R}}$.

L^d is locally self-similar with parameter $\widetilde{H}$, that is, for every fixed $t \in R$,

$$\lim_{\epsilon\to 0}\left\{\frac{L^d_{t+\epsilon x}-L^d_t}{\epsilon^{\widetilde{H}}}\right\} \stackrel{d}{=} \{Y_{\widetilde{H}}(x)\}, \quad x\in\mathbb{R},$$

here $Y_{\widetilde{H}}$ is a linear fractional stable motion with expression

$$Y_{\widetilde{H}}(t)=\frac{1}{\Gamma(d)}\int_{\mathbb{R}}[(t-s)_+^{\widetilde{H}-\frac{1}{\alpha}}-(-s)_+^{\widetilde{H}-\frac{1}{\alpha}}]L_\alpha(ds),$$

where L_α is symmetric α-stable Lévy process (see Samorodnitsky and Taqqu[12]).

The following two Lemma gives an integral relationship between fractional lévy processes and integral with Lévy processes and the second-order property of the stochastic integral respect to fractional lévy processes.

Lemma2.1. *(Marquardt[10]) Let $g\in H$, H is the completion of $L^1(R)\cap L^2(R)$ with respect to the norm $\|g\|^2_H = E[L(1)^2]\int_R (I^d_- g)^2(u)du$, then*

$$\int_R g(s)dL^d_s = \int_R (I^d_- g)(u)dL_u \tag{2.2}$$

where the equality holds in the L^2 sense and $I^d_- g$ denotes the Riemann-Liouville fractional integrals defined by

$$(I^d_- g)(x)=\frac{1}{\Gamma(d)}\int_x^\infty g(t)(t-x)^{d-1}dt.$$

Lemma2.2. *(Marquardt[10]) Let $|f|,|g|\in H$. The*

$$E\left[\int_R g(s)dL^d_s\int_R f(s)dL^d_t\right]=\frac{\Gamma(1-2d)E[L(1)^2]}{\Gamma(d)\Gamma(1-d)}\int_R\int_R f(t)g(s)|t-s|^{2d-1}dsdt. \tag{2.3}$$

$$\varphi(h_t,g_t)=\varphi(h_0,g_0)+\int_0^t\frac{\partial\varphi}{\partial h}(h_u,g_u)dh_u+\int_0^t\frac{\partial\varphi}{\partial g}(f_u,g_u)dg_u, \qquad 0\le t\le T. \tag{2.4}$$

In particular, using the Young integral of (2.4), we can rewrite the solution of (1.5) as

$$X_t=\frac{\mu}{\theta}(1-e^{-\theta t})+e^{-\theta t}\int_0^t e^{\theta u}dL^d_u. \tag{2.5}$$

Furthermore, we have

$$\begin{aligned}\int_0^T X_t dt &= \int_0^T \left[\frac{\mu}{\theta}(1-e^{-\theta t}) + e^{-\theta t}\int_0^t e^{\theta u} dL_u^d\right] dt \\ &= \frac{\mu}{\theta}\left(T + \frac{e^{-\theta T}-1}{\theta}\right) + \int_0^T \int_0^t e^{-\theta t} e^{\theta u} dL_u^d dt \\ &= \frac{\mu}{\theta}\left(T + \frac{e^{-\theta T}-1}{\theta}\right) + \int_0^T \int_u^T e^{-\theta t} dt e^{\theta u} dL_u^d \\ &= \frac{1}{\theta}(\mu T - \frac{\mu}{\theta} + L_T^d) + e^{-\theta T}\left[\frac{\mu}{\theta^2} - \frac{\int_0^T e^{\theta u} dL_u^d}{\theta}\right].\end{aligned} \tag{2.6}$$

According to (1.9), (1.10) and (2.5), we can rewrite $\hat{\theta}_T$ and $\hat{\mu}_T$ as

$$\begin{aligned}\hat{\theta}_T &= \frac{X_T \int_0^T X_t dt - T\int_0^T X_t\left[(\mu - \theta X_t)dt + dL_t^d\right]}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\ &= \frac{\theta T \int_0^T X_t^2 dt - T\int_0^T X_t dL_t^d - (\theta\int_0^T X_t dt - L_T^d)\int_0^T X_t dt}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} \\ &= \theta + \frac{L_T^d \int_0^T X_t dt - T\int_0^T X_t dL_t^d}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2},\end{aligned} \tag{2.7}$$

and

$$\hat{\mu}_T = \mu + \frac{L_T^d\int_0^T X_t^2 dt - \int_0^T X_t dt \int_0^T X_t dL_t^d}{T\int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}. \tag{2.8}$$

3. Asymptotic behavior of the least squares estimator

In this section, let $\theta < 0$, we consider the strong consistency of $\hat{\theta}$ and $\hat{\mu}$. Moreover, we also investigate the asymptotic of the estimator for the long term mean.

Theorem 3.1. *Assume $d \in (0, \frac{1}{2})$, $\theta < 0$. The*

$$\hat{\theta}_T \xrightarrow{a.s.} \theta, \quad T \to \infty, \tag{3.1}$$

and

$$\hat{\mu}_T \xrightarrow{a.s.} \mu, \quad T \to \infty, \tag{3.2}$$

For prove Theorem 3.1, we need the following lemma.

Assume $d \in (0, \frac{1}{2}),\ \theta < 0$. Let φ_T be defined as $\varphi_T = \int_0^T e^{\theta s} dL_s^d$. Then $\varphi_\infty = \int_0^\infty e^{\theta s} dL_s^d$ is well-defined, and as $T \to \infty$

$$\varphi_T \to \varphi_\infty, \quad \text{almost surely.}$$

Proof. Using the Young integral and definition of φ_T, we can write

$$\varphi_T = e^{\theta T} L_T^d - \theta \int_0^T L_s^d e^{\theta s} ds. \tag{3.3}$$

Hence φ_T is well-defined, since

$$\int_0^\infty e^{\theta s} E|L_s^d| ds \le \sqrt{c} \int_0^\infty s^{d+\frac{1}{2}} e^{\theta s} < \infty.$$

Moreover, for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n\ge 0} P\left(\sup_{n\le t\le n+1} \left| \int_t^\infty e^{\theta s} L_s^d ds \right| \ge \varepsilon \right) \\
&\quad \le \varepsilon^{-1} \sum_{n\ge 0} E\left(\sup_{n\le t\le n+1} \left| \int_t^\infty e^{\theta s} L_s^d ds \right| \right) \\
&\quad \le \varepsilon^{-1} \sum_{n\ge 0} E\left(\left| \int_n^\infty e^{\theta s} |L_s^d| ds \right| \right) \\
&\quad \le \varepsilon^{-1} \sqrt{c} \sum_{n\ge 0} \int_0^\infty e^{\theta s} s^{d+\frac{1}{2}} ds \\
&\quad \le \varepsilon^{-1} \sqrt{c} \sum_{n\ge 0} e^{\frac{\theta}{2} n} \int_0^\infty e^{\frac{\theta}{2} s} s^{d+\frac{1}{2}} ds \\
&\quad = \varepsilon^{-1} \sqrt{c}\, \Gamma(d+\frac{2}{3})(\frac{2}{\theta})^{d+\frac{2}{3}} \sum_{n\ge 0} e^{\frac{\theta}{2} n} < \infty.
\end{aligned}$$

By using Borel-Cantell's lemma, we can obtain $\varphi_T \xrightarrow{a.s.} \varphi_\infty$, as $T \to \infty$. This completes the proof.

Proof of Theorm 3.1 . By (2.7), we have

$$\hat{\theta}_T - \theta = \frac{L_T^d \int_0^T X_t dt - T \int_0^T X_t dL_t^d}{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2} = \frac{\frac{L_T^d \int_0^T X_t dt - T \int_0^T X_t dL_t^d}{Te^{-2\theta T}}}{\frac{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}{Te^{-2\theta T}}}. \tag{3.4}$$

We first consider the term

$$\frac{T \int_0^T X_t^2 dt - (\int_0^T X_t dt)^2}{Te^{-2\theta T}}.$$

From (2.6) and L'Hôspital's rule, we can get

$$\lim_{T\to\infty} \int_0^T e^{\theta T} X_t dt = \frac{\mu - \theta\varphi_\infty}{\theta^2}, \tag{3.5}$$

And

$$\lim_{T\to\infty} \int_0^T e^{2\theta T} X_t^2 dt = -\frac{(\mu - \theta\varphi_\infty)^2}{2\theta^3}. \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\lim_{T\to\infty}\frac{T\int_0^T X_t^2dt-(\int_0^T X_tdt)^2}{Te^{-2\theta T}}=-\frac{(\mu-\theta\varphi_\infty)^2}{2\theta^3}. \tag{3.7}$$

Next we consider term

$$\frac{L_T^d\int_0^T X_tdt-T\int_0^T X_tdL_t^d}{Te^{-2\theta T}}.$$

From (3.5), we get

$$\lim_{T\to\infty}\frac{L_T^d\int_0^T X_tdt}{Te^{-2\theta T}}=0. \tag{3.8}$$

By Young integral together with L'Hôspital's rule, we get

$$\begin{aligned}\lim_{T\to\infty}\frac{T\int_0^T X_tdL_t^d}{Te^{-2\theta T}}&=\lim_{T\to\infty}\frac{\int_0^T[\frac{\mu}{\theta}(1-e^{-\theta t})+e^{-\theta t}\int_0^t e^{\theta u}dL_u^d]dL_t^d}{e^{-2\theta T}}\\&=\lim_{T\to\infty}\frac{\frac{\mu}{\theta}L_T^d-\frac{\mu}{\theta}\int_0^T e^{-\theta t}dL_t^d+\int_0^T e^{-\theta t}\int_0^t e^{\theta u}dL_u^ddL_t^d}{e^{-2\theta T}}\\&=-\frac{\mu}{\theta}\lim_{T\to\infty}\frac{\int_0^T e^{-\theta t}dL_t^d}{e^{-2\theta T}}+\lim_{T\to\infty}\frac{\int_0^T e^{-\theta t}\int_0^t e^{\theta u}dL_u^ddL_t^d}{e^{-2\theta T}}\\&=-\frac{\mu}{\theta}\lim_{T\to\infty}\frac{e^{-\theta T}L_T^d+\theta\int_0^T L_t^de^{-\theta t}dt}{e^{-2\theta T}}+\lim_{T\to\infty}\frac{\int_0^T e^{-\theta t}\varphi_tdL_t^d}{e^{-2\theta T}}\\&=\lim_{T\to\infty}\frac{\frac{1}{2}e^{-2\theta T}{\varphi_T}^2+\theta\int_0^T {\varphi_T}^2e^{-2\theta T}dt}{e^{-2\theta T}}\\&=\frac{\varphi_\infty^2}{2}-\lim_{T\to\infty}\frac{\theta\int_0^T {\varphi_T}^2e^{-2\theta T}dt}{e^{-2\theta T}}\\&=\frac{\varphi_\infty^2}{2}-\frac{\varphi_\infty^2}{2}=0.\end{aligned} \tag{3.9}$$

So, we can easy get

$$\lim_{T\to\infty}\frac{L_T^d\int_0^T X_tdt-T\int_0^T X_tdL_t^d}{Te^{-2\theta T}}=0. \tag{3.10}$$

Combining (3.7) and (3.8), we obtain

$$\hat\theta_T\xrightarrow{a.s.}\theta,\qquad T\to\infty.$$

Next, using the similar method as above, from (2.8), we have

$$\hat\mu_T-\mu=\frac{L_T^d\int_0^T X_t^2dt-\int_0^T X_tdt\int_0^T X_tdL_t^d}{T\int_0^T X_t^2dt-(\int_0^T X_tdt)^2}. \tag{3.11}$$

Moreover, we can easily obtain

$$\lim_{T\to\infty}\frac{L_T^d\int_0^T X_t^2dt-\int_0^T X_tdt\int_0^T X_tdL_t^d}{Te^{-2\theta T}}=0. \tag{3.12}$$

So,

$$\hat{\mu}_T\xrightarrow{a.s.}\mu,\qquad T\to\infty.$$

This completes the proof.

It is easy to see that the long term mean of X_t is $\frac{\mu}{\theta}$. It follows from (1.9) and (1.10), we have

$$\frac{\hat{\mu}_T}{\hat{\theta}_T}=\frac{X_T\int_0^T X_t^2dt-\int_0^T X_tdX_t\int_0^T X_tdt}{X_T\int_0^T X_tdt-T\int_0^T X_tdX_t}. \tag{3.13}$$

Corollary 3.1. Assume $d\in(0,\frac{1}{2})$, $\theta<0$. The

$$\frac{\hat{\mu}_T}{\hat{\theta}_T}\xrightarrow{a.s.}\frac{\mu}{\theta},\qquad T\to\infty. \tag{3.14}$$

By the Young integral, we have

$$\frac{\hat{\mu}_T}{\hat{\theta}_T}=\frac{\frac{e^{\theta T}}{T}\int_0^T X_t^2dt-\frac{X_T}{2T}e^{\theta T}\int_0^T X_tdt}{\frac{e^{\theta T}}{T}\int_0^T X_tdt-\frac{X_T}{2}e^{\theta T}}. \tag{3.15}$$

It follows from (2.5) and (3.5), we obtain

$$\begin{aligned}\lim_{T\to\infty}\left(\frac{e^{\theta T}}{T}\int_0^T X_tdt-\frac{X_T}{2}e^{\theta T}\right)&=\lim_{T\to\infty}-\frac{e^{\theta T}}{2}\left(\frac{\mu}{\theta}(1-e^{-\theta t})+e^{-\theta t}\int_0^t e^{\theta u}dL_u^d\right)\\&=\frac{\mu}{2\theta}-\lim_{T\to\infty}\frac{1}{2}\varphi_T=\frac{\mu}{2\theta}-\frac{1}{2}\varphi_\infty.\end{aligned} \tag{3.16}$$

Using (2.5) and Young integral we have

$$\begin{aligned}\frac{e^{\theta T}}{T}\int_0^T X_t^2dt&=\frac{e^{\theta T}}{T}\int_0^T\left[\frac{\mu}{\theta}(1-e^{-\theta t})+e^{-\theta t}\int_0^t e^{\theta u}dL_u^d\right]^2dt\\&=(\frac{\mu}{\theta})^2\left[e^{\theta T}(1-\frac{3}{2\theta T})-\frac{1}{2\theta T}e^{-\theta T}+\frac{2}{\theta T}\right]\\&\quad+\frac{2\mu\varphi_T}{T\theta^2}(e^{-\theta t}-1)+\frac{e^{\theta T}}{T}\int_0^T e^{-2\theta t}\varphi_t^2dt,\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\frac{X_T}{2T}e^{\theta T}\int_0^T X_t dt &= \frac{1}{2T}\left[\frac{\mu}{\theta}(1-e^{-\theta t}) + e^{-\theta t}\int_0^t e^{\theta u} dL_u^d\right] \\ &\quad \times e^{\theta T}\int_0^T \left[\frac{\mu}{\theta}(1-e^{-\theta t}) + e^{-\theta t}\int_0^t e^{\theta u} dL_u^d\right] \\ &= \frac{1}{2T}\left(\frac{\mu}{\theta} - \frac{\mu}{\theta}e^{-\theta T} + e^{-\theta T}\varphi_T\right)\cdot\left(\frac{\mu}{\theta}e^{\theta T}T + \frac{\mu}{\theta^2} - \frac{\mu}{\theta^2}e^{\theta T} - \frac{\varphi_T}{\theta}\right).\end{aligned} \tag{3.18}$$

Combining\eqref sec3-eq3.12 with\eqref sec3-eq3.13, we have

$$\lim_{T\to\infty} = \frac{e^{\theta T}}{T}\int_0^T X_t^2 dt - \frac{X_T}{2T}e^{\theta T}\int_0^T X_t dt = \frac{1}{2}(\frac{\mu}{\theta})^2 - \frac{\mu}{2\theta}\varphi_\infty. \tag{3.19}$$

Finally, by (3.16) and (3.19) we can obtain the conclusion This completes the proof.

Competing Interests

The author declares that no competing interests exist.

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