Cela, F. (2018). Some Generalizations of Green's Relations in Rings and Modules. Journal of Advances in Mathematics, 14(2), 7937-7945. https://doi.org/10.24297/jam.v14i2.7772.This work is licensed under a Creative Commons Attribution 4.0 International License.

SOME GENERALIZATIONS OF GREEN'S RELATIONS IN RINGS AND MODULES

Florion ÇELA
Faculty of Natural and Sciences. University of Tirana.
florion.cela@fshn.edu.al

Abstract
In semigroups theory Green's relations, introduced by J. Green, are a very important and useful tool for developing the semigroup theory. They characterise the element of a semigroup or a ring in terms of the principal ideals they generate.
In contrast to early semigroup theory , where, as we have seen, ideas from rings were applied to semigroups, Greens relations have also been applied to rings (Hollings, 2014). In ring theory Greens relations are introduced by (Petro,2002) In this paper at first we generalize Green's relations in rings.
After this we notice that there exist an one to one correspondence between the ideals of a ring and this type of new relations we introduced.Then we compare them with Green's relations in rings. At last we define some new relations in module theory, which mimic Green's relations in rings, as an attempt to get tools in studying modules.

Keywords: Rings, Semigroups, Green's relation, module
Subject Classification: 16-XX, 20M25
Website: https://cirworld.com

1 Introduction

We give some notions and present some auxiliary result that will be used throughout the paper. For all unexplained concepts and propositions the reader may refer to [1],[2],[5].
[3] Let U be a ring and let $a \in U$. The principal left(right) ideal $(a)_l((a)_r)$ generated by a is $Ia + Ua(Ia + aU)$, were I denotes the ring of all integers. Green's relation $\mathcal{L}$ and $\mathcal{R}$ in the ring U are denoted by :

$$a\mathcal{L}b \Leftrightarrow (a)_l = (b)_l,$$
$$a\mathcal{R}b \Leftrightarrow (a)_r = (b)_r.$$

$\mathcal{L}$ and $\mathcal{R}$ are equivalence relations. $\mathcal{L}_a(\mathcal{R}_a)$ denote the equivalence class of $Umod\mathcal{L}(mod\mathcal{R})$ containing the element $a \in U$.

[1] Let Y be a non-empty subset of $(X, \leq)$ if d is the greatest lower bound or meet of Y we write

$$d = \bigwedge\{y : y \in Y\}$$

if c is the least upper bound or join we write

$$c = \vee\{y : y \in Y\}$$

[7] Let ϵ, η be equivalence relations on a set X, the *join* $\epsilon \vee \eta$ is the smallest equivalence relation on X containing ϵ and η.
Since the universal relation ω is an equivalence relation containing ϵ and η on X, therefore $\epsilon \vee \eta$ exist's and it is the intersection of all equivalence relations containing ϵ and η.
An equivalence relation γ on a grupoid C will be called a right (left) congruence on C if $a\gamma b(a, b \in C)$ implies $ac\gamma bc(ca\gamma cb)$ for every $c \in C$.
By a congruence on C means an equivalence relation on C which is both a right and a left congruence.

2 Main Results

Let U be a ring and K an ideal in U. We define relations $\mathcal{R}_K, \mathcal{L}_K$ as following :

$$a\mathcal{R}_K b \Leftrightarrow a - (zb + bs), b - (z'a + as') \in K \text{ for any } (z, z') \in \mathbb{Z}^2, (s, s') \in U^2$$
$$a\mathcal{L}_K b \Leftrightarrow a - (zb + sb), b - (z'a + s'a) \in K \text{ for any } (z, z') \in \mathbb{Z}^2, (s, s') \in U^2$$

It is easy to verify that the relations $\mathcal{R}_K, \mathcal{L}_K$are equivalence relations in the ring U.
For the sake of simplicity we denote the element $a - (zb + bs)$ and $a - (zb + sb)$ respectively by $a - b(z, s)$ and $a - (z, s)b$.
The intersection $\mathcal{H}_K = \mathcal{L}_K \cap \mathcal{R}_K$ of the equivalence relations $\mathcal{L}_K, \mathcal{R}_K$ on the ring U is an equivalence relation on U.

Proposition 2.1 *Let I, J be ideals in U then the following propositions are equivalent::*

$$a)\ \mathcal{R}_I \leq \mathcal{R}_J,\ b)\ \mathcal{L}_I \leq \mathcal{L}_J,\ c)\ \mathcal{H}_I \leq \mathcal{H}_J,\ c)\ I \subseteq J.$$

Proof. Let I and J be ideals such that $I \subseteq J$ and $a\mathcal{R}_I b$ for $a, b \in U$, then there exist $\exists z, z' \in \mathbb{Z}\ s, s' \in U$ such that $a - b(z, s) \in I \subseteq J$ and $b - a(z', s') \in I \subseteq J$, so we have that $a\mathcal{R}_J b$ hence $\mathcal{R}_I \leq \mathcal{R}_J$.
To prove the converse, let i be an arbitrary element of I. Then it is true that $i\mathcal{R}_I 0$. By using the fact that $\mathcal{R}_I \leq \mathcal{R}_J$ we get that $i\mathcal{R}_J 0$, thus $i \in J$. We can prove analogously the other cases.

Greens Relations $\mathcal{R}, \mathcal{L}, \mathcal{H}$ in rings are a special case of the above relations. If $K = \{0\}$ then we have

$$\mathcal{R}_{\{0\}} = \mathcal{R}, \mathcal{L}_{\{0\}} = \mathcal{L}, \mathcal{H}_{\{0\}} = \mathcal{H}$$

Thus the following proposition hold.

Proposition 2.2 *Let K be an arbitrary ideal of the ring U then the following inequalities are true.*

$$\mathcal{R} = \bigwedge \{ \mathcal{R}_K | K \text{ ideal of } U \},$$
$$\mathcal{L} = \bigwedge \{ \mathcal{L}_K | K \text{ ideal of } U \},$$
$$\mathcal{H} = \bigwedge \{ \mathcal{H}_K | K \text{ ideal of } U \}.$$

Leme 2.1 *The Green's relations $\mathcal{R}_K, \mathcal{L}_K$ are such for each $a, b, s \in U$ and $m \in \mathbb{Z}$ we have*

$$a\mathcal{L}_K b \Rightarrow a(m,s)\mathcal{L}_K b(m,s),$$
$$a\mathcal{R}_K b \Rightarrow (m,s)a\mathcal{R}_K (m,s)b.$$

Proof. Suppose $a\mathcal{L}_K b$ then there are $z, z' \in \mathbb{Z}$, $y, y' \in U$ such that $a - (z,y)b \in K$, $b - (z',y')a \in K$.
Equalities :

$$b(m,s) - (z',y')(a(m,s)) = m(b - (z',y')a) + (b - (z',y')a)s \in K$$

are true, the same hold $a(m,s) - (z,y)(b(m,s)) \in K$, so $a(m,s)\mathcal{L}_K b(m,s)$. The other implication is proved in on analogues manner.

Leme 2.2 *For every ideal K of the ring U the Green's relations $\mathcal{L}_K$ and $\mathcal{R}_K$ commute.*

Proof. It suffices to show that $\mathcal{L}_K \circ \mathcal{R}_K \subseteq \mathcal{R}_K \circ \mathcal{L}_K$.
Let a and b be element of the ring such that $a\mathcal{L}_K \circ \mathcal{R}_K b$, there exist $c \in U$ such that

$$a\mathcal{L}_K c \text{ and } c\mathcal{R}_K b.$$

So $c - (z_1, y_1)a \in K$ and $b - c(z_2, y_2) \in K$.
Let $d = a(z_2, y_2)$, since $a\mathcal{L}_K c$ then

$$a(z_2, y_2)\mathcal{L}_K c(z_2, y_2)\ (1).$$

Since $b\mathcal{R}_K c$ then $b - c(z_2, y_2) \in K$.
Take $b = c(z_2, y_2) + k$ for any $k \in K$,
We get that $b\mathcal{L}_K c(z_2, y_2)$.
and by using (1) we have that $b\mathcal{L}_K a(z_2, y_2)$.
Now let us show that $a\mathcal{R}_K a(z_2, y_2)$. For this we take $a(z_2, y_2) - a(z_2, y_2) \in K$ (2).
We calculate $a - (a(z_2, y_2))(z_2', y_2')$.The following equalities hold

$$\begin{aligned}
(a(z_2, y_2))(z_2', y_2') &= (((z_1', y_1')c + k')(z_2, y_2))(z_2', y_2') \text{ for any } k' \in K \\
&= (z_1', y_1')((c(z_2, y_2))(z_2', y_2')) \\
&= (z_1', y_1')((b - k)(z_2', y_2')) \text{ for any } k \in K \\
&= (z_1', y_1')(b(z_2', y_2') - k(z_2', y_2')) \\
&= (z_1', y_1')(c + k_1) \text{ where } k_1 \in K \\
&= (z_1', y_1')c + (z_1', y_1')k_1 = a + k_2 \text{ where } k_2 \in K.
\end{aligned}$$

Since $a-(a(z_2,y_2))(z_2',y_2') = a-(a+k_2) \in K$, and by (2) we get that $a\mathcal{R}_{\mathcal{K}}a(z_2,y_2)$. Finally we have that $a\mathcal{R}_{\mathcal{K}}a(z_2,y_2)$ and $b\mathcal{L}_{\mathcal{K}}a(z_2,y_2)$ this means that $a\mathcal{R}_{\mathcal{K}} \circ \mathcal{L}_{\mathcal{K}}b$, so $\mathcal{L}_{\mathcal{K}} \circ \mathcal{R}_{\mathcal{K}} \subseteq \mathcal{R}_{\mathcal{K}} \circ \mathcal{L}_{\mathcal{K}}$. ■

We denote $\mathcal{D} = \mathcal{R}_{\mathcal{K}} \vee \mathcal{L}_{\mathcal{K}}$. From the above lemma we actually get that $\mathcal{D} = \mathcal{R}_{\mathcal{K}} \circ \mathcal{L}_{\mathcal{K}}$.
We can notice that there exists an one to one correspondence between the relations R_K and the ideals of the ring.
This is shown in the following proposition.

Proposition 2.3 *There exists an one to one correspondence between the relations of "types" $\mathcal{R}_K, \mathcal{L}_K, \mathcal{H}_{\mathcal{K}}, \mathcal{D}_K$. and the ideals of the ring U.*

Proof. Let I , be an arbitrary ideal of the ring U. Now we take into consideration the relation $\mathcal{R}_I$.
Let J be another ideal of U such that $I \neq J$ which implies that $(I-J) \neq \emptyset$ or $(J-I) \neq \emptyset$. If $(I-J) \neq \emptyset$ then it exists $i \in I-J$, hence $\overline{i\mathcal{R}_J 0}$, but we have that $i\mathcal{R}_I 0$, so $\mathcal{R}_I \neq \mathcal{R}_J$. We can prove analogously the other Greens relations . ■

Proposition 2.4 *Let U be a ring, then the following propositions are equivalent.*

$$\begin{gathered} a)\ U \text{ is a simple ring,} \\ b)\ \mathcal{R} = \bigvee\{\mathcal{R}_K | K - ideal, K \neq U\}, \\ c)\ \mathcal{L} = \bigvee\{\mathcal{L}_K | K - ideal, K \neq U\}. \end{gathered}$$

where $\mathcal{R}, \mathcal{L}$ are Greens relations in rings.

Proof. If U is a simple ring then its only ideal $K \neq U$ is $\{0\}$,so propositons b) and c) hold.
Conversely, for any $K \neq U$ from proposition 2 we get that $\mathcal{R} \leq \mathcal{R}_{\mathcal{K}}$, if $\mathcal{R} = \vee\mathcal{R}_K$ then $\mathcal{R} = \mathcal{R}_K$, by using the above corespondence we get that $K = \{0\}$. So U is a simple ring. The same hold for the relation $\mathcal{L}$. ■

If K is an ideal it can be verified that the equivalence relation defined as following, $a\rho_K b \Leftrightarrow a-b \in K$ is less or equal than ther relation $\mathcal{R}_K$ because

$$\text{For all } k \in K, a\mathcal{R}_K(a+k) \ (3)$$

So we get that that $a+K \subseteq \mathcal{R}_K^a$,
The following propositon holds.

Proposition 2.5 *For any ideal K, we have that $\rho_K \leq \mathcal{L}_K \cap \mathcal{R}_K = \mathcal{H}_K$. Where ρ_K is the relation defined above.*

Proposition 2.6 *Let U be a finite ring and K an ideal in U. Then the cardinal of K is a divisor of the cardinal of classes of the relation $\mathcal{R}_K$*

Proof. For all $a, b \in U$ such that $aR_K b$ we have that $a+K \subseteq \mathcal{R}_K^a$ and $b+K \subseteq \mathcal{R}_K^b$ hence $\mathcal{R}_K^a$ is a union of the equivalence classes of the relation K.
Since each two classes according relation ρ_K has the same number of elements (since the ring is finite) then we get that $|K|$ is a divisor of $|\mathcal{R}_K|$.

Proposition 2.7 *The relation $\mathcal{R}_K(\mathcal{L}_K)$ is a left congruence (right congruence) according to the multiplicative semigroup of the ring.*

The proof is analogous as in the case of Green's relation in rings [3]

Proposition 2.8 *Let K be an ideal of the ring U dhe $a,b \in U$ then*

$$\begin{gathered} 1\text{-}a\mathcal{R}_K b \text{ in } U \Leftrightarrow \overline{a}\mathcal{R}\overline{b} \text{ in the quotient ring } U/K \\ 2\text{-}a\mathcal{L}_K b \text{ in } U \Leftrightarrow \overline{a}\mathcal{L}\overline{b} \text{ in the quotient ring } U/K \\ 3\text{-}a\mathcal{H}_K b \text{ in } U \Leftrightarrow \overline{a}\mathcal{H}\overline{b} \text{ in the quotient ring } U/K \end{gathered}$$

Symbolically we denote $\mathcal{R}_K/K = \mathcal{R}, \mathcal{L}_K/K = \mathcal{L}, \mathcal{H}_K/K = \mathcal{H}$ where $\mathcal{R}, \mathcal{L}, \mathcal{H}$ are Green's relations in ring.

Proof. It is sufficient to prove only the first equivalence because the other cases can be proved analogously. If $a\mathcal{R}_K b$ then it exists $((z,z'),(s,s')) \in \mathbb{Z}^2 \times U^2$ such that $a - b(z,s) in K$ and $b - a(z',s') \in K$, which imply that $\overline{a}\mathcal{R}\overline{b}$. In a similar way we can prove also the converse.

Taking into consideration the connection between the ideals of a ring and the kernel of rings homomorphism the above proposition can be restatedas following.

Proposition 2.9 *If $f : U \to V$ is an epimorphism then $\mathcal{R}_{kerf}/kerf = \mathcal{R}$ where $\mathcal{R}$ is Green's relation in V.*
$\mathcal{L}_{kerf}/kerf = \mathcal{L}$

The proof is analogous as in the above proposition.

Definition 2.1 *Let M be a right U- module. Then in M we define the relation $\mathcal{R}$ as following*

$$m_1 \mathcal{R} m_2 \Leftrightarrow (m_1) = (m_2), m_1, m_2 \in M$$

For the left modules we denote the relation by $\mathcal{L}$.
For the element m_1, m_2 in the definition we say that there exist $a_1, a_2 \in U$, $z_1, z_2 \in \mathbb{Z}$ such that :

$$m_1 = m_2 z_2 + m_2 a_2,\ m_2 = m_1 z_1 + m_1 a_1.$$

Or simbolically $m_2 = m_1(z_1, a_1)$, $m_1 = m_2(z_2, a_2)$.

It can be proved easily that the above relation is an equivalence relations in M.

Proposition 2.10 *Let M be a right U-module and let $a,b \in U$ be arbitrary elements such that $a\mathcal{R}b$. Then $ma\mathcal{R}mb$, for all $m \in M$*

Proof. If $a\mathcal{R}b$ then there exist $((z,z'),(s,s')) \in \mathcal{Z}^2 \times U^2$ such that $a = b(z,s)$ and $b = a(z',s')$. Hence we get:

$$ma = m(b(z,s)) = mb(z,s)$$
$$mb = m(a(z',s')) = ma(z',s')$$

so $ma\mathcal{R}mb$ in M.

Proposition 2.11 *Let M be a nonzero U-module. Then M is irreducible if and only if $|M/\mathcal{R}| = 2$*

Proof. Since M is an irreducible module then for all $m \neq 0$ we get that $mU = M$. Because $MU \neq 0$, so for all $m, n \neq 0, m\mathcal{R}n$ and consequently $|M/\mathcal{R}| = 2$.
For the converse, let $N \neq 0$ be submodule of M, for any $m \in M - \{0\}$, and for any $n \in N - \{0\}$, $\overline{m}, \overline{n} \neq \overline{0}$. By using the fact that $|M/\mathcal{R}| = 2$ we conclude that $\overline{m} = \overline{n}$, so $N = M$ and M is an ireducible module as nontrivial modules.■

Proposition 2.12 *Let M be a completely reducible module, then for each two elements $m = m_1 + .. + m_n, m' = m'_1 + .. + m'_k$ such that $m\mathcal{R}m'$ we have that $n = k$, and $m_i\mathcal{R}m'_i$ where m_i are unique representation of m as a sum of elements in an irreducilble modules.*

Proof. Let $M = \sum_{i\in I} M_i$, where M_i are irreducible modules.
If $n > k$, we can write m, m' as $m = m_1 + m_2 + ...m_n, m' = m'_1 + m'_2 + ...m'_n$ where some of m_i, m'_i can be zero and m_i, m'_i can be in the same component M_i,
Since $m\mathcal{R}m'$ then $m = m'(z_1, a_1)$ for $(z_1, a_1) \in \mathbb{Z} \times U$
Since $M_i \cap (\sum_{j\neq i, j\in I} M_j) = 0$ we get that $m_i - m'_i(z_1, a_1) = 0$ thus $m'_i = m_i(z_1, a_1)$.
In a similar way we can prove that $m'_i = m_i(z_2, a_2)$ where $m' = m(z_2, a_2)$.
Hence $m_i\mathcal{R}m'_i$. For $m_i \neq 0$ we have that $m'_i \neq 0$, so $n = k$ ■
We let as an open problem the converse although it might not be true.

References

[1] J.A. *Fundamentals of Semigroup Theory*, preprint, 1999.

[2] N. Jacobson, *Structure of Rings*, Transl. Amer. Math. Soc.(1984).

[3] P. Petro, *Green's relations and minimal quasi-ideals in rings.*, Comuniations in Algebra, Vol 30, 2002, Issue 10 .

[4] M. Hazewinkel, V.V Kirichenko. *Algebras, Rings and Modules: Volume 1.*, Mathematics and its applications, Springer 2005.

[5] F.Cela, P.Petro. *On Green's relations in ring's and their multiplicative semigroups*, Journal of Natural Sciences and Mathematics of UT, Vol.2, No.4,2017.

[6] C. Hollings. *Mathematics across the iron curtain* , History of Mathematics. American Mathematical Society Vol.41, 2014.

[7] O, Steinfeld. *Quasi-ideals in rings and semigroups* , 10. Akadmiai Kiad, Budapest, 1978. xi+154 pp.