Attracting sets of nonlinear difference equations with time-varying delays

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Abstract

In this paper, a class of nonlinear difference equations with time-varying delays is considered.Based on a generalized discrete Halanay inequality, some sufficient conditions for the attracting set and the global asymptotic stabilityof the nonlinear difference equations with time-varying delays are obtained

Keywords: Halanay inequality, Nonlinear difference equations, Attractingset, Global asymptotic stabilityare.

Subject Classification: Mathematics (2010):39A22, 39A30.

Subject Classification: Image Processing

Language: English

Date of Publication: 20-09-2018

DOI: 10.24297/jam.v14i2.7668

ISSN: 2347 - 1921

Volume: 14 Issue: 02

Journal: Journal Of Advances In Mathematics

Website: https://cirworld.com

1 Introduction

Delay difference equations are an important class of discrete-time dynamical systems whose future states depend on not only the present states,but also the past states. In the past few decades, delay difference equations have attracted considerable research interestbecause these equations play an essential role in discrete analogues and numerical solutions of delay differential equations.A massive literature on the stability analysis delay difference equations is available [1-12]. It should be noticed that the equilibrium point sometimes does not exist in many real physical systems, especially innonlinear delay difference equations. Therefore, it is more interesting for nonlinear delay difference equations to study the attracting set than to study the stability. However, not much has been developed in the study of attracting setsfor the nonlinear difference equations with time-varying delays. Motivated by the above discussions, the main aim of this paper is to study the attracting setof the nonlinear difference equations with time-varying delays. Based on a generalized discrete Halanay inequality, some sufficient conditions for the attracting set and the global asymptotic stabilityof the nonlinear difference equations with time-varying delays are obtained.with no subheadings.

2 Model description and preliminaries

Let R, R^+ and R_0^+ denote the set of real numbers, positive realnumbers, nonnegative real numbers, respectively. Let $\|\bullet\|$ denotethe Euclidean norm, Z and Z^+ represent the set of integers and positive integers, respectively. $\mathrm{Z}^{-r} = \{z \in Z : z \geq -r\}$. For simplicity, we denote $x(m)$ and $x_j(m)$ by x_m and $x_{j,m}$, respectively. For a sequence of realnumber $\{x_m\}$, the difference operator Δ on x_m is defined as $\Delta x_m = x_{m+1} - x_m$.

Consider the following nonlinear difference equations with time-varying delays:

$$\begin{cases} x_i(m+1) = c_i x_i(m) + \sum_{j=1}^{n} a_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} b_{ij} f_j(m, x_{j,m}, x_{j,m-h_1}, \ldots, x_{j,m-h_r}) + I_i, m \in Z^+ \\ x_i(m) = \phi_i(m), m = -h_r, -h_{r+1}, \ldots, 0, i = 1,2,\ldots,n, \end{cases} \tag{1}$$

where c_i, a_{ij}, b_{ij}, I_i $(i, j = 1,2,\ldots,n)$ are real constants, $0 = h_0 < h_1 < \ldots < h_r$.

For convenience, we shall rewrite (1) in the vector form:

$$\begin{cases} x_{m+1} = Cx_m + Af(x_m) + Bg(m, x_m, x_{m-h_1}, \ldots, x_{m-h_r}) + I, m \in Z^+ \\ x_m = \phi_m, m = -h_r, -h_{r+1}, \ldots, 0, i = 1,2,\ldots,n, \end{cases} \tag{2}$$

where $x_m = (x_{1,m}, x_{2,m}, \ldots, x_{n,m})^T$, $C = diag\{c_1, \ldots, c_n\}$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $I = (I_1, I_2, \ldots, I_n)^T$, $\phi_m = (\phi_{1,m}, \ldots, \phi_{n,m})^T$. Throughout this paper, we assume that Eq. (2) has at least one solution x_m for any given initial function ϕ .

Definition 2.1. The set $\mathrm{S} \subset \mathrm{R}^n$ is called a global attracting set of (2), if for any initial function ϕ, the solution x_m satisfies dist$(x_m, S) \to 0$ as $m \to \infty$, where dist$(\varphi, S) = \inf_{\varphi \in S} \rho(\varphi, \psi)$ for $\varphi \in R^n$, $\rho(\bullet,\bullet)$ is any distance in R^n .

Lemma 2.1. [13] For any $\varepsilon > 0$, $x, y \in \mathrm{R}^n$,

$$2X^TY \le \varepsilon X^T X + \varepsilon^{-1} Y^T Y .$$

Lemma 2.2. [8] Let $q_i, \gamma \in R_0^+$, $h_i \in Z^+$, $i = 1,\ldots,r$; where $0 = h_0 < h_1 < \ldots < h_r$ and $\sum_{i=0}^{r} q_i < p \le 1$, and let $\{x_n\}_{n \in Z^{-h_r}}$ be a sequence of real numbers satisfying the inequality

$$\Delta x_n \le -p x_n + \sum_{i=0}^{r} q_i x_{n-h_i} + \gamma ,\ n \in Z^0 . \tag{3}$$

Then there exists $\lambda_0 \in (0,1)$ such that

$$x_n \le \max\{0, x_0, x_{-h_1}, \ldots, x_{-h_r}\}\lambda_0^n + \Lambda,\ \ n \in Z^0, \tag{4}$$

where $\Lambda = (p - \sum_{i=0}^{r} q_i)^{-1}\gamma > 0$. Moreover, λ_0 may be chosen as the smallest root of the polynomial

$$P(\lambda) = \lambda^{h_{r+1}} - (1 - p + q_0)\lambda^{h_r} - q_1\lambda^{h_r - h_1} - \ldots - q_{r-1}\lambda^{h_r - h_{r-1}} - q_r , \tag{5}$$

which lies in the interval $(0,1)$.

3 Main results

In this section, we will obtain several sufficient conditions for the attracting set of (2) by Lemma 2.2. The following assumptions are needed for our discussion.

(A_1) For any $x_m \in R^n$, there exists positive definite symmetric matrix J such that

$$f^T(x_m) f(x_m) \le x_m^T J x_m . \tag{6}$$

(A_2) There exist positive definite symmetric matrices L_i such that

$$g^T(m, x_m, x_{m-h_1}, \ldots, x_{m-h_r}) g(m, x_m, x_{m-h_1}, \ldots, x_{m-h_r}) \le \sum_{i=0}^{r} x_{m-h_i}^T L_i x_{m-h_i} , \tag{7}$$

for all $(m, x_m, x_{m-h_1}, \ldots, x_{m-h_r}) \in Z^0 \times R^{r+1}$.

Theorem 3.1 Suppose that (A_1) and (A_2) hold. If there exists a positive definite symmetric matrix P such that $\sum_{i=0}^{r} \beta_i < -(\gamma_1 + \gamma_2) \le 1$, then every solution $x(m)$ of (2) satisfies

$$\|x_m\| \le \max\{0, \|x_0\|, \|x_{-h_1}\|, \ldots, \|x_{-h_r}\|\}(\sqrt{\lambda_0})^n + ((-(\gamma_1 + \gamma_2) - \sum_{i=0}^{r} \beta_i)^{-1}\gamma_3)^{\frac{1}{2}} ,$$

where

$$\gamma_1 = \frac{\lambda_{\max}(C^T PC + \varepsilon_1^{-1} C^T PAA^T PC + \varepsilon_2^{-1} C^T PBB^T PC + \varepsilon_3^{-1} C^T PPC + \varepsilon_1 J}{\lambda_{\min}(P)}$$

$$\gamma_2 = \frac{\lambda_{\max}(A^T PA + \varepsilon_4^{-1} A^T PPA + \varepsilon_5^{-1} A^T PBB^T PA)\lambda_{\max}(J) - \lambda_{\min}(P)}{\lambda_{\min}(P)}$$

$$\beta_i = \frac{[\varepsilon_2 + \varepsilon_5 + \lambda_{\max}(B^T PB + \varepsilon_6^{-1} B^T PPB)]\lambda_{\max}(L_i)}{\lambda_{\min}(P)}$$

$$\gamma_3 = \frac{(\varepsilon_3 + \varepsilon_4 + \varepsilon_6) I^T I + I^T PI}{\lambda_{\min}(P)}$$

and the constant $\lambda_0 \in (0,1)$ may be chosen as the smallest root of the polynomial

$$P(\lambda) = \lambda^{h_{r+1}} - (1 + (\gamma_1 + \gamma_2) + \beta_0)\lambda^{h_r} - \beta_1 \lambda^{h_r - h_1} - \cdots - \beta_{r-1}\lambda^{h_r - h_{r-1}} - \beta_r . \tag{8}$$

As a consequence,

$$S = \left\{ \phi \in R^n \left\| \phi \right\| \le ((-(\gamma_1 + \gamma_2) - \sum_{i=0}^{r} \beta_i)^{-1} \gamma_3)^{\frac{1}{2}} \right\} \tag{9}$$

is a positive attracting set of (2).

Proof. Choosing the Lyapunov functional candidate

$$V(x) \le x^T Px .$$

Obviously, we have

$$\lambda_{\min}(P)\|x\| \le V(x) \le \lambda_{\max}(P)\|x\|. \tag{10}$$

Then, it follows that

$$\begin{aligned} V(x_{m+1}) &= x_{m+1}{}^T P x_{m+1} \\ &= [Cx_m + Af(x_m) + Bg(m, x_m, x_{m-h_1}, \ldots, x_{m-h_r}) + I]^T P[Cx_m \\ &+ Af(x_m) + Bg(m, x_m, x_{m-h_1}, \ldots, x_{m-h_r}) + I] \\ &= x_m^T C^T PCx_m + x_m^T C^T PAf(x_m \end{aligned}$$

$$
\begin{aligned}
&+x_m^T C^T PBg(m,x_m,x_{m-h_1},\dots,x_{m-h_r})+x_m^T C^T PI+f^T(x_m)A^T PCx_m \\
&+f^T(x_m)A^T PAf(x_m)+f^T(x_m)A^T PBg(m,x_m,x_{m-h_1},\dots,x_{m-h_r}) \\
&+f^T(x_m)A^T PI+g^T(m,x_m,x_{m-h_1},\dots,x_{m-h_r})B^T PCx_m \\
&+g^T(m,x_m,x_{m-h_1},\dots,x_{m-h_r})B^T PAf(x_m) \\
&+g^T(m,x_m,x_{m-h_1},\dots,x_{m-h_r})B^T PBg(m,x_m,x_{m-h_1},\dots,x_{m-h_r}) \\
&+g^T(m,x_m,x_{m-h_1},\dots,x_{m-h_r})B^T PI+I^T PCx_m+I^T PAf(x_m) \\
&+I^T PBg(m,x_m,x_{m-h_1},\dots,x_{m-h_r})+I^T PI.
\end{aligned}
\tag{11}
$$

From (11) and Lemma 2.1, we have

$$
\begin{aligned}
V(x_{m+1}) = {} & x_m^T[C^T PC+\varepsilon_1^{-1}C^T PAA^T PC+\varepsilon_2^{-1}C^T PBB^T PC+\varepsilon_3^{-1}C^T PPC]x_m \\
&+\varepsilon_1 f^T(x_m)f(x_m)+\varepsilon_2 g^T(m,x_m,x_{m-h_1},\dots,x_{m-h_r})g(m,x_m,x_{m-h_1},\dots,x_{m-h_r}) \\
&+f^T(x_m)[A^T PA+\varepsilon_4^{-1}A^T PPA+\varepsilon_5^{-1}A^T PBB^T PA]f(x_m) \\
&+\varepsilon_5 g^T(m,x_m,x_{m-h_1},\dots,x_{m-h_r})g(m,x_m,x_{m-h_1},\dots,x_{m-h_r}) \\
&+g^T(m,x_m,x_{m-h_1},\dots,x_{m-h_r})[B^T PB+\varepsilon_6^{-1}B^T PPB]g(m,x_m,x_{m-h_1},\dots,x_{m-h_r}) \\
&+(\varepsilon_3+\varepsilon_4+\varepsilon_6)I^T I+I^T PI
\end{aligned}
\tag{12}
$$

Utilizing (12) and Assumptions (A_1) and (A_2), we obtain

$$
\begin{aligned}
V(x_{m+1}) = {} & x_m^T[C^T PC+\varepsilon_1^{-1}C^T PAA^T PC+\varepsilon_2^{-1}C^T PBB^T PC+\varepsilon_3^{-1}C^T PPC+\varepsilon_1 J]x_m \\
&+\lambda_{\max}(A^T PA+\varepsilon_4^{-1}A^T PPA+\varepsilon_5^{-1}A^T PBB^T PA)x_m^T Jx_m \\
&+[\varepsilon_2+\varepsilon_5+\lambda_{\max}(B^T PB+\varepsilon_6^{-1}B^T PPB)](\sum_{i=0}^{r}x_{m-h_i}^T L_i x_{m-h_i}) \\
&+(\varepsilon_3+\varepsilon_4+\varepsilon_6)I^T I+I^T PI.
\end{aligned}
\tag{13}
$$

From (10) and (13) we have

$$
\begin{aligned}
\lambda_{\min}(P)\|x_{m+1}\|^2 \le {} & V(x_{m+1}) \\
= {} & \lambda_{\max}(C^T PC+\varepsilon_1^{-1}C^T PAA^T PC+\varepsilon_2^{-1}C^T PBB^T PC+\varepsilon_3^{-1}C^T PPC+\varepsilon_1 J)\|x_m\|^2 \\
&+\lambda_{\max}(A^T PA+\varepsilon_4^{-1}A^T PPA+\varepsilon_5^{-1}A^T PBB^T PA)\lambda_{\max}(J)\|x_m\|^2 \\
&+[\varepsilon_2+\varepsilon_5+\lambda_{\max}(B^T PB+\varepsilon_6^{-1}B^T PPB)](\sum_{i=0}^{r}\lambda_{\max}(L_i)\|x_{m-h_i}\|^2) \\
&+(\varepsilon_3+\varepsilon_4+\varepsilon_6)I^T I+I^T PI.
\end{aligned}
\tag{14}
$$

Let $y_m=\|x_m\|^2$, we have

$$\Delta y_m \le (\gamma_1+\gamma_2)y_m + (\sum_{i=0}^{r}\beta_i y_{m-h_i}) + \gamma_3 \,. \tag{15}$$

Therefore, by Lemma 2.2, we obtain

$$y_m \le \max\{0, y_0, y_{-h_1}, \ldots, y_{-h_r}\}\lambda_0^n + (-(\gamma_1+\gamma_2) - \sum_{i=0}^{r}\beta_i)^{-1}\gamma_3 \tag{16}$$

This means

$$\|x_m\| \le \max\{0, \|x_0\|, \|x_{-h_1}\|, \ldots, \|x_{-h_r}\|\}(\sqrt{\lambda_0})^n + ((-(\gamma_1+\gamma_2) - \sum_{i=0}^{r}\beta_i)^{-1}\gamma_3)^{\frac{1}{2}}, \tag{17}$$

where $\lambda_0 \in (0,1)$ may be chosen as the smallest root of the polynomial (\ref{qq8}). The proof is completed.

Corollary 3.1 Suppose that (A_1) and (A_2) hold. If exists a positive definite symmetric matrix P such that $\sum_{i=0}^{r}\beta_i < -(\gamma_1+\gamma_2) \le 1$, then the equation (2) with $I=0$ is globally exponentially stable with the exponential convergence rate $r=\sqrt{\lambda_0}$, where $\lambda_0 \in (0,1)$ may be chosen as the smallest root of the polynomial (8).

Proof. Substituting $I=0$ into γ_3 yields $\gamma_3 = 0$. Therefore, from (17) we have

$$\|x_m\| \le \max\{0, \|x_0\|, \|x_{-h_1}\|, \ldots, \|x_{-h_r}\|\}(\sqrt{\lambda_0})^n, \tag{18}$$

where $\lambda_0 \in (0,1)$ may be chosen as the smallest root of the polynomial (8). The proof is completed.

Let $\varepsilon_i (i=1,2,\ldots,6)=1$ and P be the n-dimensional unit matrix, then from Theorem 3.1 and Corollary 3.1 we obtain the following Corollary 3.2 and Corollary 3.3, respectively.

Corollary 3.2 Suppose that (A_1) and (A_2) hold. If there exists a positive definite symmetric matrix P such that $\sum_{i=0}^{r}\hat{\beta}_i < -(\hat{\gamma}_1+\hat{\gamma}_2) \le 1$, then every solution $x(m)$ of (2) satisfies

$$\|x_m\| \le \max\{0, \|x_0\|, \|x_{-h_1}\|, \ldots, \|x_{-h_r}\|\}(\sqrt{\lambda_0})^n + ((-(\hat{\gamma}_1+\hat{\gamma}_2) - \sum_{i=0}^{r}\hat{\beta}_i)^{-1}\hat{\gamma}_3)^{\frac{1}{2}},$$

where

$$\hat{\gamma}_1 = \lambda_{\max}(C^TPC + \varepsilon_1^{-1}C^TPAA^TPC + \varepsilon_2^{-1}C^TPBB^TPC + \varepsilon_3^{-1}C^TPPC + \varepsilon_1 J$$

$$\hat{\gamma}_2 = \lambda_{\max}(A^T PA + \varepsilon_4^{-1} A^T PPA + \varepsilon_5^{-1} A^T PBB^T PA)\lambda_{\max}(J) - 1$$

$$\hat{\beta}_i = [\varepsilon_2 + \varepsilon_5 + \lambda_{\max}(B^T PB + \varepsilon_6^{-1} B^T PPB)]\lambda_{\max}(L_i)$$

$$\hat{\gamma}_3 = (\varepsilon_3 + \varepsilon_4 + \varepsilon_6) I^T I + I^T PI$$

and the constant $\lambda_0 \in (0,1)$ may be chosen as the smallest root of the polynomial

$$P(\lambda) = \lambda^{h_{r+1}} - (1 + (\hat{\gamma}_1 + \hat{\gamma}_2) + \hat{\beta}_0)\lambda^{h_r} - \hat{\beta}_1 \lambda^{h_r - h_1} - \cdots - \hat{\beta}_{r-1}\lambda^{h_r - h_{r-1}} - \hat{\beta}_r. \quad (19)$$

As a consequence,

$$S = \left\{ \phi \in R^n \Big| \|\phi\| \le ((-(\hat{\gamma}_1 + \hat{\gamma}_2) - \sum_{i=0}^{r} \hat{\beta}_i)^{-1} \hat{\gamma}_3)^{\frac{1}{2}} \right\} \quad (20)$$

is a positive attracting set of (2).

Corollary 3.3 Suppose that (A_1) and (A_2) hold. If exists a positive definite symmetric matrix P such that $\sum_{i=0}^{r} \hat{\beta}_i < -(\hat{\gamma}_1 + \hat{\gamma}_2) \le 1$, then the equation (2) with $I = 0$ is globally exponentially stable with the exponential convergence rate $r = \sqrt{\lambda_0}$, where $\lambda_0 \in (0,1)$ may be chosen as the smallest root of the polynomial (19).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Funding Statement

The work is supported by the National Natural Science Foundation of China under Grants 11501518 and the Zhejiang Natural Science Foundation of China under Grants LQ16A010004.

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