Approximation of General Form for a Sequence of Linear Positive Operators Based on Four Parameters

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Abstract

In the present paper, we define a generalization sequence of linear positive operators based on four parameters which is reduce to many other sequences of summation–integral older type operators of any weight function (Bernstein, Baskakov, Szász or Beta). Firstly, we find a recurrence relation of the m-th order moment and study the convergence theorem for this generalization sequence. Secondly, we give a Voronovaskaja-type asymptotic formula for simultaneous approximation. Finally, we introduce some numerical examples to view the effect of the four parameters of this sequence.

Keywords: Sequences based on parameters, Korovkin theorem, $\boldsymbol{m}$-th order moment, Voronovaskaja –type asymptotic formula, simultaneous approximation

Date of Publication: 2018-09-30

DOI: https://doi.org/10.24297/jam.v14i2.7573

ISSN: 2347-1921

Volume: 14 Issue: 02

Journal: Journal of Advances in Mathematics

Website: https://cirworld.com

1. Introduction

In the past three years, many researchers have been built and studied sequences of summation-integral type operators based on parameters. These sequences give us some older sequence of summation-integral type operators, when we give suitable values for the parameters This paper is a continuation of the work of previous papers [1, 2, 3, 4, 5, 6, 8 and 11]. There are, introduce and studies general forms for sequences based on less than or equal four parameters, which are reduces to some knew sequences. Indeed, this paper is a generalization of these papers introduced in [10, 12 and 13].

The reader should be know the following sequences of linear positive operators:

- Summation integral Szász-Szász type sequence [10]

$$S_n(f,x) = n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} q_{n,k}(t)f(t)dt \quad ,n \in N, \qquad x \in [0,\infty)$$

where

$$q_{n,k}(x) = \frac{(nx)^k}{k!}e^{-nx}.$$

- Summation integral Baskakov-Szász type sequence [12]

$$V_n(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} q_{n,k}(t)f(t)dt\,,$$

where

$p_{n,k}(x) = \binom{n+k-1}{k}x^k(1+x)^{-n-k}.$

- Summation integral Beta-Szász type sequence [13]

$$M_n(f,x) = \sum_{k=0}^{\infty} \beta_{n,k}(x)\int_0^{\infty} q_{n,k}(t)f(t)dt \quad ,x \in [0,\infty),$$

where

$$\beta_{n,k}(x) = \frac{1}{\mathrm{B}(k+1,n)}x^k(1+x)^{-n-k-1},$$

$\mathrm{B}(k+1,n)$ being the Beta function given by $\frac{\Gamma(k+1)\,\Gamma(n)}{\Gamma(n+k+1)}$.

Suppose that $C[0,\infty)$ denotes the space of all continuous real-valued functions on the interval $[0,\infty)$ the subspace $C_\gamma[0,\infty)$ of the space $C[0,\infty)$ is defined as:

$C_\gamma[0,\infty) := \{f \in C[0,\infty): |f(t)| = O(e^{\gamma t}),$ for some $\gamma > 0\}$.

The space $C_\gamma[0,\infty)$ in normed by the norm: $\|f\|_{C_\gamma} = \sup_{t\in[0,\infty)} \frac{|f(t)|}{e^{\gamma t}}, f \epsilon C_\gamma[0,\infty)$.

We define and study our sequence based on four parameters $\rho > 0$, $c \in N^0 \cup \{-1\}$ and $r \in N^0 = N \cup \{0\}$ as follows:

$$H_{\alpha}^{\rho}(f,x,c,r)=\sum_{k=0}^{\infty}p_{\alpha,k}(x,c,r)\int_{0}^{\infty}\theta_{\alpha,k}^{\rho}(t)f(t)dt\,,x\in[0,\infty),\qquad(1.1)$$

where $p_{\alpha,k}(x,c,r)=\frac{(-x)^k}{k!}\phi_{\alpha,c,r}^{(k)}(x)$,

$$\phi_{\alpha,c,r}(x)=\begin{cases}e^{-\alpha x}, & c=r=0\\(1+cx)^{-\frac{\alpha}{c}-\frac{r}{c}}, & \text{otherwise}\end{cases}$$

and

$$\theta_{\alpha,k}^{\rho}(t)=\frac{\alpha\rho}{\Gamma(k\rho+1)}e^{-\alpha\rho t}(\alpha\rho t)^{k\rho}.$$

Sometime, we write the operators $H_{\alpha}^{\rho}(f,x,c,r)$ as $H_{\alpha}^{\rho}(f,x,c,r)=\int_0^{\infty}W_{\alpha}^{\rho}(t,x)f(t)dt$, where $W_{\alpha}^{\rho}(t,x)$ is called the kernel of the operators $H_{\alpha}^{\rho}(f,x,c,r)$ and define as:

$$W_{\alpha}^{\rho}(t;x)=\sum_{k=0}^{\infty}p_{\alpha,k}(x,c,r)\,\theta_{\alpha,k}^{\rho}(t)$$

where $\delta(t)$ being the Dirac-delta function.

Our new sequence is give us many other older sequences when $\alpha=n\in N$ for examples

- $H_n^1(f,x,0,0)=S_n(f,x)$;
- $H_n^1(f,x,0,1)=V_n(f,x)$;
- $H_n^1(f,x,1,1)=M_n(f,x)$.

In addition, from our sequence, we can get summation integral Bernstein-Szász type sequence. Here, we refer to the sequence getting by putting $\alpha=n$, $\rho=1$, $c=-1$ and $r=0$, i.e.

$$H_n^1(f,x,-1,0)=B_n(f,x)=n\sum_{k=0}^{n}b_{n,k}(x)\int_0^{\infty}q_{n,k}(t)f(t)dt\,,$$

where

$$b_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k},x\in[0,1]$$

During this paper, we assume that M is a positive real constant not necessarily the same in different cases.

2. Preliminary Results

First, we need to introduce some properties of the classical following sequence:

$$Y_{\alpha}(f,x,c,r)=\sum_{k=0}^{\infty}p_{\alpha,k}(x,c,r)f(\frac{k}{\alpha})\,,x\in[0,\infty)$$

where $f\in C_{\gamma}[0,\infty)$ and $m\in N^0$.

The m-th order moment of the sequence $Y_{\alpha}(f,x,c,r)$ is defined as

$$\mu_{\alpha,m}(x)=Y_{\alpha}((t-x)^m;x,c,r)=\sum_{k=0}^{\infty}p_{\alpha,k}(x,c,r)\left(\frac{k}{\alpha}-x\right)^m.$$

Lemma 2.1. The function $\mu_{\alpha,m}(x)$ defined above has the folloing properties:

$\mu_{\alpha,0}(x) = 1$, $\mu_{\alpha,1}(x) = \frac{rx}{\alpha}$, $\mu_{\alpha,2}(x) = \frac{x(1+cx)}{\alpha} + \frac{rx(1+cx)}{\alpha^2} + \frac{r^2x^2}{\alpha^2}$ and has the following recurrence relation

$$\alpha\mu_{\alpha,m+1}(x) = x(1+cx)\left[\mu'_{\alpha,m}(x) + m\mu_{\alpha,m-1}(x)\right] + rx\mu_{\alpha,m}(x), m \geq 1. \quad (2.1)$$

Further, the following consequences of $\mu_{\alpha,m}(x)$ are hold:

(i) $\mu_{\alpha,m}(x)$ is polynomial in x of degree at most m;

(i) For every $x \in [0,\infty)$, $\mu_{\alpha,m}(x) = O(\alpha^{-\left[\frac{m+1}{2}\right]})$.

Proof: It is clear that the relation is true at $x = 0$. Now, for $x \in [0,\infty)$, we have:

$$\mu_{\alpha,m}(x) = \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\left(\frac{k}{\alpha} - x\right)^m$$

$$\mu'_{\alpha,m}(x) = \sum_{k=0}^{\infty} p'_{\alpha,k}(x,c,r)\left(\frac{k}{\alpha} - x\right)^m - m\sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\left(\frac{k}{\alpha} - x\right)^{m-1}$$

$$x(1+cx)\mu'_{\alpha,m}(x) = -mx(1+cx)\mu_{\alpha,m-1}(x) + \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\left(\frac{k}{\alpha} - x\right)^m [k - (\alpha+r)x]$$

$$x(1+cx)\left[\mu'_{\alpha,m}(x) + m\mu_{\alpha,m-1}(x)\right] = \alpha\sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\left(\frac{k}{\alpha} - x\right)^{m+1} - rx\sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\left(\frac{k}{\alpha} - x\right)^m.$$

Form which (2.1) is immediate. Form the values of $\mu_{\alpha,0}(x)$, $\mu_{\alpha,1}(x)$, using the indication on m and the recurrence relation above, we can easily prove that $\mu_{\alpha,m}(x) = O(\alpha^{-\left[\frac{m+1}{2}\right]})$.

Lemma 2.2. For $H_\alpha^\rho(t^m; x,c,r)$ and $m \in N^0$ the following conditions are hold

$H_\alpha^\rho(1; x,c,r) = 1$;

$$H_\alpha^\rho(t; x,c,r) = x + \frac{rx}{\alpha} + \frac{1}{(\alpha\rho)};$$

$$H_\alpha^\rho(t^2; x,c,r) = x^2 + \frac{x(1+cx)}{\alpha} + \frac{rx(1+cx)}{\alpha^2} + \frac{3x}{\alpha\rho} + \frac{3rx}{\alpha^2\rho} + \frac{r^2x^2}{\alpha^2} + \frac{2rx^2}{\alpha} + \frac{2}{(\alpha\rho)^2}.$$

Hence, by applying Korovkin theorem [7] for $H_\alpha^\rho(f; x,c,r)$, we have that

$$H_\alpha^\rho(f(t), x,c,r) \to f(x) \text{ as } \alpha \to \infty.$$

In the same manner, we define the m-th order moment $T_{n,m}(x)$ for the sequence $H_\alpha^\rho(.; x,c,r)$ by:

$$T_{\alpha,m}(x) = H_\alpha^\rho((t-x)^m; x,c,r) = \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\int_0^\infty \theta_{\alpha,k}^\rho(t)(t-x)^m dt.$$

Lemma 2.3. For the function $T_{\alpha,m}(x)$, we have:

$T_{\alpha,0}(x) = 1;$

$$T_{\alpha,1}(x) = \frac{rx}{\alpha} + \frac{1}{\alpha\rho};$$

$$T_{\alpha,2}(x) = \frac{x(1+cx)}{\alpha} + \frac{rx(1+cx)}{\alpha^2} + \frac{r^2x^2}{\alpha^2} + \frac{x}{\alpha\rho} + \frac{3rx}{\alpha^2\rho} + \frac{2}{(\alpha\rho)^2}$$

also, we have the following recurrence relation

$$\alpha T_{\alpha,m+1}(x)$$

$$= x(1+cx)T'_{\alpha,m}(x) + mx\left[\frac{1}{\rho} + (1+cx)\right]T_{\alpha,m-1}(x) + \left[\frac{m+1}{\rho} + rx\right]T_{\alpha,m}(x). \qquad (2.2)$$

In addition, the function $T_{\alpha,m}(x)$ is a polynomial in x of degree at most m and $T_{\alpha,m}(x) = O(\alpha^{-\left[\frac{m+1}{2}\right]})$, where $\left[\frac{m+1}{2}\right]$ denotes the integer part of $\frac{m+1}{2}$.

Proof: By direct computation, the values $T_{\alpha,0}(x)$, $T_{\alpha,1}(x)$ and $T_{\alpha,2}(x)$ can be easily follow, we prove (2.2). For

$x = 0$ it clearly holds. For $x \in (0,\infty)$, we have:

Next,

$$T'_{\alpha,\mathrm{m}}(x) = \sum_{k=0}^{\infty} p'_{\alpha,k}(x,c,r)\int_0^{\infty}\theta_{\alpha,k}^{\rho}(t)(t-x)^m dt - m\sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\int_0^{\infty}\theta_{\alpha,k}^{\rho}(t)(t-x)^{m-1}dt.$$

Using the equation $x(1+cx)p'_{\alpha,k}(x,c,r) = (k-(\alpha+r)x)p_{\alpha,k}(x,c,r)$, we get

$$x(1+cx)T'_{\alpha,m}(x) = \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)(k-(\alpha+r)x)\int_0^{\infty}\theta_{\alpha,k}^{\rho}(t)\,(t-x)^m dt - mx(1+cx)T_{\alpha,m-1}(x).$$

Let $k-(\alpha+r)x = \left(k-\alpha t+\frac{1}{\rho}\right) - \frac{1}{\rho} + \alpha(t-x) - rx.$

Since $\frac{1}{\rho}\frac{d}{dt}\left(t\theta_{\alpha,k}^{\rho}(t)\right) = \left(k-\alpha t+\frac{1}{\rho}\right)\theta_{\alpha,k}^{\rho}(t).$

$$x(1+cx)T'_{\alpha,m}(x)$$

$$= \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\int_0^{\infty}\frac{1}{\rho}\frac{d}{dt}\left(t\theta_{\alpha,k}^{\rho}(t)\right)(t-x)^m dt - \frac{1}{\rho}\sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\int_0^{\infty}\theta_{\alpha,k}^{\rho}(t)\,(t-x)^m dt + \alpha T_{\alpha,m+1}(x)$$

$$-rxT_{\alpha,m}(x) - mx(1+cx)T_{\alpha,m-1}(x).$$

Since

$$\int_0^{\infty}\frac{d}{dt}\left(t\theta_{\alpha,k}^{\rho}(t)\right)(t-x)^m dt = -m\int_0^{\infty} t\theta_{\alpha,k}^{\rho}(t)(t-x)^{m-1}dt.$$

The identity $t = (t-x) + x$

$$x(1+cx)T'_{\alpha,m}(x)=\frac{-m}{\rho}\sum_{k=0}^{\infty}p_{\alpha,k}(x,c,r)\int_0^{\infty}\theta_{\alpha,k}^{\rho}(t)(t-x)^m dt-\frac{mx}{\rho}\sum_{k=0}^{\infty}p_{\alpha,k}(x,c,r)\int_0^{\infty}\theta_{\alpha,k}^{\rho}(t)(t-x)^{m-1}dt$$

$$-\frac{1}{\rho}\sum_{k=0}^{\infty}p_{\alpha,k}(x,c,r)\int_0^{\infty}\theta_{\alpha,k}^{\rho}(t)(t-x)^m dt-rxT_{\alpha,m}(x)-mx(1+cx)T_{\alpha,m-1}(x)+\alpha T_{\alpha,m+1}(x).$$

Finally, we get the recurrence relation state above.

From which (2.2) is immediate.

From the values of $T_{\alpha,0}(x)$, $T_{\alpha,1}(x)$, using the induction on m and the recurrence relation above, we can easily prove that $T_{\alpha,m}(x)=O\left(\alpha^{-\left[\frac{m+1}{2}\right]}\right)$ for every $x\in[0,\infty)$.

Lemma 2.4. Let γ and δ be any two positive real number and $[a,b]\subset[0,\infty]$ be any bounded interval. Then, for any $m>0$ there exists a constant M depending on m only and α independent of M.

We have

$$\left\|\int_{|t-x|\geq\delta}W_{\alpha}(t,x)e^{\gamma t}dt\right\|_{C[a,b]}=O(\alpha^{-m}),$$

where $\|.\|_{C[a,b]}$ means the sup norm in the space $C[a,b]$.

Proof: Using the Schwartz inequality for integration and then for summation, the proof this lemma is easily follows and the details are omitted.

Lemma 2.5. For every $x\in(0,\infty)$ and$s\in N^0$, there exist polynomials $Q_{i,j,r}(x,c,r)$ in x independent of α and k

$$\big(x(1+cx)\big)^s p_{\alpha,k}^{(s)}(x,c,r)=\sum_{\substack{2i+j\,\leq\,s\\ i,j\geq 0}}\alpha^i(k-(\alpha+r)x)^j Q_{i,j,r}(x,c,r)\,p_{\alpha,k}(x,c,r).$$

Proof: For the cases $c=1$ and $r=0$ the proof of this relation can be seen in [9].

3. Main Results

In this section, we introduce a Voronovskaja type asymptotic formula for the sequence $H_{\alpha}^{\rho}(f;x,c,r)$, we show that the s-th derivative $\frac{d^s}{dw^s}H_{\alpha}^{\rho}(f;w,c,r)\Big|_{w=x}$ is an approximation process for $f^{(s)}(x), s=1,2,\ldots$.

Theorem 3.1. For $\gamma>0$, $f\in C_{\gamma}[0,\infty)$ and $f^{(s)}$ exists at a point $x\in(0,\infty)$, then we have

$$\lim_{\alpha\to\infty}\frac{d^s}{dw^s}H_{\alpha}^{\rho}(f;w,c,r)\Big|_{w=x}=f^{(s)}(x). \qquad (3.1)$$

Further, if $f^{(s)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty),\eta>0$, then (3.1) holds uniformly in $[a,b]$.

Proof: by Taylor's expansion, we have

$$f(t)=\sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!}(t-x)^i+\psi(t,x)(t-x)^s\,,t\in[0,\infty)$$

where $\psi(t,x) \to 0$ as $t \to x$ hence,

$$\frac{d^s}{dw^s} H_\alpha^\rho(f(t); w, c, r)\Big|_{w=x} = \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} \frac{d^s}{dw^s} H_\alpha^\rho((t-x)^i; w, c, r)\Big|_{w=x} + \frac{d^s}{dw^s} H_\alpha^\rho(\psi(t,x)(t-x)^s; w, c, r)\Big|_{w=x}$$

$$:= I_1 + I_2.$$

$$I_1 = \frac{f^{(s)}(x)}{s!} \frac{d^s}{dw^s} H_\alpha^\rho((t-x)^s; w, c, r)\Big|_{w=x}$$

$$= \frac{f^{(s)}(x)}{s!} \frac{d^s}{dw^s} H_\alpha^\rho\Big(\sum_{j=0}^{s} \binom{s}{j} t^j (-x)^{s-j}; w, c, r\Big)\Bigg|_{w=x}$$

$$= \frac{f^{(s)}(x)}{s!} \frac{d^s}{dw^s} H_\alpha^\rho(t^s; w, c, r)\Big|_{w=x}$$

Since

$$H_\alpha^\rho(t^s; x, c, r) = \frac{\prod_{i=0}^{s-1}(\alpha + r + ci)}{(\alpha\rho)^s}(\rho x)^s + \left(\frac{s(s-1)\rho}{2} + \frac{s(s+1)}{2}\right)\frac{\prod_{i=0}^{s-2}(\alpha + r + ci)}{(\alpha\rho)^s}(\rho x)^{s-1} + O(\alpha^{-2}).$$

Then

$$\frac{d^s}{dx^s} H_\alpha^\rho(t^s; x, c, r) = \frac{\prod_{i=0}^{s-1}(\alpha + r + ci)}{(\alpha)^s}\, s!\,.$$

Therefore $I_1 = f^{(s)}(x)$ as $\alpha \to \infty$

$$I_2 = \frac{d^s}{dw^s} H_\alpha^\rho(\psi(t,x)(t-x)^s; w, c, r)\Big|_{w=x}$$

$$= \frac{d^s}{dw^s} \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r) \int_0^\infty \theta_{\alpha,k}^\rho(t)\psi(t,x)(t-x)^s dt\Bigg|_{w=x}.$$

From Lemma (2.5)

$$(x(1+cx))^s p_{\alpha,k}^{(s)}(x,c,r) = \sum_{\substack{2i+j \le s \\ i,j \ge 0}} \alpha^i (k - (\alpha + r)x)^j Q_{i,j,s}(x,c,r)\, p_{\alpha,k}(x,c,r)$$

$$\le \sum_{\substack{2i+j \le s \\ i,j \ge 0}} \frac{|Q_{i,j,s}(x,c,r)|}{(x(1+cx))^s} \alpha^i \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\, |k - (\alpha + r)x|^j \int_0^\infty \theta_{\alpha,k}^\rho(t)|\psi(t,x)||t-x|^s dt$$

Since $\psi(t,x) \to 0$ as$t \to x$, then for a given $\epsilon > 0$ there exists $\delta > 0$ such that $|\psi(t,x)| < \varepsilon$ whenever $|t-x| < \delta$. For $|t-x| \ge \delta$, we have$|\psi(t,x)(t-x)| \le Me^{\gamma t}$, for some $M > 0$. Thus,

$$I_3 = M \sum_{\substack{2i+j \le s \\ i,j \ge 0}} \alpha^i \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\, |k - (\alpha + r)x|^j \left\{\varepsilon \int_{|t-x|<\delta} \theta_{\alpha,k}^\rho(t)|t-x|^s dt + \int_{|t-x|\ge\delta}^{\infty} \theta_{\alpha,k}^\rho(t)e^{\gamma t} dt\right\}$$

$$:= I_4 + I_5,$$

where $M = \sup_{\substack{2i+j \le s \\ i,j \ge 0}} \frac{|Q_{i,j,s}(x,c,r)|}{(x(1+cx))^s}$.

Now, applying Schwarz inequality for integration, summation, w conclude

$$I_4 = \varepsilon M \sum_{\substack{2i+j \le s \\ i,j\ge 0}} \alpha^i \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\,|k-(\alpha+r)x|^j \left(\int_0^{\infty} \theta_{\alpha,k}^{\rho}(t)|t-x|^{2s}dt\right)^{\frac{1}{2}} \left(\int_0^{\infty} \theta_{\alpha,k}^{\rho}(t)dt\right)^{\frac{1}{2}}$$

Since $\int_0^{\infty} \theta_{\alpha,k}^{\rho}(t)dt = 1$

$$\le \varepsilon M \sum_{\substack{2i+j \le s \\ i,j\ge 0}} \alpha^i \left(\sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\,|k-(\alpha+r)x|^{2j}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\int_0^{\infty} \theta_{\alpha,k}^{\rho}(t)|t-x|^{2s}dt\right)^{\frac{1}{2}}$$

$$\le \varepsilon M\, O\left(\alpha^{-\frac{s}{2}}\right) \sum_{\substack{2i+j \le s \\ i,j\ge 0}} \alpha^i\, O\left(\alpha^{-\frac{j}{2}}\right) = \varepsilon\, O(1).$$

Since $\varepsilon > 0$ is arbitrary then $I_4 \to 0$ as $\alpha \to \infty$.

$$I_5 = M \sum_{\substack{2i+j \le s \\ i,j\ge 0}} \alpha^i \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\,|k-(\alpha+r)x|^j \int_{|t-x|\ge\delta} \theta_{\alpha,k}^{\rho}(t)e^{\gamma t}dt$$

$$\le M \sum_{\substack{2i+j \le s \\ i,j\ge 0}} \alpha^i \sum_{k=0}^{\infty} p_{\alpha,k}(x,c,r)\,|k-(\alpha+r)x|^j \left(\int_{|t-x|\ge\delta} \theta_{\alpha,k}^{\rho}(t)(t-x)^{2\gamma}dt\right)^{\frac{1}{2}} \left(\int_{|t-x|\ge\delta} \theta_{\alpha,k}^{\rho}(t)dt\right)^{\frac{1}{2}}$$

$$\le M \sum_{\substack{2i+j \le s \\ i,j\ge 0}} \alpha^i \left(\sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\,|k-(\alpha+r)x|^{2j}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\int_{|t-x|\ge\delta} \theta_{\alpha,k}^{\rho}(t)(t-x)^{2\gamma}dt\right)^{\frac{1}{2}}$$

$$= \sum_{\substack{2i+j \le r \\ i,j\ge 0}} \alpha^i\, O\left(\alpha^{\frac{2i+j}{2}}\right) O\left(\alpha^{\frac{-s}{2}}\right) = O(1)$$

$I_5 \to 0$ as $\alpha \to \infty$.

Hence, $I_2 = O(1)$ as $\alpha \to \infty$.

Combining the estimates of I_1 and I_2we obtain (3.1).

Theorem 3.2. Let $f \in C_\gamma[0,\infty)$ For some $\gamma > 0$. if f admits a derivative of order $(s+2)$ at a fixed point, $x \in (0,\infty)$, then we have

$$\lim_{\alpha\to\infty} \alpha\left(\frac{d^s}{dw^s} H_\alpha^{\rho}(f;w,c,r) - f^s(x)\Big|_{w=x}\right) = \left(\frac{s(s-1)c}{2} + sr\right) f^{(s)}(x)$$

$$+\left(x(r+cs) + \frac{s(\rho+1)}{2\rho} + \frac{1}{\rho}\right) f^{(s+1)}(x) + \left(\frac{x}{2\rho}[(\rho+1)+c\rho x]\right) f^{(s+2)}(x). \qquad (3.2)$$

Further, if $f^{(s+2)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty),\eta>0$, then (3.2) holds uniformly in $[a,b]$.

Proof: By Taylor's expansion, we have

$$f(t)=\sum_{i=0}^{s+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\psi(t,x)(t-x)^{s+2}.$$

Where $\psi(t,x)\to 0$ as $t\to x$ hence,

$$\frac{d^s}{dw^s}H_\alpha^\rho(f(t);w,c,r)\Big|_{w=x}=\sum_{i=0}^{s+2}\frac{f^{(i)}(x)}{i!}\frac{d^s}{dw^s}H_\alpha^\rho((t-x)^i;w,c,r)\Big|_{w=x}+\frac{d^s}{dw^s}H_\alpha^\rho(\psi(t,x)(t-x)^{s+2};w,c,r)\Big|_{w=x}$$

$$\lim_{\alpha\to\infty}\alpha\left(\frac{d^s}{dw^s}H_\alpha^\rho(f;w,c,r)-f^{(s)}(x)\Big|_{w=x}\right)=\lim_{\alpha\to\infty}\alpha\left(\sum_{i=0}^{s+2}\frac{f^{(i)}(x)}{i!}\left(\frac{d^s}{dw^s}H_\alpha^\rho((t-x)^i;w,c,r)\right)_{w=x}-f^{(s)}(x)\right)$$

$$+\lim_{\alpha\to\infty}\alpha\left(\frac{d^s}{dw^s}H_\alpha^\rho(\psi(t,x)(t-x)^{s+2};w,c,r)\Big|_{w=x}\right)$$

$:=I_1+I_2.$

$$I_1=\alpha\left\{\frac{f^{(s)}(x)}{s!}\frac{d^s}{dw^s}H_\alpha^\rho(t^s;w,c,r)\Big|_{w=x}+\frac{f^{(s+1)}(x)}{(s+1)!}\left[(s+1)(-x)\frac{d^s}{dw^s}H_\alpha^\rho(t^s;w,c,r)\Big|_{w=x}\right.\right.$$

$$\left.+\frac{d^s}{dw^s}H_\alpha^\rho(t^{s+1};w,c)\Big|_{w=x}\right]+\frac{f^{(s+2)}(x)}{(s+2)!}\left[\frac{(s+1)(s+2)}{2}x^2\frac{d^s}{dw^s}H_\alpha^\rho(t^s;w,c,r)\Big|_{w=x}\right.$$

$$\left.\left.+(s+2)(-x)\frac{d^s}{dw^s}H_\alpha^\rho(t^{s+1};w,c,r)\Big|_{w=x}+\frac{d^s}{dw^s}H_\alpha^\rho(t^{s+2};w,c,r)\Big|_{w=x}\right]-f^{(s)}(x)\right\}$$

$$=\alpha f^{(s)}(x)\left[\frac{\prod_{i=0}^{s-1}(\alpha+r+ci)}{(\alpha)^s}-1\right]+\alpha\frac{f^{(s+1)}(x)}{(s+1)!}\left[(s+1)!\,(-x)\frac{\prod_{i=0}^{s-1}(\alpha+r+ci)}{(\alpha)^s}\right.$$

$$\left.+\frac{\prod_{i=0}^{s}(\alpha+r+ci)}{(\alpha)^{s+1}}(s+1)!\,x+(s+1)!\left(\frac{s}{2}\rho+\frac{(s+2)}{2}\right)\frac{\prod_{i=0}^{s-1}(\alpha+r+ci)}{\rho(\alpha)^{s+1}}\right]$$

$$+\alpha\frac{f^{(s+2)}(x)}{(s+2)!}\left[\frac{(s+2)!}{2}\frac{\prod_{i=0}^{s-1}(\alpha+r+ci)}{(\alpha)^s}x^2+(s+2)!\frac{\prod_{i=0}^{s}(\alpha+r+ci)}{(\alpha)^{s+1}}(-x^2)\right.$$

$$+\frac{(s+2)!}{2}\frac{\prod_{i=0}^{s+1}(\alpha+r+ci)}{(\alpha)^{s+2}}x^2+(s+2)!\left(\frac{s}{2}\rho+\frac{(s+2)}{2}\right)\frac{\prod_{i=0}^{s-1}(\alpha+r+ci)}{\rho(\alpha)^{s+1}}(-x)$$

$$\left.+\left(\frac{(s+1)}{2}\rho+\frac{(s+3)}{2}\right)(s+2)!\frac{\prod_{i=0}^{s}(\alpha+r+ci)}{\rho(\alpha)^{s+2}}x\right]$$

$:=J_1+J_2+J_3.$

$$J_1=\left(\frac{s(s-1)c}{2}+sr\right)f^{(s)}(x);$$

$$J_2=f^{(s+1)}(x)\left(x(r+cs)+\frac{s(\rho+1)}{2\rho}+\frac{1}{\rho}\right);$$

$$J_3 = f^{(s+2)}(x)\left(\frac{x}{2\rho}[(\rho+1)+c\rho x]\right).$$

Since $I_2 \to 0$ as $\alpha \to \infty$.

Thus, we obtain (3.2).

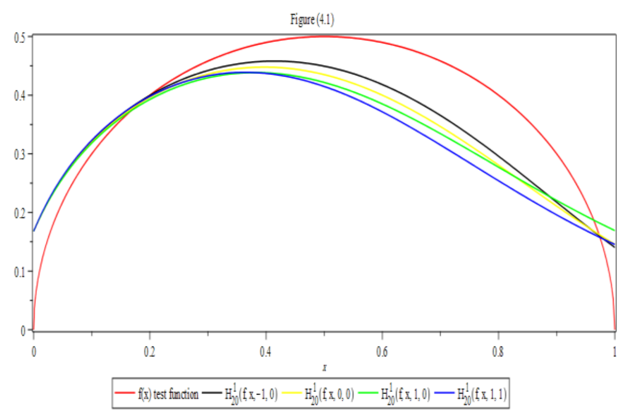
4. Numerical Examples

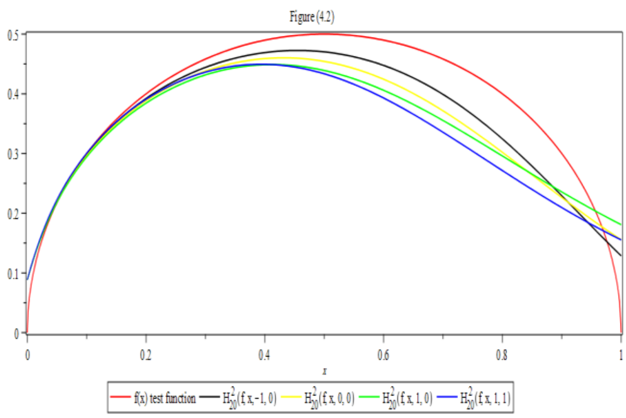
In this section, we introduce some numerical examples for the sequence $H_\alpha^\rho(f(t);x,c,r)$ by taking two test functions and show the approximation by this sequence $H_\alpha^\rho(f(t);x,c,r)$ and its derivatives to the function being approximate. The results are explain by graphs and the error functions occur between the test functions and the approximations.

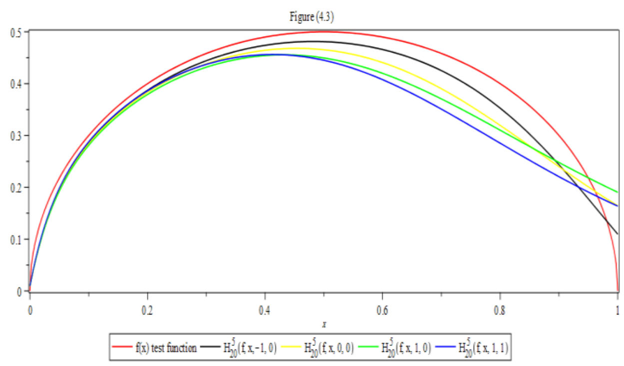
Suppose that g is an integrable function on interval $[a,b]$ and h is an approximate to the function g in the interval $[a,b]$. We define the error E as follows:

$$E = \int_a^b |g(t) - h(t)|\, dt.$$

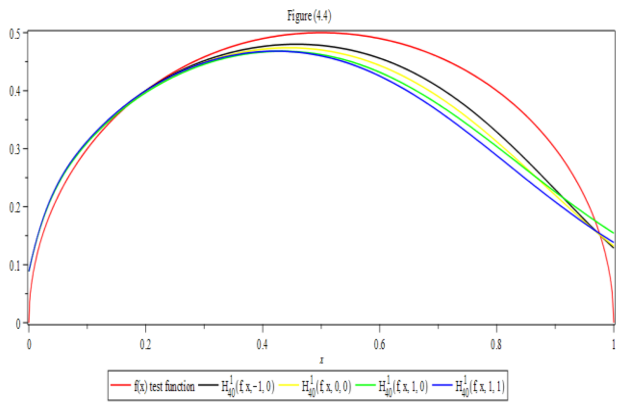
Example 4.1. Put $\alpha = 20$ and $\rho = 1, 2, 5$ for the test function $f(x) = \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}$, $x \in [0.1]$, we get the figures (4.1)-(4.3) respectively.

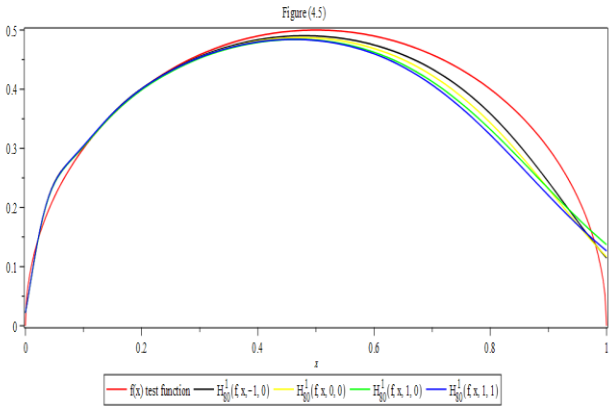


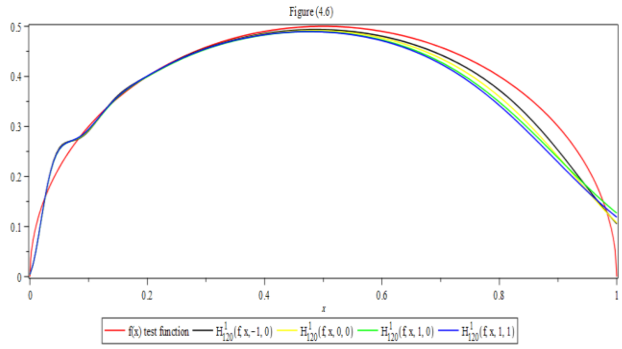




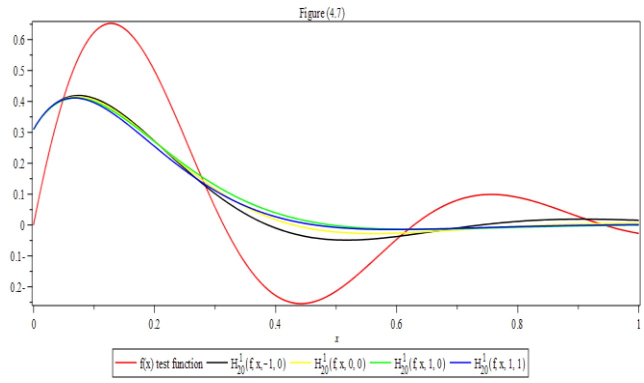
Example 4.2. Put $\alpha = 40,\ 80,\ 120$ and $\rho = 1$ for the test function $f(x) = \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}, x \in [0.1]$, we get the figures (4.4)-(4.6) respectively.

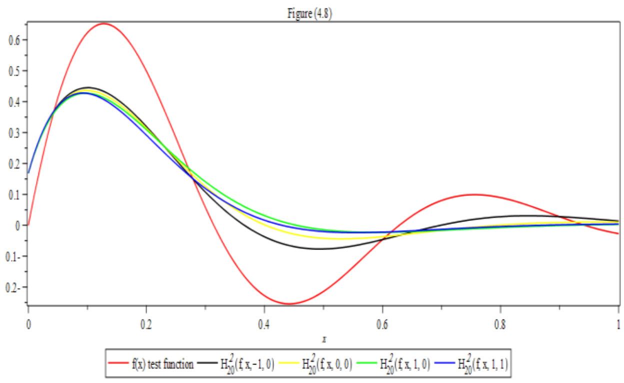


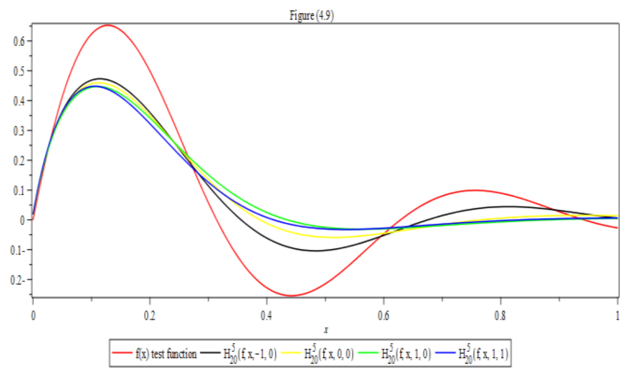




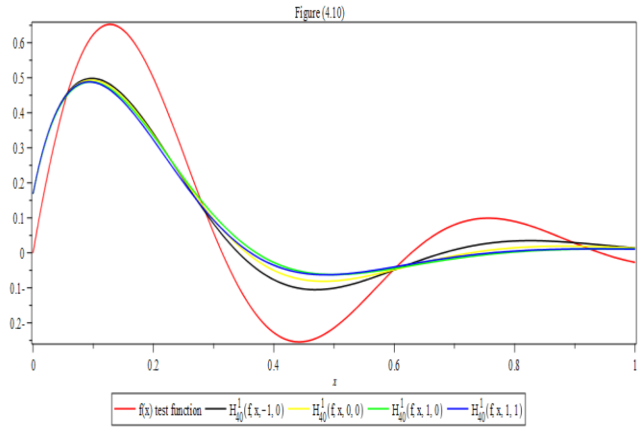
Example 4.3. Put $\alpha = 20$ and $\rho = 1, 2, 5$ for the test function $f(x) = \sin(10x)\, e^{-3x}, x \in [0,1]$, we get the figures (4.7)-(4.9) respectively.

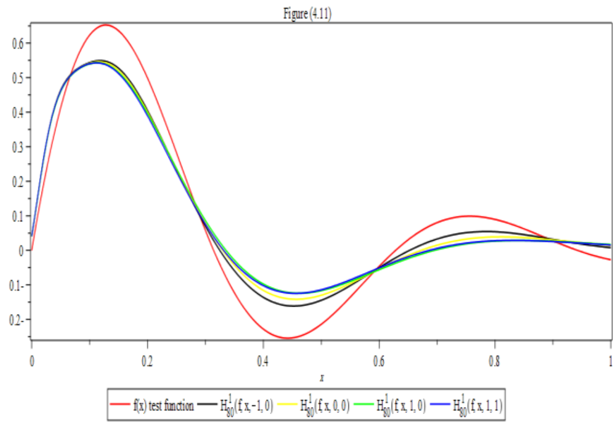


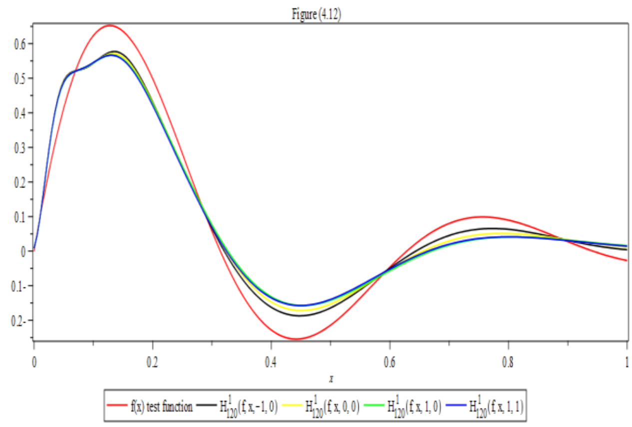




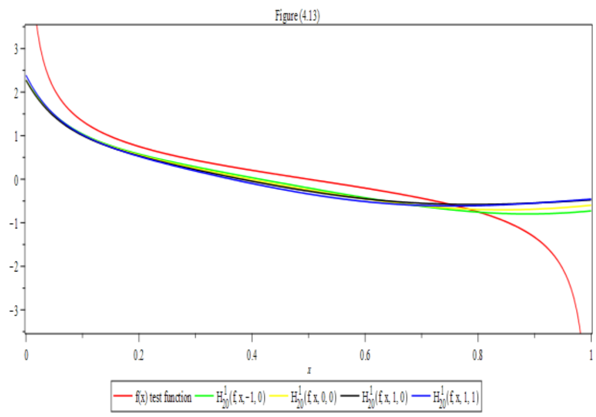
Example 4.4. Put $\alpha = 40, 80, 120$ and $\rho = 1$ for the test function $f(x) = \sin(10x)\, e^{-3x}, x \in [0,1]$, we get the figures (4.10)-(4.12) respectively.

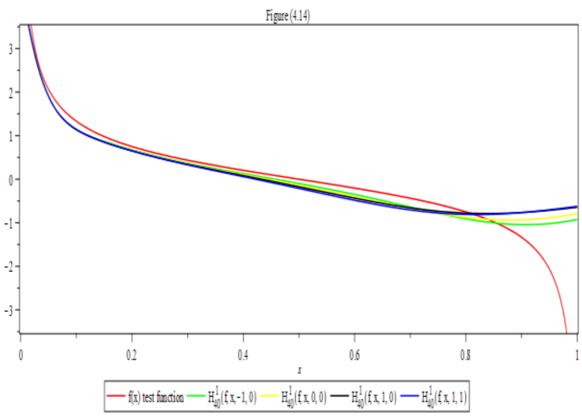




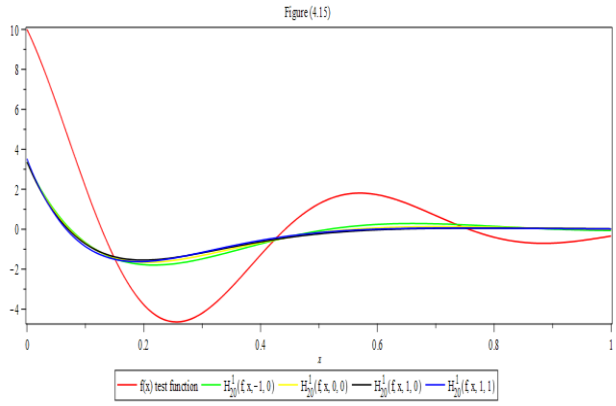


Exempla 4.5. Put $\alpha = 20, 40$ and $\rho = 1$ the convergence of the sequence $\frac{d}{dx} H_\alpha^\rho(f(t); x, c, r)$ to the function $\frac{d}{dx} f(x) = \frac{d}{dx}(\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2})$, $x \in [0,1]$, we get the figures (4.13)-(4.14) respectively.





Exempla 4.6. Put $\alpha = 20, 40$ and $\rho = 1$ the convergence of the sequence $\frac{d}{dx} H_{\alpha}^{\rho}(f(t); x, c, r)$ to the function $\frac{d}{dx} f(x) = \frac{d}{dx}(\sin(10x)\, e^{-3x}),\ x \in [0,1]$, we get the figures (4.15)-(4.16) respectively.



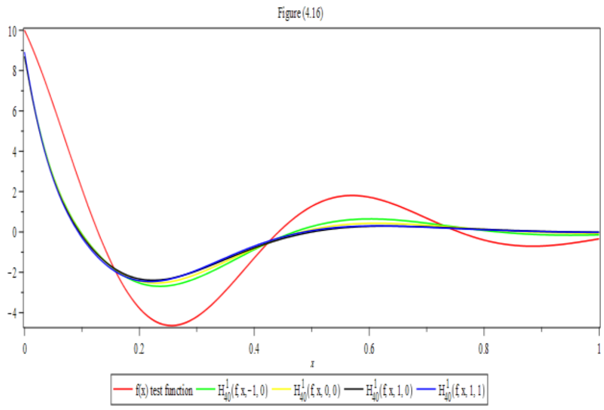


Table 4.1. Shows the value of the error function E by the parameter values α, ρ of the sequence $H_{\alpha}^{\rho}(f, x, c, r)$ for function the test $f(x) = \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}$

The sequence	$\rho = 1$	$\rho = 1.5$	$\rho = 2$
$H_{20}^{\rho}(f, x, -1,0)$	$0.5231988685e - 1$	$0.4035461651e - 1$	$0.3401310448e - 1$
$H_{40}^{\rho}(f, x, -1,0)$	$0.3098022066e - 1$	$0.2340296322e - 1$	$0.2027696454e - 1$
$H_{40.5}^{\rho}(f, x, -1,0)$	$0.3037282140e - 1$	$0.2305826975e - 1$	$0.2010000244e - 1$
$H_{20}^{\rho}(f, x, 0,0)$	$0.6127126920e - 1$	$0.5073029289e - 1$	$0.4553608828e - 1$
$H_{40}^{\rho}(f, x, 0,0)$	$0.3803138880e - 1$	$0.3128938169e - 1$	$0.2874132397e - 1$
$H_{40.5}^{\rho}(f, x, 0,0)$	$0.3819753878e - 1$	$0.3148135068e - 1$	$0.2893827328e - 1$
$H_{20}^{\rho}(f, x, 1,0)$	$0.6658951445e - 1$	$0.5729171730e - 1$	$0.5299131396e - 1$
$H_{40}^{\rho}(f, x, 1,0)$	$0.4250067617e - 1$	$0.3640006110e - 1$	$0.3426809680e - 1$
$H_{40.5}^{\rho}(f, x, 1,0)$	$0.4260499198e - 1$	$0.3652658173e - 1$	$0.3439444871e - 1$
$H_{20}^{\rho}(f, x, 1,1)$	$0.7520779764e - 1$	$0.6518004854e - 1$	$0.5992990033e - 1$
$H_{40}^{\rho}(f, x, 1,1)$	$0.4762326366e - 1$	$0.4109374653e - 1$	$0.3853042539e - 1$
$H_{40.5}^{\rho}(f, x, 1,1)$	$0.4776705635e - 1$	$0.4125428055e - 1$	$0.3870234942e - 1$

Table (4.1)

Table 4.2. Shows the value of the error function by parameter values α, ρ for the sequence $H_{\alpha}^{\rho}(f, x, c, r)$ for function the test $f(x) = \sin(10x)\, e^{-3x}$

The sequence	$\rho = 1$	$\rho = 1.5$	$\rho = 2$
$H_{20}^{\rho}(f, x, -1,0)$	0.1153700137	0.1026977270	$0.9480519059e - 1$
$H_{40}^{\rho}(f, x, -1,0)$	$0.7965284146e - 1$	$0.06704181936e - 1$	$0.5991964902e - 1$
$H_{40.5}^{\rho}(f, x, -1,0)$	$0.7888248106e - 1$	$0.6635425644e - 1$	$0.5929919364e - 1$
$H_{20}^{\rho}(f, x, 0,0)$	0.1248684215	0.1156266940	0.1101446950
$H_{40}^{\rho}(f, x, 0,0)$	$0.9010708858e - 1$	$0.08001907225e - 1$	$0.7452265499e - 1$
$H_{40.5}^{\rho}(f, x, 0,0)$	$0.8948387285e - 1$	$0.7941388577e - 1$	$0.7393553518e - 1$
$H_{20}^{\rho}(f, x, 1,0)$	0.1311079457	0.1238737934	0.1197078019
$H_{40}^{\rho}(f, x, 1,0)$	$0.9809666512e - 1$	$0.8957011436e - 1$	$0.8501446977e - 1$
$H_{40.5}^{\rho}(f, x, 1,0)$	$0.9749668073e - 1$	$0.8897284530e - 1$	$0.8442355702e - 1$
$H_{20}^{\rho}(f, x, 1,1)$	0.1300450161	0.1223008671	0.1177245063
$H_{40}^{\rho}(f, x, 1,1)$	$0.9742135996e - 1$	$0.8859542122e - 1$	$0.8381471380e - 1$
$H_{40.5}^{\rho}(f, x, 1,1)$	$0.9683332206e - 1$	$0.8801244668e - 1$	$0.8324007680e - 1$

Table (4.2)

Table 4.3. Shows the value of the error function E by parameter values α, ρ for the sequence $\frac{d}{dx} H_{\alpha}^{\rho}(f, x, c, r)$ for function the test $\frac{d}{dx} f(x) = \frac{d}{dx}\left(\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}\right)$

The sequence	$\rho = 1$	$\rho = 1.5$
$\frac{d}{dx} H_{20}^{\rho}(f, x, -1,0)$	0.5165158746	0.4315109183
$\frac{d}{dx} H_{40}^{\rho}(f, x, -1,0)$	0.3698853954	0.3196723100
$\frac{d}{dx} H_{20}^{\rho}(f, x, 0,0)$	0.5535021159	0.4855199331
$\frac{d}{dx} H_{40}^{\rho}(f, x, 0,0)$	0.4017649664	0.3643542686
$\frac{d}{dx} H_{20}^{\rho}(f, x, 1,0)$	0.5909387065	0.5263493438
$\frac{d}{dx} H_{40}^{\rho}(f, x, 1,0)$	0.4358381024	0.3984571765

$\frac{d}{dx}H_{20}^{\rho}(f,x,1,1)$	0.6107287290	0.5468538012
$\frac{d}{dx}H_{40}^{\rho}(f,x,1,1)$	0.4515302033	0.4174223054

Table (4.3)

Table 4.4. Shows the value of the error function E by parameter values α, ρ for the sequence $\frac{d}{dx}H_{\alpha}^{\rho}(f,x,c,r)$ for function the test $\frac{d}{dx}f(x) = \frac{d}{dx}(\sin(10x)\,e^{-3x})$

The sequence	1	1.5
$\frac{d}{dx}H_{20}^{\rho}(f,x,-1,0)$	1.558174800	1.357430849
$\frac{d}{dx}H_{40}^{\rho}(f,x,-1,0)$	1.045151546	0.8463874786
$\frac{d}{dx}H_{20}^{\rho}(f,x,0,0)$	1.637468279	1.474332363
$\frac{d}{dx}H_{40}^{\rho}(f,x,0,0)$	1.151186273	0.9814928752
$\frac{d}{dx}H_{20}^{\rho}(f,x,1,0)$	1.687781683	1.543273588
$\frac{d}{dx}H_{40}^{\rho}(f,x,1,0)$	1.227638145	1.074604477
$\frac{d}{dx}H_{20}^{\rho}(f,x,1,1)$	1.684164816	1.535270668
$\frac{d}{dx}H_{40}^{\rho}(f,x,1,1)$	1.222612969	1.070498968

Table (4.4)

Conclusions

The numerical results show the effect of parameter ρ, c and r values in the approximation by our sequence. It turns out that,

- The approximation increases whenever ρ increase
- The best approximation occur in our examples when $c = -1$ and $r = 0$
- When α is sufficiently large the effect of the parameters ρ, c and r is vanish

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