Global existence and uniqueness of the solution to a nonlinear parabolic equation

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Abstract

Consider the equation

$$u'(t) - \Delta u + |u|^\rho u = 0, \quad u(0) = u_0(x), (1),$$

where $u' := \frac{du}{dt}$, $\rho = const > 0$, $x \in \mathbb{R}^3$, $t > 0$.

Assume that u_0 is a smooth and decaying function,

$$\|u_0\| = \sup_{x\in\mathbb{R}^3, t\in\mathbb{R}_+} |u(x,t)|.$$

It is proved that problem (1) has a unique global solution and this solution satisfies the following estimate

$$\|u(x,t)\| < c,$$

where $c > 0$ does not depend on x, t.

Mathematics Subject Classification: MSC 2010, 35K55.

Keywords: nonlinear parabolic equations; global solutions.

1 Introduction

Let

$$u' - \Delta u + |u|^\rho u = 0, \quad u(0) = u_0; \quad u' := \frac{du}{dt}, \tag{1}$$

where $\rho > 0$, $t \in \mathbb{R}_+ = [0, \infty)$, $x \in \mathbb{R}^3$, X is a Banach space of real-valued functions with the norm $\|u(x,t)\| := \sup_{x\in\mathbb{R}^3, t\in R_+} |u(x,t)|$. We assume that

$$\|u\| \le c. \tag{2}$$

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ISSN: 2347-1921
Volume:14 Issue: 2
DOI:10.24297/jam.v14i2.7503
Journal: Journal of Advances in Mathematics

We say that u is a global solution to (1) if u exists $\forall t \geq 0$.

Our result is formulated in Theorem 1. Our method is simple and differs from the published results, see [1], [2] and references there.

The novel points in this work are:

a) There is no restriction on the upper bound of ρ.

In [1], (Section 1.1) a nonlinear hyperbolic equation with the same nonlinearity is studied in a bounded domain, uniqueness of the solution is proved only for $\rho \leq 2/(n-2)$, and existence is proved by a different method. The contraction mapping theorem is not used.

In [2] the quasi-linear problems for parabolic equations are studied in Chapter 5 in a bounded domain and under the assumptions different from ours. There are many papers and books on non-linear problems for parabolic equations (see the bibliography in [1], [2].

b) Existence of the global solution is proved.

c) Method of the proof differs from the methods in the cited literature.

Our result is formulated in Theorem 1:

Theorem 1. *Problem* (1) *has a unique global solution in* X *for any* $u_0 \in X$.

2 Proofs

Let $g(x,t) = \frac{e^{-|x|^2}}{(4\pi t)^{3/2}}$. If u solves (1) then

$$u(t) = -\int_0^t d\tau \int g(x-y, t-\tau)|u|^\rho u dy + \int g(x-y,t)u_0(y)dy := A(u) + F := Q(u), \tag{3}$$

where $\int := \int_{\mathbb{R}^3}$. Let X be the Banach space of continuous in $\mathbb{R}^3 \times R_+$ functions, $\mathbb{R}_+ := [0,\infty)$, $\|u\| := \max_{x\in\mathbb{R}^3, t\in[0,T]} |u(x,t)|$. If $\|u\| \leq R$ then $\|A(u)\| \leq TR^{\rho+1}$, where the identity $\int g(x-y,t-\tau)dy = 1$ was used. From (3) one gets

$$\|u\| \leq T\|u\|^{\rho+1} + \|F\|. \tag{4}$$

Thus, Q maps the ball $B(R) = \{u : \|u\| \leq R\}$ into itself if T is such that

$$TR^{\rho+1} + \|F\| \leq R. \tag{5}$$

The Q is a contraction on $B(R)$ if

$$\|Q(u) - Q(v)\| \leq T(\rho+1)R^\rho \|u-v\| \leq q\|u-v\|, \quad 0 < q < 1.$$

Thus, if

$$T(\rho+1)R^\rho \leq q < 1, \tag{6}$$

then Q is a contraction in $B(R)$ in the Banach space X_T with the norm $\|\cdot\|$, $t \in [0,T]$. We use the same notations for the norms in X_T and in X_∞.

We have proved that

For T satisfying (5)- (6) *there exists and is unique the solution to* (1), *and this solution can be obtained from* (3) *by iterations.*

The problem now is:

Does this solution exist and is unique on R_+?

From our proof it follows that if the solution exists and is unique in X_T, then the solution exists and is unique in X_{T_1} for some $T_1 > T$.

To prove that the solution $u(x,t)$ to (1) exists on $\mathbb{R}_+$, assume the contrary: this solution does not exist on any interval $[0,T_1)$, $T_1 > T$, where T is the maximal interval of the existence of the continuous solution. Then $\lim_{t\to T-0} u(x,t) = \infty$, because otherwise there is a sequence $t_n \to T-0$ such that $u(x,t_n) \to u(x,T)$ and one may construct the solution defined on $[T,T_1]$, $T_1 > T$, by using the local existence and uniqueness of the solution to (1) with the initial value $u(x,T)$ for $t \in [T,T_1]$. This contradicts the assumption that T is the maximal interval of the existence of the continuous solution u.

Thus, if $T < \infty$ then one has $\lim_{t\to T-0} u(x,t) = \infty$. Let us prove that this also leads to a contradiction. Then we have to conclude that $T = \infty$ and Theorem 1 is proved.

We need some estimates. Multiply (1) by u, integrate over $\mathbb{R}^3$ with respect to x, and then integrate by parts the second term. The result is:

$$0.5\frac{dN(u)}{dt} + N(gradu) + \int |u|^{\rho+2}dy = 0, \tag{7}$$

where $N(u) := \int u^2 dy$. Integrate (7) with respect to time over $[0,T]$ and get

$$0.5N(u(T)) + \int_0^T \left(N(gradu) + \int |u|^{\rho+2}dy\right) d\tau = 0.5N(u(0)). \tag{8}$$

Therefore,

$$N(u(t)) \le c, \ \forall t \in [0,T], \quad \int_0^T N(gradu)d\tau \le c, \quad \int_0^T d\tau \int |u|^{\rho+2}dy \le c, \tag{9}$$

where $c = 0.5N(u_0)$.

Lemma 1. *From* (9) *and* (3) *it follows that*

$$\|u(x,t)\| < \infty \quad \forall t \in [0,T]. \tag{10}$$

If (10) is proved then T is not the maximal interval of the existence of the solution to (1). This contradiction proves Theorem 1.

Proof of Lemma 1. One uses the Hölder inequality twice and gets

$$\int_0^T d\tau \int g(x-y,t-\tau)|u|^{\rho+1}dy \leq$$
$$\left(\int_0^T d\tau \int |u|^{\rho+2}dy\right)^{(\rho+1)/(\rho+2)} \left(\int_0^T d\tau \int g^{\rho+2}dy\right)^{1/(\rho+2)} \leq \qquad (11)$$
$$\left(\int_0^T d\tau \int |u|^{\rho+2}dy\right)^{(\rho+1)/(\rho+2)} \left(\int_0^T d\tau \int g^{\rho+2}dy\right)^{1/(\rho+2)}.$$

By the last inequality (9) it follows that $\int_0^T d\tau \int |u|^{\rho+2}dy < c \ \forall T > 0$, where $c > 0$ is a constant independent of T. The last integral in (11) is also bounded independently of T. It can be calculated analytically.

Thus, inequalities (11), (9) and equation (3) imply (10).

Lemma 1 is proved. □

Therefore Theorem 1 is proved. □

The ideas related to the ones used in this paper were developed and used in [3]–[5].

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doi:10.3390/math1020046

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