Estimates of solutions to nonlinear evolution equations

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Abstract

Consider the equation

$$u'(t) = A(t, u(t)), \quad u(0) = u_0; \quad u' := \frac{du}{dt} \quad (1).$$

Under some assumptions on the nonlinear operator $A(t, u)$ it is proved that problem (1) has a unique global solution and this solution satisfies the following estimate

$$\|u(t)\| < \mu(t)^{-1} \quad \forall t \in \mathbb{R}_+ = [0, \infty).$$

Here $\mu(t) > 0$, $\mu \in C^1(\mathbb{R}_+)$, is a suitable function and the norm $\|u\|$ is the norm in a Banach space X with the property $\|u(t)\|' \leq \|u'(t)\|$.

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1 Introduction

Let

$$u' = A(t, u(t)), \quad u(0) = u_0; \quad u' := \frac{du}{dt}, \tag{1}$$

where $t \in \mathbb{R}_+ = [0, \infty)$, $A(t, u)$ is a locally continuous map from $\mathbb{R}_+ \times X$ into X, where X is a Banach space of functions with the norm $\| \cdot \|$, such that $\|u(t)\|' \leq \|u'(t)\|$ if $u(t)$ is continuously differentiable with respect to t. If $u(t) \in X$ is a function then $|u(t)|$ and $\|u(t)\|$ make sense. We assume that if $|u| \leq |v|$ then $\|u\| \leq \|v\|$. For the spaces of continuous functions and L^p spaces this assumption holds.

We assume that

$$\|A(t, u) - A(t, v)\| \leq k\|u - v\|, \tag{2}$$

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where $k > 0$ is a constant which may depend on R, $\|u\| \le R$, $\|v\| \le R$, and on T, $t \in [0, T]$.

If $A(t, u)$ is a function with values in $\mathbb{R}$ and $\|A(t, u)\| = |A(t, u)|$, then (1) is a nonlinear ordinary differential equation and condition (2) guarantees local existence and uniqueness of its solution on an interval $[0, T]$ where T is a sufficiently small number. If $T = \infty$ then the solution $u(t)$ is called global.

The map $A(t, u)$ may be of the form

$$A(t, u) = \int_0^t a(t, s, u(s))ds, \tag{3}$$

where $a(t, s, u)$ is a locally continuous function on $\mathbb{R}_+ \times R_+ \times X$, locally Lipschitz with respect to u.

The following assumptions will be valid throughout this paper:

There exists a $C^1(\mathbb{R}_+)$ function $\mu(t) > 0$ such that

$$\|A(t, \frac{w}{\mu(t)})\| \le \left(\frac{1}{\mu(t)}\right)', \tag{4}$$

where $\|w\| = 1$, $w \in X$ is an arbitrary element,

$$\|A(t, u)\| \le \|A(t, v)\| \quad if \quad |u| \le |v|, \tag{5}$$

and

$$\|u(0)\| < \frac{1}{\mu(0)}. \tag{6}$$

Theorem 1. *Under the above assumptions the solution to* (1) *exists globally, is unique, and satisfies the following estimate:*

$$\|u(t)\| < \frac{1}{\mu(t)}, \quad \forall t \in \mathbb{R}_+. \tag{7}$$

Remark 1. Some conditions on $A(t, u)$ of the type (4)- (6) are necessary for the global existence of the solution.

Consider the following example: $u' = u^2$, $u(0) = 1$. This problem is equivalent to the equation $u = 1 + \int_0^t u^2(s)ds$. The solution to this problem is $u(t) = (1 - t)^{-1}$, so it tends to ∞ as $t \to 1$. The solution is smooth on $[0, \lambda]$, where $0 < \lambda < 1$ is arbitrary.

2 Proofs

The proof of Theorem 1 consists of several parts. We start with the part dealing with the inequality

$$\|u(t)\|' \le \|u'(t)\|. \tag{8}$$

We assume throughout that $u(t)$ is continuously differentiable with respect to t.

2.1. Inequality (8) holds if $X = H$, where H is a Hilbert space. The inner product in H is denoted as usual (u, v). A simple proof of (8) goes as follows. Start with the inequality

$$\frac{\|u(t+h)\| - \|u(t)\|}{h} \le \|\frac{u(t+h)-u(t)}{h}\| \tag{9}$$

and let $h \to 0$. The result is (8). Indeed, the limit of the right side does exist and is equal to $\|u'(t)\|$. To calculate the limit of the left side in (9) consider the identity

$$h^{-1}(\|u(t+h)\| - \|u(t)\|)(\|u(t+h)\| + \|u(t)\|) =$$

$$h^{-1}(u(t+h)-u(t), u(t+h)) + h^{-1}(u(t), u(t+h)-u(t)).$$

Clearly, the limit of the right side exists and is equal to $2Re(u'(t), u(t))$. One has $\lim_{h\to 0}(\|u(t+h)\| + \|u(t)\|) = 2\|u(t)\|$. Assuming that $\|u(t)\| > 0$ one concludes that

$$\|u(t)\|' = \lim_{h\to 0} h^{-1}(\|u(t+h)\| - \|u(t)\|) = Re(u'(t), u(t))/\|u(t)\| \le \|u'(t)\|.$$

If $\|u(t)\| = 0$, then $\|u(t)\|' = \lim_{h\to 0} h^{-1}\|u(t+h)\|$. One has $\|u(t+h)\|^2 = (u(t+h), u(t+h)) = h^2\|u'(t)\|^2 + o(h^2)$. Thus, $\|u(t+h)\| = |h|\|u'(t)\| + o(h)$. Therefore $\|u(t)\|' = \lim_{h\to 0} h^{-1}|h|\|u'(t)\| = \operatorname{sign} h\|u'(t)\| \le \|u'(t)\|$. Formula (8) is proved for $X = H$. □

If $X = \mathbb{R}$ the proof of (8) is left for the reader. One gets $|\ |u(t)|'\ | \le |u'(t)|$.

2.2. Let us study problem (1) assuming that $X = \mathbb{R}$, $w = 1$ in (4) and $\|u(t)\| = |u(t)|$. Assumption (2) guarantees local existence and uniqueness of the solution to (1). We want to prove that assumptions (4)–(6) guarantee the global existence of the solution $u(t)$ and estimate (7). If (6) holds, then, by continuity, there exists a small $\delta > 0$ such that

$$|u(t) < \frac{1}{\mu(t)}, \quad 0 \le t \le \delta. \tag{10}$$

This and (5) imply

$$|A(t, u(t))| \le |A(t, \frac{1}{\mu(t)})|, \quad 0 \le t \le \delta. \tag{11}$$

Take the absolute value of (1), use (7), (11) and (4) to get

$$|u(t)|' \le |A(t, u(t))| \le |A(t, \frac{1}{\mu(t)})| \le \left(\frac{1}{\mu(t)}\right)', \quad 0 \le t \le \delta. \tag{12}$$

Integrating (12) with respect to t one gets

$$|u(t)| - |u(0)| \leq \frac{1}{\mu(t)} - \frac{1}{\mu(0)}, \quad 0 \leq t \leq \delta. \tag{13}$$

This and (6) imply (7) for $t \in [0, \delta]$. Define T as follows:

$$T = \sup\{\delta : |u(t) < \frac{1}{\mu(t)}, \quad 0 \leq t \leq \delta\}. \tag{14}$$

Let us prove that $T = \infty$.

Assuming the contrary, i.e., $T < \infty$, one uses the local existence of the solution to (1) taking as initial value $u(T)$ and as the interval of the existence of the solution $[T, T+h]$, where $h > 0$ is a sufficiently small number. Then inequality (7) holds for $t \in [0, T+h]$. This contradicts to the definition (14) of T. So, one gets a contradiction which proves that $T = \infty$ and estimate (7) holds for all $t \in \mathbb{R}_+$. Theorem 1 is proved for $X = \mathbb{R}$. □

2.3. Consider the nonlinear Volterra equation:

$$u(t) = \int_0^t a(t, s, u(s))ds + f(t). \tag{15}$$

Assume that $a(t, s, u)$ and $a_t := \frac{\partial a}{\partial t}$ are continuous functions on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, locally Lipschitz with respect to u. Differentiate (15) with respect to t and get

$$u' = a(t, t, u(t)) + \int_0^t a_t(t, s, u(s))ds + f'(t) := A_1(t, u(t)). \tag{16}$$

Assume that $A_1(t, u)$ satisfies conditions (4)–(6) with $w = 1$, and $\|u(t)\| = |u(t)|$.Then the argument used in scetion **2.2.** proves Theorem 1 with $A_1(t, u)$ replacing $A(t, u)$.

Example 1. The aim of this example is to derive sufficient conditions on $a(t, s, u)$ for the assumptions (4)–(6) to hold. Let

$$\begin{aligned} |a(t, s, u)| + |a_t(t, s, u)| \leq ce^{-b(t+s)}(1 + |u|^{2m}), \quad m > 1, \\ |f(t)| + |f'(t)| \leq ce^{-bt}, \end{aligned} \tag{17}$$

where $c, b > 0$ are constants. We assume that a and a_t are Lipschitz functions with respect to u. Assume that

$$|a(t, t, |u|)| \leq |a(t, t, |v|)| \quad if \quad |v| \geq |u|, \tag{18}$$

$$|a_t(t, t, |u|)| \leq |a_t(t, t, |v|)| \quad if \quad |v| \geq |u|. \tag{19}$$

Let

$$\mu(t) = c_0 e^{-at}, \quad a > 0. \tag{20}$$

Note that $\left(\frac{1}{\mu(t)}\right)' = ac_0^{-1}e^{at}$. If (17) holds, then the following two inequalities

$$|f'(t)| + |a(t,t,c_0^{-1}e^{at})| \le ce^{-bt} + ce^{-2bt}(1 + c_0^{-2m}e^{2mat}) \le 0.5ac_0^{-1}e^{at} = 0.5\left(\frac{1}{\mu(t)}\right)', \quad (21)$$

$$\int_0^t |a_t(t,s,c_0^{-1}e^{as})|ds \le \int_0^t ce^{-b(t+s)}(1 + e^{2mas}/c_0^{2m})ds \le ce^{-bt}[(1-e^{-bt})/b + (1-e^{-(b-2ma)t})/[c_0^{2m}(b-2ma)]. \quad (22)$$

and conditions (4)–(5) hold provided that

$$c/b + 1/[c_0^{2m}(b-2ma)] \le a/(2c_0), \quad b > 2ma, \quad (23)$$

where b is sufficiently large and c is sufficiently small. If in addition (6) holds, i.e., $cc_0 < 1$, then $u(t)$ exists globally and the estimate $|u(t)| < c_0^{-1}e^{at} \quad \forall t \in \mathbb{R}_+$ holds. □

2.4. Consider equation (1) in X. Assume that conditions (2), (4)– (6) and (8) hold. Then there is a unique local solution to (1) continuous with respect to t in X. It follows from (4)-(6) that

$$\|u(t)\|' \le \|A(t,u(t)\| \le \|A(t,w/\mu(t))\| < (1/\mu(t))', \quad 0 \le t \le \delta. \quad (24)$$

Here $\delta > 0$ is sufficiently small so that $\|u(t)\| < 1/\mu(t)$ for $0 \le t \le \delta$. Integrate (22) on any interval $[0,T]$ on which the solution $u(t)$ exists one gets $\|u(t)\| < 1/\mu(t)$ for $t \in [0,T]$. As in section **2.3** we prove that $T = \infty$. Therefore problem (1) has a unique global solution in X and estimate (7) holds.

Theorem 1 is proved. □

The ideas close to the ones used in this paper were developed and used in [1]–[3].

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