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On Group Von Neumann Algebras with Vector-Valued Functions

Essé Julien ATTO* and V.S. Kofi ASSIAMOUA**

Department of Mathematics - FaST, University of Kara, BP 404 Kara-Togo*

Department of Mathematics - FDS, University of Lomé, BP 1515 Lomé,-Togo**

E-mail : attoej@yahoo.fr

Abstract

Let G be a locally compact group equipped with a normalized Haar measure μ, $A(G)$ the Fourier algebra of G and $VN(G)$ the von Neumann algebra generated by the left regular representation λ of G. In this paper, we introduce the space $VN(G,\mathcal{A})$ associated with the Fourier algebra $A(G,\mathcal{A})$ for vector-valued functions on G, where $\mathcal{A}$ is a H^*-algebra. Some basic properties are discussed in the category of Banach space, and also in the category of operator space.

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1. Introduction

The theory of rings of operators called today *von Neumann algebra* was first introduced and developed by Murray and von Neumann in 1936 [10], with the aim of developing a suitable mathematical framework for quantum mechanics. Today, it extends into the larger theory of noncommutative geometry and intervenes in various fields such as the theory of representations and the L_2-invariants theory.

In mathematics, one can assign to a locally compact group G an operator algebra such that representations of the algebra are related to representations of the group. Any space constructed in this way is called group algebra.

Let $L_p(G)(1 \leq p < \infty)$ be the set of all functions $f: G \to \mathbb{C}$, such that $\int_G |f(x)|^p < \infty$, and $C(G)$ the set of all continuous complex-valued functions on G. These spaces form Banach algebras under usual operations and convolution. In [13], D. Z. Spicer extended the group algebras $L_p(G)$and $C(G)$ to group algebras of vector-valued functions respectively denoted $B_p(G,A)$and $C(G,A)$. Mainly, $B_p(G,A)$ is the space of all continuous functions $f: G \to A$ such that $\int_G \|f(x)\|_A^p dx < \infty$ (usually denoted $L_p(\mathrm{G},\mathrm{A})$)), and $C(G,A)$ is the space of all continuous functions from G to A, where A is a Banach algebra.

As far as we know, the space $VN(G)$ is associated with the space of complex-valued continuous functions on G with compact support and there is no analog for vector-valued functions yet. In this paper, we want to extend this definition in the case of Banach algebra-valued functions with additional conditions.

Section 2 deals with preliminaries.

In Section 3, we introduce a vector-valued analog of the group C^*-algebra $C^*(G)$ and the reduced group C^*-algebra $C_r^*(G)$ which will be denoted respectively by $C^*(\mathrm{G},\mathcal{A})$ and $C_r^*(\mathrm{G},\mathcal{A})$ where $\mathcal{A}$ is assumed to be an H^*-algebra.

Now, we deal with one of the main results of our paper in Section 4: the generalization of the space $V(G)$ in the case of vector-valued functions. The vector-valued von Neumann algebra $VN(G,\mathcal{A})$ is the weak operator topology closure of $C^*(\mathrm{G},\mathcal{A})$. We discuss some basic properties of this space.

Finally, in Section 5, the spaces $VN(G)$ (resp. $\mathrm{A}(G)$) and $VN(G,\mathcal{A})$ (resp. $\mathrm{A}(\mathrm{G},\mathcal{A})$)) are equipped with their natural operator space structure. We then study some properties of isomorphisms and isometries in the category of operator spaces. A characterization of completely bounded multiplier on a specific dense subspace of by $A(G,\mathcal{A})$ is established.

2. Preliminaries

In this section, we recall some notations and results related to locally compact groups and operator spaces.The reader is referred to P. Eymard [6], Effros and Ruan [5] for more details. Through this paper, we shall assume that G is a locally compact Hausdorff topological group endowed with its left Haar measure μ normalized so that $\mu(G) = 1$.

Let $B(G)$ be the Fourier-Stieltjes algebra of G, then the Fourier algebra $A(G)$ is defined as the Banach subalgebra of $B(G)$ generated by the continuous functions of positive type with compact support. $A(G)$ is identified with the space

$$\{f * \tilde{g} : f, g \in L_2(G)\}(\text{see Eymard [6]})$$

where $f * g(s) = \int_G f(st^{-1})g(t)dt$ is the convolution product and $\tilde{f}: t \mapsto \overline{f(t^{-1})}$.

$A(G)$ is equipped with the norm

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$$\|u\|_{A(G)} = \inf_{u=f*g} \|f\|_{L_2(G)} \|g\|_{L_2(G)}$$

and is known to be a subalgebra of $C_0(G)$ (the space of decreasing continuous functions on G, vanishing at infinity), so a commutative Banach algebra with respect to the pointwise multiplication.

Let $C_c(G)$ be the space of complex-valued continuous functions on G with compact support; this acts on $L_2(G)$ by left convolution, and forms a $*$-subalgebra of $Hom(L_2(G))$: $\{\Lambda_f : L_2(G) \ni g \longmapsto f * g \in L_2(G), f \in C_c(G)\}$, , which closure is $C_r^*(G)$, *the reduced group* C^**-algebra*. *The group* C^**-algebra*$C^*(G)$ is obtained by taking the supremum over all C^*-norms. The weak operator topology closure of $C_r^*(G)$ is called *the group von Neumann algebra* of G, denoted $VN(G)$. Equivalently, if $\mathcal{B}(L_2(G))$ denotes the space of all bounded linear maps on $L_2(G)$, we have:

$$VN(G) = \{\lambda(s): s \in G\}$$

where $\lambda: G \to (\mathcal{B}L_2(G))$ is the left regular representation of G.

$$\begin{array}{rcl} \lambda(s): L_2(G) & \longrightarrow & L_2(G) \\ f & \longmapsto & g, \quad g(t) = f(s^{-1}t). \end{array}$$

$A(G)$ is the predual of $VN(G)$.

An *operator space* is a closed subspace of the space $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H. In other words, it is a Banach space given together with an isometric linear embedding into the space $\mathcal{B}(H)$. An abstract characterization of operator spaces was given by Ruan in [5]. A complex vector space E is an operator space if and only if for each integer $n \geq 1$, there is a complete norm $\|.\|_n$ on $M_n(E)$, the space of $n \times n$ matrices with entries in E, such that the following properties are satisfied:

$$\forall u \in M_n(E), v \in M_m(E), \alpha, \beta \in M_n,$$

(i) $\|u \oplus v\|_{n+m} = max\{\|u\|_n, \|v\|_m\}$,

(ii) $\|\alpha u \beta\|_n \leq \alpha \|u\|_n \beta$.

A linear map $\phi: E_1 \subset B(H_1) \longrightarrow E_2 \subset B(H_2)$ between two operator spaces is said to be *completely bounded* (c.b. in short) if the linear maps

$$\begin{array}{rcl} \phi_n: M_n(E_1) & \longrightarrow & M_n(E_2) \\ (a_{ij})_{1\leq i,j\leq n} & \longmapsto & (\phi(a_{ij}))_{1\leq i,j\leq n} \end{array}$$

are such that $\sup_{n\geq 1} \|\phi_n\| < \infty$. The completely bounded norm is denoted by $\|\phi\|_{cb} = \sup_{n\geq 1} \|\phi_n\|$. The space of all completely bounded maps from E_1 into E_2 is denoted $cb(E_1, E_2)$ and simply $cb(E_1)$ if $E_1 = E_2$.

We give the following definition about H^**-algebras* as introduced by Ambrose in [1]:

An involutive Banach algebra $\mathcal{A}$ over $\mathbb{C}$ with involution

$$\begin{array}{rcl} * : \mathcal{A} & \rightarrow & \mathcal{A} \\ x & \mapsto & x^* \end{array}$$

is called an H^**-algebra* if $\mathcal{A}$ admits an inner product $(\cdot,\cdot)$ satisfying the following postulates:

(i) The underlying Banach space of $\mathcal{A}$ is a Hilbert space (of arbitrary dimension);

(ii) For each $x \in \mathcal{A}$, there is an element in $\mathcal{A}$ denoted by x^* and called an *adjoint* of x, such that for all $y, z \in \mathcal{A}$, we have both $(xy, z) = (y, x^*z)$and $(yx, z) = (y, zx^*)$.

3. The generalized group C^*-algebras $C_r^*(G, \mathcal{A})$ and $C^*(G, \mathcal{A})$

In the sequel, $\mathcal{A}$ will denote an H^*-algebra and G a compact topological group with Haar measure μ , normalized so that $\mu(G) = 1$. For $1 \leq p < \infty$, $L_p(G, \mathcal{A})$ is the space of all equivalence classes (modulo null functions) of all measurable functions $f: G \to \mathcal{A}$ such that $\int_G \|f(x)\|_{\mathcal{A}}^p d\mu(x) < \infty$, and $C_c(G, \mathcal{A})$ will denote the space of all continuous functions from G to $\mathcal{A}$ with compactly support. The space $L_p(G, \mathcal{A})$ (resp. $C_c(G, \mathcal{A})$) equipped with the norm $\|f\|_p = \|f(x)\|_{\mathcal{A}}^p d\mu(x)$(resp. $\|f\|_\infty = \sup_{x \in G} \|f(x)\|_{\mathcal{A}}$) is a Banach space.

The Fourier algebra $A(G, \mathcal{A})$ on G associated with functions $f: G \to \mathcal{A}$ is defined as the usual one:

$$A(G, \mathcal{A}) \coloneqq \{f * \tilde{g} : f, g \in L_2(G, \mathcal{A})\},$$

where $\tilde{f}(t) = \left(f(t^{-1})\right)^{*_{\mathcal{A}}}$ and $*_{\mathcal{A}}$ is the involution in $\mathcal{A}$. Equipped with the norm

$$\|u\|_{A(G,\mathcal{A})} := \inf_{u = f * \tilde{g}} \{\|f\|_2 \|g\|_2 : f, g \in L_2(G, \mathcal{A})\},$$

it becomes a Banach space.

If we set $f_\vee(t) = f(t^{-1})$, then $\tilde{f}(t) = \left(f_\vee(t)\right)^{*_{\mathcal{A}}}$.

The completion of $C_c(G, \mathcal{A})$ in the $L_1(G, \mathcal{A})$-norm is isomorphic to the space $L_1(G, \mathcal{A})$.

In this section we will generalize the group algebras $C_r^*(G)$ and $C^*(G)$ of complex-valued functions to those of vector-valued functions denoted $C_r^*(G, \mathcal{A})$ and $C^*(G, \mathcal{A})$, then we will study some of their properties. Recall that since $\mathcal{A}$ is an H^*-algebra, so is $L_2(G, \mathcal{A})$. Set $\langle , \rangle_{L_2}$(resp.$\langle , \rangle_{\mathcal{A}}$) the inner product associated with $L_2(G, \mathcal{A})$ (resp. with $\mathcal{A}$) as a Hilbert space. We have:

$$\langle g, h \rangle_{L_2} = \int_G \langle g(x), h(x) \rangle_{\mathcal{A}} dx$$

Proposition 3.*1* *Let G be a locally compact group and $\mathcal{A}$ be an H^*-algebra.*

(i) The space $C_c(G, \mathcal{A})$ acts boundedly on on $L_2(G, \mathcal{A})$ by left convolution.

(ii) The space $\mathcal{T}(G, \mathcal{A}) = \{\Lambda_f : L_2(G, \mathcal{A}) \longrightarrow L_2(G, \mathcal{A}), f \in C_c(G, \mathcal{A})\}$ of operators such that

$$\Lambda_f(g) = f * g, \forall g \in L_2(G, \mathcal{A})$$

is a $*$-subalgebra of $\mathcal{B}(L_2(G, \mathcal{A}))$.

Proof.

(i) For all $f \in C_c(G, \mathcal{A}), g \in L_2(G, \mathcal{A})$, we have:

$$\left(\int_G \|(f * g)(x)\|_{\mathcal{A}}^2 dx\right)^{1/2} = \left(\int_G \left\|\int_G f(y) g(y^{-1}x) dy\right\|_{\mathcal{A}}^2 dx\right)^{1/2}$$

$$\leq \left(\int_G \left(\int_G \|f(y)g(y^{-1}x)\|_{\mathcal{A}} dy\right)^2 dx\right)^{1/2}$$

$$\leq \int_G \|f(y)\|_{\mathcal{A}} \left(\int_G \|g(y^{-1}x)\|_{\mathcal{A}}^2 dx\right)^{1/2} dy \quad \text{(by Minkowski)}$$

$$= \int_G \|f(y)\|_{\mathcal{A}} \left(\int_G \|g(x)\|_{\mathcal{A}}^2 dx\right)^{1/2} dy$$

$$\leq \|f\|_\infty \|g\|_2$$

Thus, $f * g \in L_2(G, \mathcal{A})$ and $\exists C > 0, \|f * g\|_2 \leq C\|g\|_2$.

(ii) From (i), it is clear that for each $f \in C_c(G, \mathcal{A}), \Lambda_f \in \mathcal{B}(L_2(G, \mathcal{A}))$.

-*Step 1: $C_c(G, \mathcal{A})$ is a $*$-algebra*

It is easy to check that, $C_c(G, \mathcal{A})$endowed with the convolution product is an algebra. Set $*_{\mathcal{A}}$ the involution in $\mathcal{A}$, then the mapping $\tilde{}: f \mapsto \tilde{f}$ such that $\tilde{f}(s) = \left(f(s^{-1})\right)^{*_{\mathcal{A}}}$ is an involution of $C_c(G, \mathcal{A})$. In fact,$\forall \lambda \in \mathbb{C}$, $\forall f, g \in C_C(G, \mathcal{A}), \forall x \in G$,

$$\begin{aligned}
\widetilde{f * g}(x) &= \left((f * g)(x^{-1})\right)^{*_{\mathcal{A}}} \\
&= \left(\int_G f(y)g(y^{-1}x^{-1})\, dy\right)^{*_{\mathcal{A}}} \\
&= \int_G \left(f(y)g(y^{-1}x^{-1})\right)^{*_{\mathcal{A}}} dy \\
&= \int_G \left(g(y^{-1}x^{-1})\right)^{*_{\mathcal{A}}} \left(f(y)\right)^{*_{\mathcal{A}}} dy \\
&= \int_G \left(\tilde{g}(xy)\right)\left(\tilde{f}(y^{-1})\right) dy \\
&= \int_G \left(\tilde{g}(z)\right)\left(\tilde{f}(z^{-1}x)\right) dz \\
&= \tilde{g} * \tilde{f}(x) \\
&\Longrightarrow (\widetilde{f * g}) = \tilde{g} * \tilde{f}.
\end{aligned}$$

Trivially,

$$(\widetilde{\lambda f + g}) = \bar{\lambda}\tilde{f} + \tilde{g}, \qquad (\tilde{\tilde{f}}) = f.$$

-*Step 2: The space $\mathcal{B}(L_2(G, \mathcal{A}))$ is a $*$-algebra*

Like $C_c(G, \mathcal{A})$, the space $L_2(G, \mathcal{A})$ is a $*$-algebra under the convolution product and the involution denoted by $\tilde{}$. Moreover $\mathcal{B}(L_2(G, \mathcal{A}))$ is a -algebra if endowed with:

-the inner product $T_1 \circ T_2: f \mapsto T_1(T_2 f)$,

-and the involution $^{\star}: T \mapsto T^{\star}$ such that $\langle T^{\star}g, h\rangle_{L_2} = \langle g, Th\rangle_{L_2}$, for all $g, h \in L_2(G, \mathcal{A})$.

-*Step 3: The space $\mathcal{T}(G, \mathcal{A})$ is a $*$-subalgebra of $\mathcal{B}(L_2(G, \mathcal{A}))$*

$$(\lambda\Lambda_{f_1}+\Lambda_{f_2})(g)=\lambda f_1 * g+f_2 * g$$

$$=(\lambda f_1+f_2) * g$$

$$=\Lambda_{\lambda f_1+f_2}(g)$$

$$\Longrightarrow \lambda\Lambda_{f_1}+\Lambda_{f_2}=\Lambda_{\lambda f_1+f_2}\in\mathcal{T}(G,\mathcal{A}).$$

$$\left\langle(\Lambda_f)^{\star}(g),h\right\rangle_{L_2(G,\mathcal{A})}=\left\langle g,\Lambda_f h\right\rangle_{L_2(G,\mathcal{A})}$$

$$=\langle g,f * h\rangle_{L_2(G,\mathcal{A})}$$

$$=\left\langle\tilde{f} * g,h\right\rangle_{L_2(G,\mathcal{A})}$$

$$=\left\langle\Lambda_{\tilde{f}}(g),h\right\rangle_{L_2(G,\mathcal{A})}$$

$$\left\langle(\Lambda_f)^{\star}\ (g),h\right\rangle_{L_2(G,\mathcal{A})}=\left\langle\Lambda_{\tilde{f}}(g),h\right\rangle_{L_2(G,\mathcal{A})}\Longrightarrow(\Lambda_f)^{\star}=\Lambda_{\tilde{f}}\in\mathcal{T}(G,\mathcal{A}).$$

$$(\lambda\Lambda_{f_1}+\Lambda_{f_2})^{\star}=\Lambda_{(\widetilde{\lambda f_1+f_2})}$$

$$=\Lambda_{\bar{\lambda}\widetilde{f_1}+\widetilde{f_2}}$$

$$=\bar{\lambda}\Lambda_{\widetilde{f_1}}+\Lambda_{\widetilde{f_2}}$$

$$\Longrightarrow(\lambda\Lambda_{f_1}+\Lambda_{f_2})^{\star}=\bar{\lambda}(\Lambda_{f_1})^{\star}+(\Lambda_{f_2})^{\star}.$$

Since,

$$\begin{aligned}(\Lambda_{f_1}\circ\Lambda_{f_2})g &= \Lambda_{f_1}(\Lambda_{f_2}g)\\ &= f_1 * (f_2 * g)\\ &= (f_1 * f_2) * g\\ &= \Lambda_{f_1 * f_2}g\end{aligned}$$

$$\Longrightarrow \Lambda_{f_1}\circ\Lambda_{f_2}=\Lambda_{f_1 * f_2}\,,$$

,

then,

$$(\Lambda_{f_1}\circ\Lambda_{f_2})^{\star}=\Lambda_{(\widetilde{f_1 * f_2})}$$

$$=\Lambda_{\widetilde{f_2} * \widetilde{f_1}}$$

$$=\Lambda_{\widetilde{f_2}}\circ\Lambda_{\widetilde{f_1}}$$

.

The rest of the proof is obvious. ■

Remark 3.2 The *previous proposition is always true, if* $C_c(G,\mathcal{A})$ *is replaced by* $L_1(G,\mathcal{A})$.

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Corollary 3.3 *If $\mathcal{A}$ is an H^*-algebra endowed with its natural operator space structure, then $\forall f \in L_1(G,\mathcal{A})$, Λ_f is completely bounded. More precisely, $\mathcal{T}(G,A) \subset cb\big(L_1(G,\mathcal{A})\big)$.*

Proof. For $n \in \mathbb{N}^*$, consider the mapping

$$\begin{array}{rcl} \Lambda_f^{(n)}: M_n(L_2(G,\mathcal{A})) & \longrightarrow & M_n(L_2(G,\mathcal{A})) \\ \left(g_{ij}\right)_{1\le i,j\le n} & \longmapsto & \left(\Lambda_f g_{ij}\right)_{1\le i,j\le n} \end{array}$$

$$\begin{array}{rcl} \|\Lambda_f^{(n)}\| & = & \sup\left\{\left\|\Lambda_f^{(n)}\left(\left(g_{ij}\right)_{1\le i,j\le n}\right)\right\|_{M_n(L_2(G,\mathcal{A}))} : \left\|\left(g_{ij}\right)_{1\le i,j\le n}\right\|_{M_n(L_2(G\mathcal{A}))} \le 1\right\} \\ & = & \sup\left\{\left\|\left(\Lambda_f g_{ij}\right)_{1\le i,j\le n}\right\|_{M_n(L_2(G,\mathcal{A}))} : \sup_{1\le i,j\le n}\left\|g_{ij}\right\|_{L_2(G,\mathcal{A})} \le 1\right\} \\ & = & \sup_{1\le i,j\le n}\left\{\left\|\Lambda_f g_{ij}\right\|_2 : \|g_{ij}\|_2 \le 1\right\} \end{array}$$

$$\begin{array}{rcl} \|\Lambda_f^{(n)}\| & = & \sup_{1\le i,j\le n}\left\{\left\|f * g_{ij}\right\|_2 : \|g_{ij}\|_2 \le 1\right\} \\ & \le & \sup_{1\le i,j\le n}\left\{\|f\|_1 \left\|g_{ij}\right\|_2 : \|g_{ij}\|_2 \le 1\right\} \\ & \le & \|f\|_1 \end{array}$$

So $\sup_{n\ge 1}\|\Lambda_f^{(n)}\| \le \|f\|_1 < \infty$,

And Λ_f is completely bounded. ■

Definition 3.4 *Assume $\mathcal{A}$ is an H^*-algebra.*

For a locally compact group G, we denote by $C^*(G,\mathcal{A})$ the (vector-valued) C^*-algebra of G, which is G the C^*-envelopping algebra of $L_1(G,\mathcal{A})$, i.e. the completion of $C_c(G,\mathcal{A})$ with respect to the largest C^*-norm

$$\| f \|_{*\infty} = \sup_{\pi}\| \pi(f) \|,$$

where π ranges over all non-degenerates $*$-representations of $C_c(G,\mathcal{A})$ on Hilbert spaces.

Definition 3.5 Let *G be a locally compact group and $\mathcal{A}$ an H^*-algebra.*

The (vector-valued) reduced group C^*-algebra $C_r^*(G,\mathcal{A})$ is the completion of $C_c(G,\mathcal{A})$ with respect to the norm

$$\sup_{g\in L_2(G,A)}\left\{\|f * g\|_{L_2(G,\mathcal{A})} : \|g\|_2 \le 1\right\}.$$

Proposition 3.6 *The space $C_c(G,\mathcal{A})$ is isometrically isomorphic to the space $\mathcal{T}(G,\mathcal{A})$.*

Proof. The operators Λ_f determine the bijective linear map

$$\begin{array}{rcl} \Lambda: C_c(G,\mathcal{A}) & \longrightarrow & \mathcal{T}(G,\mathcal{A}) \subset \mathcal{B}\big(L_2(G,\mathcal{A})\big) \\ f & \longmapsto & \Lambda_f. \end{array}$$

Moreover, for any $f \in C_c(G,\mathcal{A})$,

$$\begin{aligned}\| \Lambda(f) \|_{\mathcal{B}(L_2(G,\mathcal{A}))} &= \|\Lambda_f\|_{\mathcal{B}(L_2(G,\mathcal{A}))} \\ &= \sup_{g\in L_2(G,\mathcal{A})} \left\{\|\Lambda_f g\|_{L_2(G,\mathcal{A})} : \|g\|_2 \leq 1\right\} \\ &= \sup_{g\in L_2(G,\mathcal{A})} \left\{\|f * g\|_{L_2(G,\mathcal{A})} : \| g \|_2 \leq 1\right\} \\ &= \|f\|_{C_r^*},\end{aligned}$$

which completes the proof. ■

Corollary 3.7 *Assume $\mathcal{A}$ is an H^*-algebra and G a locally compact group. The norm*

$$\|f\|_{C_r^*} := \sup_{g\in L_2(G,\mathcal{A})} \{\|f * g\|_2 : \|g\|_2 \leq 1\}$$

is a C^*-norm on $C_r^*(G,\mathcal{A})$.

Proof. We already know that $C_c(G,\mathcal{A})$ is a $*$-algebra, and $\left(C_r^*(G,\mathcal{A}), \|\cdot\|_{C_r^*}\right)$ is a Banach space. Moreover, using **Proposition 3.6** we have :

(i) *-Submultiplicative property:*

$$\begin{aligned}\|f_1 * f_2\|_{C_r^*} &= \|\Lambda_{f_1} \circ \Lambda_{f_2}\|_{\mathcal{B}(L_2(G,\mathcal{A}))} \\ &\leq \|\Lambda_{f_1}\|_{\mathcal{B}(L_2(G,\mathcal{A}))} \|\Lambda_{f_2}\|_{\mathcal{B}(L_2(G,\mathcal{A}))} = \|f_1\|_{C_r^*} \|f_2\|_{C_r^*}\end{aligned}$$

(ii) -$\|\cdot\|_{C_r^*}$ is a normed algebra:

$$\|\tilde{f}\|_{C_r^*} = \left\|(\Lambda_f)^*\right\|_{\mathcal{B}(L_2(G,\mathcal{A}))} = \|\Lambda_f\|_{\mathcal{B}(L_2(G,\mathcal{A}))} = \| f \|_{C_r^*},$$

(iii) *-The C^*-property:*

$$\|\tilde{f} * f\|_{C_r^*} = \left\|(\Lambda_f)^* \circ \Lambda_f\right\| = \|\Lambda_f\|^2 = \|f\|^2 . \ ■$$

Following **Proposition 3.6**, the reduced group C^*-algebra $C_r^*(G,\mathcal{A})$ can be defined equivalently as follows:

Definition 3.8 (*Definition 3.5 bis*)

Let G be a locally compact group and $\mathcal{A}$ an H^*-algebra. The (vector-valued) reduced group C^*-algebra $C_r^*(G,\mathcal{A})$ is the closure of the space $\mathcal{T}(G,\mathcal{A})$, with respect to the operator norm on $\mathcal{B}\left(L_2(G,\mathcal{A})\right)$.

Remark 3.9 *In the second definition, $C_r^*(G,\mathcal{A})$ is indeed a C^*-algebra. In fact, $C_r^*(G,\mathcal{A})$ is a closed self-adjoint subalgebra of the C^*-algebra $\mathcal{B}\left(L_2(G,\mathcal{A})\right)$ with respect to the C^*-norm of $\mathcal{B}\left(L_2(G,\mathcal{A})\right)$ (the operator norm). By following this definition, one can conclude that $C_r^*(G,\mathcal{A})$ is the C^*-algebra generated by the image of the left regular representation of $C_c(G,\mathcal{A})$ on $L_2(G,\mathcal{A})$.*

4. The generalized group von Neumann algebra $VN(G,\mathcal{A})$

Definition 4.1 *The (vector-valued) group von Neumann algebra* $VN(G,\mathcal{A})$ *of* G *is the enveloping von Neumann algebra of* $C^*(G,\mathcal{A})$*, i.e. the weak operator topology closure of* $C_r^*(G,\mathcal{A})$.

Remark 4.2 -*Considering the previous assertions in Remark* **3.9** *about the reduced* C^**-algebra, one can also define the (vector-valued) group von Neumann algebra as follows:*

$$VN(G,\mathcal{A})=\{\lambda(s): s\in G\}$$

where $\lambda: G\to\mathcal{B}\big(L_2(G,\mathcal{A})\big)$ *is the left regular representation of* G.

$$\begin{array}{rcll}\lambda(s): L_2(G,\mathcal{A}) & \longrightarrow & & L_2(G,\mathcal{A})\\ f & \longmapsto & g, & g(t)=f(s^{-1}t)\end{array}$$

-*Naturally,* $A(G,\mathbb{C})=A(G)$ *and* $VN(G,\mathbb{C})=VN(G)$.

Proposition 4.3 *The Fourier algebra* $A(G,\mathcal{A})$ *is isometrically isomorphic to the predual of the group von Neumann algebra* $VN(G,\mathcal{A})$.

Proof. Consider the mapping

$$\begin{array}{rcl}\phi: VN(G,\mathcal{A}) & \longrightarrow & \big(A(G,\mathcal{A})\big)^*\\ v & \longmapsto & \phi(v)\end{array}$$

such that if $u=f*\tilde{g}\in A(G,\mathcal{A})$, then $\phi(v)(u)=\langle\Lambda_v f,g\rangle=\int_G\langle(\Lambda_v f)(x),g(x)\rangle_{\mathcal{A}}\,\mathrm{d}\mu(\mathrm{x})$.

We know that $C_r^*(G,\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}\big(L_2(G,\mathcal{A})\big)$ with strong closure $VN(G,\mathcal{A})$, so the closed unit ball of $C_r^*(G,\mathcal{A})$ is strongly dense in the unit ball of $VN(G,\mathcal{A})$ (Kaplansky theorem of density). Thus, there exists a sequence (w_n) in $C_c(G,\mathcal{A})$ such that $\|w_n\|_{C_r^*}\le\|v\|_{C_r^*}$ and $\big(\Lambda_{w_n}\big)\underset{strongly}{\longrightarrow}\Lambda_v$. Moreover,

$$\begin{aligned}|\phi(v)(u)| &= \lim_n\big|\langle\Lambda_{w_n}f,g\rangle\big|\\ &= \lim_n\left|\int_G\langle\Lambda_{w_n}f(x),g(x)\rangle_{\mathcal{A}}d\mu(x)\right|\\ &= \lim_n\left|\int_G\langle w_n*f(x),g(x)\rangle_{\mathcal{A}}d\mu(x)\right|\\ &= \lim_n\left|\int_G\left\langle\int_G w_n(y)f(y^{-1}x)dy,g(x)\right\rangle_{\mathcal{A}}d\mu(x)\right|\\ &= \lim_n\left|\int_G\left\langle w_n(y),\int_G g(x)\big(f(y^{-1}x)\big)^{*_{\mathcal{A}}}dx\right\rangle_{\mathcal{A}}dy\right|\\ &= \lim_n\left|\int_G\left\langle w_n(y),\int_G g(yz)\big(f(z)\big)^{*_{\mathcal{A}}}dz\right\rangle_{\mathcal{A}}dy\right|\end{aligned}$$

$$= \lim_n \left| \int_G \left\langle w_n(y), \left(\int_G f(z) \big(g(yz)\big)^{*_A} dz \right)^{*_\mathcal{A}} \right\rangle_\mathcal{A} dy \right|$$
$$= \lim_n \left| \int_G \left\langle w_n(y), \left(\int_G f(z) \tilde{g}(z^{-1}y^{-1}) dz \right)^{*_\mathcal{A}} \right\rangle_\mathcal{A} dy \right|$$
$$= \lim_n \left| \int_G \langle w_n(y), \big(f * \tilde{g}(y^{-1})\big)^{*_\mathcal{A}} \rangle_\mathcal{A} dy \right|$$
$$= \lim_n \left| \int_G \langle w_n(y), \tilde{u}(y) \rangle_\mathcal{A} dy \right|$$
$$\leq \lim_n \|w_n\|_{C_r^*} \|u\|_{A(G,\mathcal{A})}$$
$$\leq \|v\|_{C_r^*} \|u\|_{A(G,\mathcal{A})}$$
$$\Longrightarrow \|\phi(v)\| \leq \|v\|$$

Moreover, we have

$$\| v \|_{C_r^*} = \sup_{f \in L_2(G,\mathcal{A})} \{\|h * f\|_2 : \|f\|_2 \leq 1\}$$
$$= \sup_{f,g \in L_2(G,\mathcal{A})} \{|\langle h * f, g\rangle_{L_2}| : \|f\|_2 \leq 1, \|f\|_2 \leq 1\}$$
$$\geq \sup_{f,g \in L_2(G,\mathcal{A})} \{|\phi(v)(u)| : \|u\| \leq 1\}$$
$$\geq \|\phi(v)\|.$$

The linearity of ϕ is obvious, let us prove the injectivity. Assume $\phi(T) = 0$, then for all $f, g \in L_2(G,\mathcal{A})$,

$$\phi(T)(f * \tilde{g}) = 0 \quad \Longrightarrow \quad \int_G \langle T(y), \big(f * \tilde{g}(y)\big)^{*_\mathcal{A}} \rangle_\mathcal{A} dy = 0$$
$$\Longrightarrow \quad T(y) = 0 \ \forall y \in G$$
$$\Longrightarrow \quad T = 0$$

Conversely, assume $\varphi \in \big(A(G,\mathcal{A})\big)^*$ and let $f, g \in L_2(G,\mathcal{A})$, then

$$|\varphi(f * \tilde{g})| \leq \|\varphi\|_{(A(G,\mathcal{A}))^*} \|f * \tilde{g}\|_{A(G,\mathcal{A})} \quad \text{(since } \varphi \text{ is continuous)}$$
$$\leq \|\varphi\| \|f\|_2 \|g\|_2$$
$$\Longrightarrow \sup_{f,g \in L_2(G\mathcal{A})} \{|\varphi(f * \tilde{g})| : \|f\|_2 \leq 1, \|g\|_2 \leq 1\} \leq \|\varphi\|_{(A(G,\mathcal{A}))^*}$$

Then, there exists a linear map $\mathcal{V}_\varphi \in \mathcal{B}\big(L_2(G,\mathcal{A})\big)$ such that $\langle \mathcal{V}_\varphi f, g\rangle = \varphi(f * \tilde{g})$ and $\|\mathcal{V}_\varphi\| \leq \|\varphi\|$.

Let us prove that $\mathcal{V}_\varphi$ commutes with convolution :

$\forall f, g \in L_2(G,\mathcal{A}), \forall h \in C_c(G,\mathcal{A})$, we have

$$\langle \mathcal{V}_\varphi(f * h), g\rangle = \varphi\big((f * h) * \tilde{g}\big) = \varphi\big(f * (h * \tilde{g})\big)$$
$$= \varphi\left(f * \widetilde{(g * \tilde{h})}\right)$$
$$= \langle \mathcal{V}_\varphi f, g * \tilde{h}\rangle$$
$$= \langle (\mathcal{V}_\varphi f) * h, g\rangle,$$

which implies that $\mathcal{V}_\varphi(f*h)=(\mathcal{V}_\varphi f)*h$, and $\mathcal{V}_\varphi$ is an element of $VN(G,\mathcal{A})$. ■

Let $C^b(G)$ be the space of all bounded continuous functions from G to $\mathbb{C}$, a function $f\in C^b(G)$ such that $\forall g\in A(G), fg\in A(G)$ is said to be *a multiplier of* $A(G)$. The space of all completely bounded multipliers on $A(G)$ is denoted by $M_{cb}(A(G))$. In a similar way, we define the space of completely bounded multipliers on $A(G,\mathcal{A})$ and the space of completely bounded vector-valued multipliers on $A(G,\mathcal{A})$.

Definition 4.4 *Let G be a locally compact group, $\mathcal{A}$ an H^*-algebra and $C^b(G,\mathcal{A})$ the space of all bounded continuous functions from G to $\mathcal{A}$. Let $V_1\subset C^b(G,\mathcal{A})$ and $V_2\subset A(G,\mathcal{A})$ two vector spaces. We denote by $M_{cb}A(G,\mathcal{A})\subset C^b(G,\mathcal{A})$ (resp. $M_{cb}V_2$) the space of completely bounded multipliers on $A(G,\mathcal{A})$ (resp. on V_2), i.e. the collection of functions $f\in C^b(G,\mathcal{A})$ (resp. on V_1) such that $fg\in A(G,\mathcal{A})$ (resp. $fg\in V_2$) for each $g\in A(G,\mathcal{A})$ (resp. for each $g\in V_2$) and the operator*

$$\begin{array}{rcl} M_f\colon A(G,\mathcal{A}) & \longrightarrow & A(G,\mathcal{A})(resp. V_2\to V_2) \\ g & \longmapsto & fg, \end{array}$$

is completely bounded,

where

$$\begin{array}{rcl} fg\colon G & \longrightarrow & \mathcal{A} \\ t & \longmapsto & \underbrace{f(t)}_{\in\mathcal{A}}\underbrace{g(t)}_{\in\mathcal{A}} \end{array}$$

Remark 4.5 *We denote by $MA(G,\mathcal{A})$ the space of all multipliers of $A(G,\mathcal{A})$. Let λ be the left regular representation of $C_c(G,\mathcal{A})$ on $L_2(G,\mathcal{A})$. As in the case of multipliers of $A(G)$ (cf [9], Introduction), each $f\in MA(G,\mathcal{A})$ generates an operator M_f on $A(G,\mathcal{A})$ whose transpose defines a σ-weakly continuous operator M_f on $VN(G,\mathcal{A})$ such that $M_f\lambda(s)=f(s)\lambda(s)$, for $s\in VN(G,\mathcal{A})$,*

Definition 4.6 We define $A^0(G,\mathcal{A})$ as the following vector space.

$$A^0(G,\mathcal{A})=\left\{\sum_{j=1}^{n}a_jg_j\colon a_j\in\mathcal{A}, g_j\in A(G), n\in\mathbb{N}^*\right\}.$$

We also define the vector space $C^{b_0}(G,\mathcal{A})$ as follows.

$$C^{b_0}(G,\mathcal{A})=\left\{\sum_{j=1}^{n}a_jg_j\colon a_j\in\mathcal{A}, g_j\in C^b(G), n\in\mathbb{N}^*\right\}.$$

Theorem 4.7 *Let G be a locally compact group and let $\mathcal{A}$ be a unital and commutative H^*-algebra. The following assertions hold:*

(i) $A^0(G,\mathcal{A})\subset A(G,\mathcal{A})$, $A^0(G,\mathcal{A})$ is dense in $A(G,\mathcal{A})$.

(ii) $C^{b_0}(G,\mathcal{A})\subset C^b(G,\mathcal{A})$, $C^{b_0}(G,\mathcal{A})$ is dense in $C^b(G,\mathcal{A})$.

Proof.

(i) Let $a\in\mathcal{A}$ and $f\in A(G)$. There exists $f_1,f_2\in L_2(G)$ such that $f=f_1*\tilde{f_2}$. Consider the function

$$\begin{array}{rcl} af\colon G & \to & L_2(G,\mathcal{A}) \\ t & \mapsto & a(f(t)). \end{array}$$

We have, $af = af_1 * \widetilde{f_2} = (af_1) * \left(1_{\mathcal{A}}\widetilde{f_2}\right) = \underbrace{(af_1)}_{\in L_2(G,\mathcal{A})} * \underbrace{(\widetilde{1_{\mathcal{A}} f_2})}_{\in L_2(G,\mathcal{A})}$ which implies that $af \in A(G,\mathcal{A})$, and finally $A^0(G,\mathcal{A}) \subset A(G,\mathcal{A})$.

Let $f = g * h \in A(G,\mathcal{A})$, with $g,h \in L_2(G,\mathcal{A})$

Set

$$L_2^0(G,\mathcal{A}) = \left\{\sum_{j=1}^{n} a_j g_j : a_j \in \mathcal{A}, g_j \in L_2(G), n \in \mathbb{N}^*\right\},$$

, It is know that $L_2^0(G,\mathcal{A})$ is dense in $L_2(G,\mathcal{A})$, then for all $\varepsilon > 0$, there exist $g_\varepsilon, h_\varepsilon \in L_2^0(G,\mathcal{A})$ such that

$$\|g - g_\varepsilon\|_2 \leq \frac{\varepsilon}{2(1+M_\varepsilon)} \quad \text{and} \quad \|h - h_\varepsilon\|_2 \leq \frac{\varepsilon}{2(1+M_\varepsilon)},$$

Where $M_\varepsilon = \sup\{\|g_\varepsilon\|_2; \|h_\varepsilon\|_2\}$.

Moreover, there exist $n,m \in \mathbb{N}^*, a_{\varepsilon,1}, a_{\varepsilon,2}, \dots, a_{\varepsilon,n}, b_{\varepsilon,1}, b_{\varepsilon,2}, \dots, b_{\varepsilon,m} \in \mathcal{A}$ and

$g_{\varepsilon,1}, g_{\varepsilon,2}, \dots, g_{\varepsilon,n}, h_{\varepsilon,1}, h_{\varepsilon,2} \dots, h_{\varepsilon,m} \in L_2(G,\mathcal{A})$ such that

$$\begin{aligned} g_\varepsilon * \tilde{h}_\varepsilon &= \left(\sum_{i=1}^{n} a_{\varepsilon,i} g_{\varepsilon,i}\right) * \left(\sum_{i=1}^{m} b_{\varepsilon,j} \tilde{h}_{\varepsilon,j}\right) \\ &= \sum_{\mathrm{i}=1}^{\mathrm{n}} \sum_{\mathrm{j}=1}^{\mathrm{m}} a_{\varepsilon,i} b_{\varepsilon,j} \underbrace{(g_{\varepsilon,i} * \tilde{h}_{\varepsilon,j})}_{\in \mathrm{A(G)}} \end{aligned}$$

which means that $g_\varepsilon * \tilde{h}_\varepsilon$ is an element of $A^0(G,\mathcal{A})$.

Set $f_\varepsilon = g_\varepsilon * \tilde{h}_\varepsilon$, we have,

$$\begin{aligned} \|f - f_\varepsilon\|_{A(G,\mathcal{A})} &= \left\|g * \tilde{h} - g_\varepsilon * \tilde{h}_\varepsilon\right\|_{A(G\mathcal{A})} \\ &\leq \left\|g_\varepsilon * (\tilde{h} - \tilde{h}_\varepsilon)\right\|_{A(G,\mathcal{A})} + \left\|(g - g_\varepsilon) * \tilde{h}\right\|_{A(G,\mathcal{A})} \\ &\leq \|g_\varepsilon\|_2 \|h - h_\varepsilon\|_2 \|g - g_\varepsilon\|_2 \|h\|_2 \\ &\leq \varepsilon\left(\frac{M_\varepsilon}{1+M_\varepsilon}\right) \\ &\leq \varepsilon \end{aligned}$$

Hence, $A^0(G,\mathcal{A})$ is dense in $A(G,\mathcal{A})$.

(ii) This follows by using the same method as in (i).

Remark 4.8 *If $\mathcal{A} = \mathbb{C}$, then $A^0(G,\mathbb{C}) = A(G)$ and $C^{b_0}(G,\mathbb{C}) = C^b(G)$.*

Corollary 4.9 *Let G be a locally compact group and let $\mathcal{A}$ be a unital commutative H^*-algebra. $A(G) \otimes \mathcal{A}$ is isometrically isomorphic to a dense subspace of $A(G,\mathcal{A})$.*

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Proof. It is obvious that the space $A^0(G,\mathcal{A})$ is isometrically isomorphic to $A(G)\otimes\mathcal{A}$, so using the previous theorem, we are done. ∎

Proposition 4.10 *Let $(\xi_j)_{j\in J}$ be an orthonormal basis of an H^*-algebra $\mathcal{A}$. For each $g\in A^0(G,\mathcal{A})$, there exists a family of functions in $A(G)$ such that*

$$g(t)=\sum_{j\in J} g_j(t)\xi_j\ .$$

Proof. Let $g\in A^0(G,\mathcal{A})$, then

$$g=\sum_{i=1}^{n} a_i h_i\ ,$$

with $a_i\in\mathcal{A}$ and $h_i\in A(G)$. Since $\mathcal{A}$ has a Hilbert space structure with $(\xi_j)_{j\in J}$ as an orthonormal basis, then for all $t\in G$ we have:

$$\begin{aligned}
g(t) &= \sum_{i=1}^{n} a_i\big(h_i(t)\big)\\
&= \sum_{i=1}^{n}\left(\sum_{j\in J}\lambda_i^j\,\xi_j\right)\big(h_i(t)\big)\text{where}\lambda_i^j\in\mathbb{C}\\
&= \sum_{j\in J}\left(\sum_{i=1}^{n}\lambda_i^j\,\big(h_i(t)\big)\right)\xi_j\\
&= \sum_{j\in J} g_j(t)\,\xi_j\text{with}g_j=\sum_{i=1}^{n}\lambda_i^j\,h_i\in A(G).\ \blacksquare
\end{aligned}$$

Lemma 4.11 *Let G be a locally compact group and let $\mathcal{A}$ be a unital and commutative H^*-algebra. A function $f=\sum_{i=1}^{n} f_i\in C^{b_0}(G,\mathcal{A})$ is a completely bounded multiplier on $A^0(G,\mathcal{A})$ if and only if for each $1\le i\le n$, f_i is a completely bounded multiplier on $A(G)$.*

Proof. Assume f is a completely bounded multiplier on $A^0(G,\mathcal{A})$, then for all $g\in A^0(G,\mathcal{A})$, $fg\in A^0(G,\mathcal{A})$,

$$i.e.\ \sum_{i=1}^{n} a_i f_i\sum_{j=1}^{m} b_j g_j=\sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j f_i g_j\in A^0(G,\mathcal{A})$$

$$i.e.\ \text{for all}\ 1\le i\le n, 1\le j\le m,\qquad f_i g_j\in A(G)\,.\quad (1)$$

Since for all $b\in\mathcal{A}$ and for all $h\in A(G)$ we have $bh\in A^0(G,\mathcal{A})$

then by setting $m=1, b_1=b$ and $g_1=h$, (1) becomes

$$\forall h\in A(G),\text{and}\ 1\le i\le n,\ \ f_i h\in A(G).$$

Moreover, since the operator,

$$\begin{aligned}
\mathcal{M}_f\colon A^0(G,\mathcal{A}) &\to A^0(G,\mathcal{A})\\
g &\mapsto fg,
\end{aligned}$$

is completely bounded, it is obvious that for each $1\le i\le n$, the operator

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$$\begin{array}{rcl} m_{f_i}: A(G) & \to & A(G) \\ h & \mapsto & f_i h, \end{array}$$

is also completely bounded. In fact, since$\mathcal{M}_f$ is $c.b.$ on $A^0(G,\mathcal{A})$, we have

$$\sup_{k\in\mathbb{N}^*} \left\| I_{M_k} \otimes \mathcal{M}_f \right\|_{\mathcal{B}\left(M_k\otimes A^0(G,\mathcal{A})\right)} < \infty \ .$$

So $\forall\alpha \in M_k, \forall h \in A(G), \forall b \in \mathcal{A}$ such that $\| \alpha \otimes bh \| \leq 1$, we have

$$\sup_{k\in\mathbb{N}^*} \left\| \alpha \otimes \left((a_i f_i)(bh)\right) \right\|_{M_k\otimes A^0(G,\mathcal{A})} < \infty \ ,$$

with $1 \leq i \leq n$. This implies that

$$\| a_i \|_{\mathcal{A}} \sup_{k\in\mathbb{N}^*} \left\| \left(I_{M_k} \otimes m_{f_i}\right)(\alpha \otimes h) \right\|_{M_k\otimes A(G)} < \infty, \forall\alpha \in M_k, \forall h \in A(G) \text{ such that } \| \alpha \otimes h \| \leq 1.$$

Hence $\sup_{k\in\mathbb{N}^*} \left\| I_{M_k} \otimes m_{f_i} \right\|_{B(M_k\otimes A(G))} < \infty$ and m_{f_i} is completely bounded on $A(G)$.

We conclude that for all $1 \leq i \leq n$, f_i is a completely bounded multiplier on $A(G)$.

Conversely, if for all $1 \leq i \leq n$, f_i is a completely bounded multiplier on $A(G)$, then for all $h \in A(G)$, $f_i h \in A(G)$.

Let $g \in A^0(G,\mathcal{A})$, there exists a family of elements $b_1, b_2, \cdots, b_m \in \mathcal{A}, g_1, g_2, \cdots, g_m \in A(G) (m \in \mathbb{N}^*)$ such that

$$g = \sum_{j=1}^{m} b_j g_j \ .$$

Thus, $f_i g_j \in A(G)$ and $a_i b_j \in \mathcal{A}$, which means that $fg \in A^0(G,\mathcal{A})$.

Now, let $k \in \mathbb{N}^*$ and $\alpha_k \in M_k$ such that $\|\alpha_k \otimes g\| \leq 1$, then $\|b_j\| \leq 1$ and $\|g_j\| \leq 1$ (for all $1 \leq j \leq m$). Set

$$\omega = \sup_{1\leq i\leq n} \{\|a_i\|_{\mathcal{A}} \| m_{f_i} \|_{cb(A(G))}\}$$

and

$$\mathcal{S} = \sup_{k\in\mathbb{N}^*} \left\{ \left\| \left(I_{M_k} \otimes \mathcal{M}_f\right)(\alpha_k \otimes g) \right\|_{M_k\otimes_{\min} A^0(G,\mathcal{A})} \right\} ,$$

we have:

$$\begin{array}{rcl}
\mathcal{S} & = & \sup_{k\in\mathbb{N}^*} \{\|\alpha_k \otimes (fg)\|_{M_k\otimes_{\min} A^0(G,\mathcal{A})}\} \\
& \leq & \sup_{k\in\mathbb{N}^*} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} \left\| \alpha_k \otimes a_i b_j f_i g_j \right\|_{M_k\otimes_{\min} A^0(G,\mathcal{A})} \right\} \\
& \leq & \sup_{k\in\mathbb{N}^*} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} \left\| a_i b_j \right\|_{\mathcal{A}} \left\| \alpha_k \otimes (f_i g_j) \right\|_{M_k\otimes_{\min} A(G)} \right\} \\
& \leq & \sum_{i=1}^{n} \sum_{j=1}^{m} \|a_i\|_{\mathcal{A}} \left\| b_j \right\|_{\mathcal{A}} \sup_{k\in\mathbb{N}^*} \left\{ \left\| \left(I_{M_k} \otimes m_{f_i}\right)(\alpha_k \otimes (g_j) \right\|_{M_k\otimes_{\min} A(G)} \right\} \\
& \leq & m \sum_{i=1}^{n} \|a_i\|_{\mathcal{A}} \sup_{k\in\mathbb{N}^*} \left\{ \left\| I_{M_k} \otimes m_{f_i} \right\|_{\mathcal{B}(M_k\otimes A(G))} \right\} \\
& \leq & m \sum_{i=1}^{n} \|a_i\| \left\| m_{f_i} \right\|_{cb(A(G))}
\end{array}$$

$$\begin{array}{rl} \mathcal{S} & \leq mn\omega \\ & < \infty \end{array}$$

which implies that $\sup_{k\in\mathbb{N}^*}\left\{\left\|\left(I_{M_k}\otimes\mathcal{M}_f\right)\right\|_{\mathcal{B}\left(M_k\otimes A^0(G,\mathcal{A})\right)}\right\}<\infty$, that is the operator

$$\begin{array}{rcl} \mathcal{M}: A^0(G,\mathcal{A}) & \rightarrow & A^0(G,\mathcal{A}) \\ g & \mapsto & fg, \end{array}$$

is completely bounded.■

We have the following theorem which is a vector-valued extension of a result given by Gilbert [7] and proved by Jolissaint in [9].

Theorem 4.12 Let *G be a locally compact group, $\mathcal{A}$ a unital and commutative H^*-algebra and let $f\in C^b(G,\mathcal{A})$. The following assertions are equivalent:*

(i) f is a completely bounded multiplier on $A^0(G,\mathcal{A})$.

(ii) there exists an integer $n\in\mathbb{N}^*$,a family $(a_i)_{i\in I}$with $a_i\in\mathcal{A}$, a Hilbert space K and two families of bounded continuous functions $(\alpha_i)_{i\in I}$, $(\beta_i)_{i\in I}$ from G to K such that for all $s,t\in G$,

$$f(t^{-1}s)=\sum_{i\in I}(\langle\alpha_i(s),\beta_i(t)\rangle_K)\,a_i\,,$$

where $\langle\cdot\,,\cdot\rangle_K$ denotes the inner-product on K and $I=\{1,2,\cdots,n\}\subset\mathbb{N}^*$.

Proof.

(i)⇒ (ii): Since $f\in C^{b_0}(G,\mathcal{A})$, there exist $n\in\mathbb{N}^*$, $a_1,a_2,\cdots,a_n\in\mathcal{A}$ and $f_1,f_2,\cdots,f_n\in A(G)$ such that

$$f=\sum_{i=1}^{n}a_if_i\,.$$

If $f=\sum_{i=1}^{n}a_if_i$ is a completely bounded multiplier on $A^0(G,\mathcal{A})$, then for each $1\leq i\leq n$, f_i is a completely bounded multiplier on $A(G)$ (**Lemma 4.11**). Using Gilbert's Theorem, we claim that for each i, there exist a Hilbert space K_i and two bounded continuous functions γ_i,δ_i from G to K_i such that

$$f_i(t^{-1}s)=\langle\alpha_i(s),\beta_i(t)\rangle\text{for all }\ s,t\in G.$$

Each γ_i (resp. δ_i) can be identified to the element

$$\alpha_i=\left(0,\dots,0,\underbrace{\gamma_i}_{i^{th}component},0,\dots,0\right)\ (\text{resp. }\ \beta_i=\left(0,\dots,0,\underbrace{\delta_i}_{i^{th}component},0,\dots,0\right))$$

of the Hilbert space

$$K=\bigoplus_{i\in I}K_i\,.$$

Finally,

$$f(t^{-1}s)=\sum_{i=1}^{n}\langle\alpha_i(s),\beta_i(t)\rangle_K\,a_i\,.$$

(ii)⇒ (i): Conversely, assume

$$f(t^{-1}s) = \sum_{i=1}^{n} \langle \alpha_i(s), \beta_i(t) \rangle_K \, a_i, \qquad \text{then } f(\mathrm{t}) = \sum_{i=1}^{n} (\langle \alpha_i(t), \beta_i(1_G) \rangle_K) \, a_i \,.$$

Let f_i be the functions from G to $\mathbb{C}$ such that $f_i(t^{-1}s) = \langle \alpha_i(s), \beta_i(t) \rangle_K$ for all $s, t \in G$, thus $f_i : t \mapsto \langle \alpha_i(t), \beta_i(1_G) \rangle_K$ are bounded on G and we have

$$f(t) = \sum_{i=1}^{n} (t) a_i \;.$$

Using **Lemma 4.11**, all we have to prove is that each f_i is a completely bounded multiplier on $A(G)$. This is obvious according to Gilbert's Theorem.■

5. Group von Neumann algebras and operator spaces

The group von Neumann algebra $VN(G)$ (resp. $VN(G, \mathcal{A})$) is a closed subspace of $\mathcal{B}(L_2(G))$ (resp. $\mathcal{B}(L_2(G, \mathcal{A}))$) and then, is an operator space. Moreover, since $A(G)$ (resp. $A(G, \mathcal{A})$) is a predual of a von Neumann algebra, it can be equipped with its canonical operator space structure.

In this section, the H^*-algebra $\mathcal{A}$ is assumed to have a dual operator space structure, *i.e.* $\mathcal{A}$ is an operator space and there exists an operator space E such that $\mathcal{A}$ is completely isometric to the dual operator E^* of E . The operator space E is called the operator predual of the dual operator space $\mathcal{A}$ and shall be denoted $\mathcal{A}_*$ (for more details about operator predual of a dual operator space, see [12]).

Theorem 5.1

(i) The space $VN(G) \otimes_{\min} \mathcal{A}$ is completely and isometrically isomorphic to the space $VN(G, \mathcal{A})$.

(ii) We have the completely isometric injection

$$A(G) \otimes_{\min} \mathcal{A} \hookrightarrow \left(VN(G) \widehat{\otimes} \mathcal{A}_*\right)^* \,.$$

Proof. The proof of this theorem will be largely analogous to that of Grothendieck's theorem [8] *(§2, section 1 théorème 2).*

(i) Consider the mapping $\mathcal{H}: VN(G) \otimes \mathcal{A} \to VN(G, \mathcal{A})$,

such that

$$(\mathcal{H}u)(t) = \sum_{k=1}^{m} u_{\mathrm{k}}(\mathrm{t}) a_k$$

$$\text{where } u = \sum_{k=1}^{m} u_{\mathrm{k}} \otimes a_k \in VN(G) \overset{\vee}{\otimes} \mathcal{A} \text{ and } t \in G.$$

$\mathcal{H}v$ is then an element of $C_c(G, \mathcal{A})$ equipped with the norm $\| \cdot \|_\infty$.

Let $n \in \mathbb{N}^*$, consider also the mapping

$$\mathcal{H}_n : M_n\left(VN(G) \overset{\vee}{\otimes} \mathcal{A}\right) \;\; \longrightarrow \;\; M_n\big(VN(G, \mathcal{A})\big)\,,$$

such that

$$(\mathcal{H}_n v)(t) = \left((\mathcal{H}v_{ij})(t)\right)_{1\le i,j\le n} = \left(\sum_{\mathrm{k}=1}^{\mathrm{m}} v_{ij}^k(\mathrm{t}) a_{ij}^k\right)_{1\le i,j\le n}$$

$$\text{where } v = (v_{ij})_{1\le i,j\le n} \in M_n\left(VN(G) \overset{\vee}{\otimes} \mathcal{A}\right),$$

$$\text{with } v_{ij} = \sum_{k=1}^{m} v_{ij}^k \otimes a_{ij}^k \in VN(G) \otimes \mathcal{A} \text{ and } t \in G.$$

,

$\mathcal{H}_n v$ is then an element of $C_c(G, M_n(\mathcal{A}))$ equipped with the norm $\|\cdot\|_\infty$.

We have to prove that $\forall n \in \mathbb{N}^*, \mathcal{H}_n$ is an isometric isomorphism and we are done.

$$\begin{aligned}
\|\mathcal{H}_n v\|_\infty &= \sup\left\{\left\|\left(\sum_{k=1}^m v_{ij}^k(g) a_{ij}^k\right)_{1\le i,j\le n}\right\|_{M_n(\mathcal{A})} : g \in G\right\} \\
&= \sup\left\{\sup_{1\le i,j\le n}\left\|\sum_{k=1}^m v_{ij}^k(g) a_{ij}^k\right\|_{\mathcal{A}} : g \in G\right\} \\
&= \sup\left\{\sup_{1\le i,j\le n}\left|a^*\left(\sum_{k=1}^m v_{ij}^k(g) a_{ij}^k\right)\right| : g \in G, a^* \in \mathcal{A}^*, \|a^*\| \le 1\right\} \\
&= \sup\left\{\sup_{1\le i,j\le n}\left|\sum_{k=1}^m v_{ij}^k(g) a^*(a_{ij}^k)\right| : g \in G, a^* \in \mathcal{A}^*, \|a^*\| \le 1\right\} \\
&= \sup\left\{\sup_{1\le i,j\le n}\left\{\sup_{g\in G}\left|\sum_{k=1}^m a^*(a_{ij}^k) v_{ij}^k(g)\right|\right\} : a^* \in \mathcal{A}^*, \|a^*\| \le 1\right\} \\
&= \sup\left\{\sup_{1\le i,j\le n}\left\|\sum_{k=1}^m a^*(a_{ij}^k) v_{ij}^k\right\|_\infty : a^* \in \mathcal{A}^*, \|a^*\| \le 1\right\} \\
&= \sup\left\{\sup_{1\le i,j\le n}\left|\varpi\left(\sum_{k=1}^m a^*(a_{ij}^k) v_{ij}^k\right)\right| : \varpi \in VN(G)^*, a^* \in \mathcal{A}^*, \|\varpi\| \le 1, \|a^*\| \le 1\right\} \\
&= \sup\left\{\sup_{1\le i,j\le n}\left|\sum_{k=1}^m a^*(a_{ij}^k) \varpi(v_{ij}^k)\right| : \varpi \in VN(G)^*, a^* \in \mathcal{A}^*, \|\varpi\| \le 1, \|a^*\| \le 1\right\} \\
&= \sup\left\{\sup_{1\le i,j\le n}\left|(\varpi \otimes a^*)\left(\sum_{k=1}^m v_{ij}^k \otimes a_{ij}^k\right)\right| : \varpi \in VN(G)^*, a^* \in \mathcal{A}^*, \|\varpi\| \le 1, \|a^*\| \le 1\right\} \\
&= \sup_{1\le i,j\le n}\left\|\sum_{k=1}^m v_{ij}^k \otimes a_{ij}^k\right\|_\vee \\
&= \|v\|
\end{aligned}$$

$$\text{that is } \|\mathcal{H}_n v\|_\infty = \|v\|.$$

We just proved that $M_n\left(VN(G) \overset{\vee}{\otimes} \mathcal{A}\right)$ is isometrically isomorphic with the closed linear subspace of $M_n(VN(G,\mathcal{A})) \cong VN(G, M_n(\mathcal{A}))$ generated by the family of functions of the form

$$\begin{array}{rcl} G & \longrightarrow & M_n(\mathcal{A}) \\ g & \longmapsto & \left(\sum_{k=1}^{m} v_{ij}^k(g) \otimes a_{ij}^k\right) \end{array}$$

where $v_{ij}^k \in VN(G)$ and $a_{ij}^k \in \mathcal{A}$ for $1 \le i,j \le n$ and $1 \le k \le m$; all we have left to do is to show that this family is dense in $VN\big(G, M_n(\mathcal{A})\big)$.

Let $f_n: G \to M_n(\mathcal{A})$ be continuous and let $\varepsilon > 0$ be given. $f_n(G)$ is compact so there are points $t_1, t_2 \cdots, t_m \in G$ such that for any $t \in G$ there's a $\ell: 1 \le \ell \le m$ for which $\| f_n(t) - f_n(t_\ell) \| \le \varepsilon/2$, say. Let $U_\ell = \{t: \|f_n(t) - f_n(t_\ell)\| \le \varepsilon\}$. Then $\{U_1, \cdots, U_m\}$ is a finite open cover of G and therefore, there is a continuous partition of unity $\{f_{n1}, f_{n2} \cdots, f_{nm}\}$ subordinate to $\{U_1, \cdots, U_m\}$, that is, there are continuous real-valued functions $f_{n1}, f_{n2} \cdots, f_{nm}$ on G each having values in [0,1] with

$$\sum_{k=1}^{m} f_{nk}(t) \equiv I_n \text{and } f_{nk}(t) = 0 \text{, when } t \text{ is outside } U_k.$$

Define $h_n: G \to M_n(A)$ by

$$h_n(t) = \sum_{\ell=1}^{m} f_{n\ell}(t) f_n(t_\ell) \ .$$

$$\text{Plainly } t = \mathcal{H}_n\left(\sum_{\ell=1}^{m} f_{n\ell} \otimes f_n(t_\ell)\right)$$

and if $t \in G$, then

$$\begin{array}{rcl} \|h_n(t) - f_n(t)\| & = & \left\| \sum_{\ell=1}^{m} f_{n\ell}(t) f_n(t_\ell) - f_n(t) \right\| \\ & = & \left\| \sum_{\ell=1}^{m} f_{n\ell}(t)[f_n(t_\ell) - f_n(t)] \right\| \\ & = & \left\| \sum_{\ell: t \in U_\ell} f_{n\ell}(t)[f_n(t_\ell) - f_n(t)] \right\| \\ & < & \varepsilon, \end{array}$$

it follows that $\|h_n - f_n\|_\infty \le \varepsilon$ and with this the density of $\mathcal{H}_n$'range is plain.

(ii) Since for any operator space X and Y, the natural embedding

$X^* \otimes_{\min} Y \hookrightarrow cb(X, Y)$ is completely isometric and we have the complete isometries

$\big(X \widehat{\otimes} Y\big)^* \cong cb(X, Y^*) \cong cb(Y, X^*)$ (Corollary 7.1.5 and Proposition 8.1.2 in [4]), we have:

$$A(G) \otimes_{\min} \mathcal{A} \hookrightarrow cb(VN(G), \mathcal{A}) \cong \big(VN(G) \widehat{\otimes} \mathcal{A}_*\big)^*. \ \blacksquare$$

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