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New High order Methods for Solving Non Linear Equations

Osama Yusuf Ababneh
Department of Mathematics, Zarqa University ,Zarqa, Jordan
ababnehukm@yahoo.com

ABSTRACT

In this paper, we present two families of third and fourth order iterative methods for solving nonlinear equations. The efficiency index of the proposed schemes is 1.442 and 1.587. In order to compare the performance with some of the existing schemes, several numerical examples are furnished here.

Indexing terms/Keywords

Newtons method; Fourth-order convergence; Third-order convergence; Non-linear equations; Root-finding; Iterative method.

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INTRODUCTION

We begin with the following iteration scheme

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+\gamma f(y_n)}\frac{f(x_n)}{f'(x_n)}, \end{aligned}\right\} \tag{1}$$

where β and γ are parameters to be determined from the following convergence theorem.

Theorem 2.1: Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \to R$ for an open interval I. If x_0 is sufficiently close to α, where $e_n = x_n - \alpha$ and $c_k = f^{(k)}(\alpha)/k!$. Then the methods defined by (1) are at least of order three for $\beta - \gamma = 1$, and of fourth-order convergence if $\beta = -1$ and $\gamma = -2$.

Proof

Using Taylor expansion of $f(x_n)$ about α and taking into account that $f'(\alpha) \neq 0$, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)]. \tag{2}$$

Furthermore, we have

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)], \tag{3}$$

and

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5). \tag{4}$$

Substituting (4) in $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ yields

$$y_n - \alpha = c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5). \tag{5}$$

Expanding $f(y_n)$ about α and using (5), we have

$$f(y_n) = f'(\alpha)[c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 5c_2^3 - 3c_4)e_n^4 + O(e_n^5)]. \tag{6}$$

From (2) and (6), we get

$$f(x_n) + \beta f(y_n) = f'(\alpha)[e_n + (1 + \beta)c_2e_n^2 + \{-2\beta c_2^2 + (1 + 2\beta)c_3\}e_n^3 + O(e_n^4)], \tag{7}$$

and

$$f(x_n) + \gamma f(y_n) = f'(\alpha)[e_n + (1+\gamma)c_2 e_n^2 + \{-2\gamma c_2^2 + (1+2\gamma)c_3\}e_n^3 + O(e_n^4)]. \tag{8}$$

Upon dividing (7) by (8) and simplifying, we get

$$\frac{f(x_n)+\beta f(y_n)}{f(x_n)+\gamma f(y_n)} = 1 + (\beta-\gamma)c_2 e_n + (\beta-\gamma)\{-(\gamma+3)c_2^2 + 2c_3\}e_n^2 + O(e_n^3). \tag{9}$$

Multiplication of (4) and (9) yields

$$\frac{f(x_n)+\beta f(y_n)}{f(x_n)+\gamma f(y_n)}\frac{f(x_n)}{f'(x_n)} = e_n - \{c_2 - (\beta-\gamma)c_2\}e_n^2 + \{(2 - 4(\beta-\gamma) - \gamma(\beta-\gamma))c_2^2\}e_n^3 + \{(7 - 4(\beta-\gamma))c_2c_3 + (5(\beta-\gamma) + \gamma(\beta-\gamma) - 4)c_2^3\}e_n^4 + O(e_n^5). \tag{10}$$

We obtain the error equation

$$e_{n+1} = [c_2 - (\beta-\gamma)c_2]e_n^2 - [(2 - 4(\beta-\gamma) - \gamma(\beta-\gamma))c_2^2]e_n^3 - [(7 - 4(\beta-\gamma))c_2c_3 + (5(\beta-\gamma) + \gamma(\beta-\gamma) - 4)c_2^3]e_n^4 + O(e_n^5). \tag{11}$$

This means that the methods defined by (1) is at least of order three for any $\beta - \gamma = 1$ to get the error equation

$$e_{n+1} = -[(2 - 4(\beta-\gamma) - \gamma(\beta-\gamma))c_2^2]e_n^3 + O(e_n^4). \tag{12}$$

Its obviously that if $\beta = -1$ and $\gamma = -2$, then the error equation should be

$$e_{n+1} = [c_2^3 - 3c_2c_3 + 3c_4]e_n^4 + O(e_n^5), \tag{13}$$

which means that the methods defined by (1) is of order four if $\beta = -1$ and $\gamma = -2$.

This completes the proof of the theorem.

Some special cases

In fact, $\beta = -1$ and $\gamma = -2$ the well-known Traub-Ostrowski method (TOM) [10] is obtained.

$$x_{n+1} = x_n - \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\frac{f(x_n)}{f'(x_n)}. \tag{14}$$

If we choose $\beta = 2$ and $\gamma = 1$, we get the third order method.

$$x_{n+1} = x_n - \frac{f(x_n) + 2f(y_n)}{f(x_n) + f(y_n)}\frac{f(x_n)}{f'(x_n)}, \tag{15}$$

which introduced by Chun in [11]

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If $\beta = 0$ and $\gamma = -1$, then we obtain from (1), we obtain a third order method

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(y_n)} \frac{f(x_n)}{f'(x_n)}. \qquad (16)$$

which was introduced by Xiaojian in [12].

If $\beta = -3$ and $\gamma = -4$, then we obtain from (1) a new third-order method

$$x_{n+1} = x_n - \frac{f(x_n) - 3f(y_n)}{f(x_n) - 4(y_n)} \frac{f(x_n)}{f'(x_n)}. \qquad (17)$$

If $\beta = 1$ and $\gamma = 0$, then we obtain from (1) a new third-order method

$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)}. \qquad (18)$$

We consider the definition of efficiency index [13] as $p^{\frac{1}{w}}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. If we assume that all the evaluations have the same cost as function one, we

have that the third order family has the efficiency index equal to $\sqrt[3]{3} \simeq 1.442$, which is better than the ones of the Newtons method $\sqrt{2} \simeq 1.414$.

New families of fourth order methods

In this section, we apply the approach used in [14] to derive new families of fourth order by using the new third order method in (1). Now, we consider the function ϕ defined in the following linear combination form

$$\phi(x) = x - \theta_1[x - \rho(x)] - \theta_2[x - \zeta(x)] - \theta_3[x - \eta(x)], \qquad (19)$$

where $\theta_i \in R$, $i = 1, 2, 3$, $\theta_1 + \theta_2 + \theta_3 = 1$.

Theorem 4.1: [14] *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \to R$ for an open interval I. Let θ, $i = 1,2,3$ be nonzero real numbers with $\theta_1 + \theta_2 + \theta_3 = 1$, and ρ, ζ and η be iteration functions of order three, then the iteration function defined by (19) is of order at least three, and the iterative method defined by $x_{n+1} = \phi(x_n)$ then satisfies the error equation*

$$e_{n+1} = \frac{1}{6}[\theta_1\rho^{(3)}(\alpha) + \theta_2\zeta^{(3)}(\alpha) + \theta_3\eta^{(3)}(\alpha)]e_n^3 + O(e_n^4). \qquad (20)$$

Furthermore, the iteration function defined by (19) is of order at least four for each triple $(\theta1, \theta2, \theta3)$ making the coefficient of e_n^3 in (20) zero, and the iterative method defined by $x_{n+1} = \phi(x_n)$ then satisfies the error equation

$$e_{n+1} = \frac{1}{24}[\theta_1\rho^{(4)}(\alpha) + \theta_2\zeta^{(4)}(\alpha) + \theta_3\eta^{(4)}(\alpha)]e_n^4 + O(e_n^5). \qquad (21)$$

To construct the fourth-order iterative method via Theorem 4.1, we consider the third-order iteration functions ρ, ζ and η defined in (16), (17) and (18).

By the help of Mathematica, we have

$$\begin{aligned} \rho^{(3)}(\alpha) &= c_2^2, \quad \rho^{(4)}(\alpha) = -3c_2^3 + 31c_2c_3 - 12c_4, \\ \zeta^{(3)}(\alpha) &= -2c_2^2, \quad \zeta^{(4)}(\alpha) = 3c_2^3 + 19c_2c_3 - 12c_4, \\ \eta^{(3)}(\alpha) &= 2c_2^2, \quad \eta^{(4)}(\alpha) = -9c_2^3 + 35c_2c_3 - 12c_4, \end{aligned} \tag{22}$$

By Theorem 4.1, we need to solve the system of equations

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 = 1, \\ \theta_1\rho^{(3)}(\alpha) + \theta_2\zeta^{(3)}(\alpha) + \theta_3\eta^{(3)}(\alpha) = 0, \end{cases} \tag{23}$$

for θ_1, θ_2 and θ_3 to construct fourth-order iteration functions via (19). Therefore, the system of equations

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 = 1, \\ \theta_1 c_2^2 - 2\theta_2 c_2^2 + \theta_3 2c_2^2 = 0, \end{cases} \tag{24}$$

to obtain

$$\theta_1 = \frac{2-4\beta}{3}, \quad \theta_2 = \frac{1+\beta}{3}, \quad \theta_3 = \beta, \tag{25}$$

where $\beta \in R$. Thus, the iteration function defined by (19) gives a family of infinitely many new fourth–order iterative methods

$$x_{n+1} = x_n - \left(\frac{2-4\beta}{3}A\right) - \left(\frac{1+\beta}{3}C\right) - (\beta D), \tag{26}$$

$$A = \frac{f(x_n)}{f(x_n)-f(y_n)}\frac{f(x_n)}{f'(x_n)},$$

$$C = \frac{f(x_n)-3f(y_n)}{f(x_n)-4f(y_n)}\frac{f(x_n)}{f'(x_n)},$$

$$D = \frac{f(x_n)+f(y_n)}{f'(x_n)},$$

with error equation

$$e_{n+1} = \left(\left(\frac{14\beta}{3} - \frac{29}{24}\right)c_2c_3 - \left(\frac{1}{3} - \frac{5\beta}{3}\right)c_4 - \left(\frac{53}{72} - \frac{31}{81}\beta\right)c_2^3\right)e^4 + O(e^5). \tag{27}$$

This fourth order family has the efficiency index equal to $\sqrt[3]{4} \simeq 1.587$, which is better than the ones of the Newtons method $\sqrt{2} \simeq 1.414$.

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From (26) we can get infinitely many forth order methods, for example for $\beta = -1$

$$x_{n+1} = x_n - \frac{f^2(x) + f^2(y)}{f'(x)\left(f(x) - f(y)\right)}, \tag{28}$$

for $\beta = \frac{1}{2}$

$$x_{n+1} = x_n - \frac{f^2(x) - 3f(x)f(y) - 2f^2(y)}{f'(x)\left(f(x) - 4f(y)\right)}. \tag{29}$$

Numerical results

All computations were done using the Mathematica package using 64 digit floating point arithmetics. We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: $/x_{n+1} - x_n/ < \epsilon$ and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. We used the fixed stopping criterion $\epsilon = 10^{-15}$.

The following test functions have been used. We employ the present methods to solve some nonlinear equations, which not only illustrate the methods practically but also serve to check the validity of theoretical results we have derived.

$$\begin{aligned}
&f_1(x) = x^3 + 4x^2 - 10, \quad \alpha = 1.3652300134140968457608068290,\\
&f_2(x) = \sin^2 x - x^2 + 1, \quad \alpha = 1.404491648215341,\\
&f_3(x) = x^2 - e^x - 3x + 2, \quad \alpha = 0.2575302854398608,\\
&f_4(x) = \cos x - x, \quad \alpha = 0.7390851332151606416553120876,\\
&f_5(x) = (x-1)^3 - 2, \quad \alpha = 2.2 : 2599210498948731647672106073,\\
&f_6(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \quad \alpha = -1.207647827130919,\\
&f_7(x) = (x+2)e^x - 1, \quad \alpha = -0.4428544010023885831413280000,
\end{aligned}$$

Displayed in Table 1 and Table 2 are the number of iterations to approximate the zero (N) and the number of function evaluations (TNFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative.

We present some numerical test results for various iterative schemes in Table 1. Compared with the Newton method (NM), the method of Chun (16)(CM), Xiaojian (XM)(16). and the methods (17)(OM1) and (18) (OM2) introduced in section 3. The test results in Table 1 show that for most of the functions we tested, the methods introduced in the present presentation have at least equal performance compared to the other third-order method, and can also compete with Newtons method.

In Table 2, we also present some numerical test results for various fourth order methods and the Newton method, were the Newton method (NM), Jarratts method (JM)[15] defined by

$$x_{n+1} = x_n - \left[1 - \frac{3}{2}\frac{f'(y) - f'(x)}{3f'(y) - f'(x)}\right]\frac{f(x)}{f'(x)}, \tag{30}$$

where $z_n = x_n - 2f(x)/f'(x)$, Traub-Ostrowski method (TM)(14), and the methods (OS1)(28), (OS2)(29) introduced in section 4.

The results presented in Table 2 show that for most of the functions we tested, the variants introduced in the present presentation have better performance as compared to the corresponding classical methods, and also converge more rapidly than Newtons method.

Table 1
Comparison of various cubically convergent iterative schemes and Newtons method

$f(x)$	N					TNFE				
	NM	CM	XM	OM1	OM2	NM	CM	XM	OM1	OM2
$f_1, x_0 = 3.0$	6	4	4	4	4	12	12	12	12	12
$f_2, x_0 = 3.0$	6	4	4	4	4	12	12	12	12	12
$f_3, x_0 = 1.0$	8	3	3	3	3	24	9	9	9	9
$f_4, x_0 = 3.0$	7	4	4	4	3	14	12	12	12	9
$f_5, x_0 = 2.5$	5	3	3	3	3	10	9	9	9	9
$f_6, x_0 = -1.0$	5	5	3	4	4	10	15	9	12	12
$f_7, x_0 = -1.0$	6	NC	4	4	6	12	NC	12	12	18

Table 2
Comparison of various fourth order schemes and Newtons method

$f(x)$	N					TNFE				
	NM	JM	TM	OS1	OS2	NM	JM	TM	OS1	OS2
$f_1, x_0 = -0.3$	53	74	60	7	44	106	222	180	21	132
$f_2, x_0 = 3.0$	6	6	3	3	3	12	12	9	9	9
$f_3, x_0 = 1.0$	4	4	2	2	2	8	12	6	6	6
$f_4, x_0 = 2.0$	4	5	3	3	3	8	10	9	9	9
$f_5, x_0 = 3.0$	6	5	3	3	6	12	10	9	9	12
$f_6, x_0 = -1.0$	5	5	3	3	3	10	15	9	9	9
$f_7, x_0 = 2.0$	6	4	3	4	4	12	12	9	12	12

Conclusions

We have proposed two families of iterative methods for solving nonlinear equations. Numerical results support the first family to be cubically convergent and show that the number of iterations of the new method are always less than that of the classical Newtons method and can be compared with other methods. We have obtained many new fourth-order methods from third-order methods. Analysis of convergence of this family of methods is supplied in Theorem 2. Analysis of efficiency shows that these methods are preferable to Newtons method and some classical fourth order methods. The number of function evaluations of the new methods are comparable.

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