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# Investigation of the method of sequential analysis of Bayesian type

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## ABSTRACT

The results of investigation of the properties of new sequential methods of testing many hypotheses based on special properties of hypotheses acceptance regions in the constrained Bayesian tasks of testing many hypotheses are offered. In particular, some relations between the errors of the first and the second kinds in constrained Bayesian task and in sequential method of Bayesian type depending on the divergence between the tested hypotheses are given. Also dependences of the Lagrange multiplier and the risk function on the probability of incorrectly accepted hypotheses are presented. Theses results are necessary for computation of errors of made decisions at testing multiple hypotheses using offered new sequential methods of testing hypotheses. Computation results of some examples confirm the rightness of theoretical researches.

**Keywords:** constrained Bayesian problem; decision rule; errors type I and type II; hypotheses testing; sequential analysis.

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## 1 Introduction

A short review of the works devoted to the classical problem of sequential analysis beginning from the fundamental work of the Wald is given in [18]. It must be noted that there are many other works where different concepts of multiple sequential comparisons are considered. For example, in [11] (Sections 6.8 and 7.5) sequential methods about multivariate parameters are described. To test a composite null hypothesis concerning a set of parameters against two tailed alternative hypothesis on the basis of sequentially obtained observations is considered in the works [4, 9, 10, 14, 15, 17 (Chapter16), 27-30, 32, 36,37 (Chapter8)]. The methods of classification of the several available sets of models are described in [1, 3, 5, 8, 13, 24, 26, 31, 33] which are extensions of the classical Wald's sequential probability ratio tests. More general problem is considered in [2, 6, 7, 12, 17 (Chapter 15), 25, 34, 35]. In particular, there are considered individual hypotheses about examined set of parameters of sequentially observed random vectors. In the present paper we bring some results of investigation of the method of sequential analysis of Bayesian type offered in [18, 19] which is quite universal approach allowing to test hypotheses of any types considered in the above mentioned works.

## 2 Constrained Bayesian problems of testing multi-hypothesis

One of possible formulations of the constrained Bayesian problem has the following form [20, 22]

$$r_\delta = \min_{\{\Gamma_j\}} \sum_{i=1}^{S} p(H_i) \sum_{j=1, j\neq i}^{S} \int_{\Gamma_j} p(\mathbf{x} \mid H_i) d\mathbf{x}, \qquad (1)$$

subject to

$$\sum_{i=1}^{S} p(H_i) \int_{\Gamma_i} p(\mathbf{x} \mid H_i) d\mathbf{x} \geq 1-\alpha, \qquad (2)$$

where $H_i : \boldsymbol{\theta} = \boldsymbol{\theta}^i$, is the hypothesis that the sample $\mathbf{x}^T = (x_1,..., x_n)$ is generated by distribution $p(\mathbf{x} \mid \boldsymbol{\theta}^i) = p(x_1,.., x_n; \theta_1^i,..., \theta_k^i) \equiv p(\mathbf{x} \mid H_i)$, $i = 1,..., S$; $p(H_i)$ is the a priori probability of hypothesis $H_i$; $\delta(\mathbf{x}) = \{\delta_1(\mathbf{x}), \delta_2(\mathbf{x}),..., \delta_S(\mathbf{x})\}$ is the decision function where $\delta_i(\mathbf{x}) = 1$ if hypothesis $H_i$ is accepted and $\delta_i(\mathbf{x}) = 0$ otherwise; $\Gamma_i$ is the region of acceptance of hypothesis $H_i$, i.e. $\Gamma_i = \{\mathbf{x} : \delta_i(\mathbf{x}) = 1\}$; $\alpha$ is the maximum allowed level of the averaged value of incorrectly accepted hypotheses.

For solving constrained optimization problem (1), (2) we shall use the method of indeterminate Lagrange multipliers, which gives [20, 22]:

$$\Gamma_j = \left\{ \mathbf{x} : \sum_{i=1, i\neq j}^{S} p(H_i) p(\mathbf{x} \mid H_i) < \lambda p(H_j) p(\mathbf{x} \mid H_j) \right\}, \quad j = 1,..., S, \qquad (3)$$

where $\lambda$, the same scalar value for all regions, is determined so that in (2) the equality takes place.

Task (1), (2) is one of possible formulations of the constrained Bayesian problem. In a similar manner, we can introduce and solve different constrained Bayesian tasks [20]. For simplicity, further we shall consider only the task (1), (2), though it is not difficult to be convinced that all results obtained below, after appropriate modifications, are true for all other tasks too.

## 3 Properties of the hypotheses acceptance regions

In conventional stat.ements of the problem of statistical hypotheses testing, their acceptance regions are not intersected, i.e. $\Gamma_i \bigcap \Gamma_j = \varnothing$, $i \neq j$, and the union of all regions of acceptance of hypotheses coincides with the observation space, i.e. $\bigcup_{i=1}^{S} \Gamma_i = R^n$ [23]. These conditions break down at consideration of above-formulated constrained Bayesian task of hypotheses testing. In particular, in [21, 22] is proved that for any value of $\alpha$ ($0 < \alpha < 1$), that is the same, for any value of $\lambda$ ($0 < \lambda < +\infty$), in the observation space $R^n$ exist: the regions of unambiguous acceptance of the tested hypotheses, the regions of the suspicion on the validity of several (more than one) tested hypotheses (corresponding to sub-regions of the intersection of the regions of acceptance of corresponding hypotheses (3)) and the region of impossibility of acceptance of the tested hypotheses (corresponding to the region of the space $R^n$, which do not belong to any of the regions of acceptance of hypotheses (3)). Accordingly, for any concrete observation result $\mathbf{x}$, on the basis of which the decision is made, in the interval $(0;1)$ there are such values $\alpha_*(\mathbf{x}) \leq \alpha^*(\mathbf{x})$ that for $\alpha \in [\alpha_*(\mathbf{x}); \alpha^*(\mathbf{x})]$ the observation result $\mathbf{x}$ belongs to only one of the regions of acceptance of hypotheses (3) and the corresponding hypothesis is accepted, respectively. At $\alpha < \alpha_*(\mathbf{x})$, the observation result $\mathbf{x}$ appears in a sub-region of intersection of two or several regions of acceptance of hypotheses (3), and it is impossible to make a simple decision. In that case, the appropriate hypotheses are suspected on the validity. At $\alpha > \alpha^*(\mathbf{x})$, the observation result $\mathbf{x}$ appears in the region of the space $R^n$ which does not belong to any of the regions of acceptance of hypotheses (3). In this case, it is impossible to make the decision on the basis of the set observation result $\mathbf{x}$.

## 4 The method of sequential analysis of Bayesian type

Using the specificity of hypotheses acceptance regions (3), *the sequential analysis method of Bayesian type* is offered in [18, 19]. In particular, there are used the following designations: $R_m^n$ is the sampling space of all possible samples of $m$ independent $n$-dimensional observation vectors $\mathbf{x}=(x_1,..., x_n)$; $R_{m,1}^n$, $R_{m,2}^n$,..., $R_{m,S}^n$, $R_{m,S+1}^n$ are the splitting of $R_m^n$ into $S+1$ disjoint sub-regions such that $R_m^n = \bigcup_{i=1}^{S+1} R_{m,i}^n$. Let $p(\mathbf{x}^1,..., \mathbf{x}^m | H_i)$ be the total probability distribution density of $m$ independent $n$-dimensional observation vectors; $m$ is sample size. Then $p(\mathbf{x}^1,..., \mathbf{x}^m | H_i) = p(\mathbf{x}^1 | H_i)... p(\mathbf{x}^m | H_i)$.

The following decision rule is determined. If the matrix of observation results $\mathbf{x}=(\mathbf{x}^1,..., \mathbf{x}^m)$ belongs to the sub-region $R_{m,i}^n$, $i=1,..., S$, then hypothesis $H_i$ is accepted, and, if $\mathbf{x}=(\mathbf{x}^1,..., \mathbf{x}^m)$ belongs to the sub-region $R_{m,S+1}^n$, the decision is not made, and the observations go on until one of the tested hypotheses is accepted.

Let us designate the population of sub-regions of intersections of acceptance regions $\Gamma_i^m$ of hypotheses $H_i$ $(i=1,..., S)$ in constrained Bayesian task of hypotheses testing with the regions of acceptance of other hypotheses $H_j$, $j=1,..., S$; $j \neq i$, by $I_i^m$. By $E_m^n = R_m^n - \bigcup_{i=1}^{S} \Gamma_i^m$, we designate the population of regions of space $R_m^n$ which do not belong to any of hypotheses acceptance regions. Then the hypotheses acceptance regions in the method of sequential analysis of Bayesian type are determined in the following way:

$$R_{m,i}^n = \Gamma_i^m / I_i^m, \; i=1,..., S;$$

$$R_{m,S+1}^n = \left( \bigcup_{i=1}^{S} I_i^m \right) \bigcup E_m^n. \qquad (4)$$

Here regions $\Gamma_i^m$, $I_i^m$, $E_m^n$, $i=1,..., S$, are defined on the basis of hypotheses acceptance regions (3).

## 5 Investigation of the method of sequential analysis of Bayesian type

### 5.1. Consistency and uniqueness

For clarity, from here on, by $\alpha_1$ and $\beta_1$, we shall designate the probabilities of errors of the first and the second kinds for sequential method of Bayesian type with $m$ sequentially obtained observation results, and, by $\alpha$ and $\beta$, the same quantities for constrained Bayesian task.

The following statements are proved in [18].

**Theorem 5.1.** *If the probability distribution $p(\mathbf{x} | H_i)$, $i=1,..., S$, is such that an increase in the sample size $m$ entails a decrease in the entropy concerning distribution parameters $\boldsymbol{\theta}$ about which the hypotheses are formulated, then infinitely increasing number of repeated observations, i.e. $m \to \infty$ in the sequential analysis method of Bayesian type, entails infinite decreasing probabilities of errors of the first and the second kinds, i.e. $\alpha_1 \to 0$ and $\beta_1 \to 0$.*

**Lemma 5.1.** *In the conditions of Theorem 5.1, at increasing divergence $J(H_i, H_j)$ between tested hypotheses $H_i$ and $H_j$, $i, j=1,..., S;$ $i \neq j$, Lagrange coefficient $\lambda$ in solution (3) decreases, and, in the limit, at $\min_{\{i,j\}} J(H_i, H_j) \to \infty$, $\lambda \to 0$ takes place for the given $\alpha$.*

**Lemma 5.2.** *In the conditions of Theorem 5.1, at infinitely decreasing divergence $J(H_i, H_j)$ between tested hypotheses $H_i$ and $H_j$, $i, j=1,..., S;$ $i \neq j$, i.e. at $\max_{\{i,j\}} J(H_i, H_j) \to 0$ Lagrange coefficient $\lambda$ in solution (3) tends to a certain value from the interval:*

$$\left\{ \min_{\{j\}} \sum_{i=1, i \neq j}^{S} p(H_i)/p(H_j), \; \max_{\{j\}} \sum_{i=1, i \neq j}^{S} p(H_i)/p(H_j) \right\} \qquad (5)$$

*depending on the value of $\alpha$.*

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**Corollary 5.1**. *In the conditions of Theorem 5.1 at the absence of a priori information about the validity of tested hypotheses, i.e. at* $p(H_i)=1/S$, $i=1,...,S$, *at* $\max_{\{i,j\}} J(H_i,H_j)\to 0$, $\lambda\to(S-1)$ *is truth.*

**Theorem 5.2.** *For any given sample size* $m$ *and as small errors of the first and the second kinds* $\alpha'$ *and* $\beta'$ *as one likes, there always exists such a positive value* $J^*$ *that, if the divergence between tested hypotheses is more than that value, i.e.* $\min_{\{i,j\}} J(H_i,H_j)>J^*$, $\alpha_1(J)<\alpha'$ *and* $\beta_1(J)<\beta'$ *hold true, i.e. the method of sequential analysis of Bayesian type rigorously surpasses the criterion with errors of the first and the second kinds equal to* $\alpha'$ *and* $\beta'$, *respectively.*

**Theorem 5.***4. For any value of* $\alpha$ *in constrained Bayesian task there always exists such an integer* $m^*$ *that if the number of repeated observations* $m$, *in the method of sequential analysis of Bayesian type, is more than this value, i.e.* $m>m^*$, *there will be accepted one of the tested hypotheses with the probability equal to unity.*

In addition to these properties the following one are also true.

**Theorem 5.3.** *At infinitely decreasing divergence between the tested hypotheses, i.e. at* $\max_{\{i,j\}} J(H_i,H_j)\to 0$, *between the probabilities of errors of the first and the second kinds in constrained Bayesian task, there exists the following ratio* $\alpha+\beta=\gamma$, *where, in the general case* $\gamma$ *is the value different from one of the following character: at* $\alpha\to 1$ *or* $\alpha\to 0$, *value* $\gamma\to 1$ *since, in these cases,* $\beta\to 0$ *or* $\beta\to 1$ *hold true, respectively.*

***Proof.*** In accordance with Lemma 5.2, at $\max_{\{i,j\}} J(H_i,H_j)\to 0$, coefficient $\lambda$ in hypotheses acceptance region (3) tends to a certain value from interval (5), i.e. its value is determined by a priori probabilities for given $\alpha$. Moreover this tendency is such that the probability of fulfillment of inequality

$$\sum_{i=1,i\neq j}^{S}\frac{p(H_i)}{p(H_j)}\frac{p(\mathbf{x}\,|\,H_i)}{p(\mathbf{x}\,|\,H_j)}<\lambda \qquad (6)$$

is equal to $(1-\alpha)$. At $\max_{\{i,j\}} J(H_i,H_j)\to 0$, $p(\mathbf{x}\,|\,H_i)/p(\mathbf{x}\,|\,H_j)\to 1$, $\forall i,j: i,j\in(1,...,S)$; $i\neq j$ takes place. Thus, for infinitesimal $J(H_i,H_j)$, the difference between $p(\mathbf{x}\,|\,H_i)$ and $p(\mathbf{x}\,|\,H_j)$ is also infinitesimal, i.e. these distributions coincide with high accuracy. Probabilities $\alpha$ and $\beta$ are the probabilities of errors of the first and the second kinds, respectively (see Fig. 1). At $p(\mathbf{x}\,|\,H_i)/p(\mathbf{x}\,|\,H_j)\to 1$, $\forall i,j: i,j\in(1,...,S)$; $i\neq j$, it is easy to guess that in interval (5) exists such a value $\lambda={}^*\lambda$ at which they become mutually complementary probabilities, satisfying condition $\alpha+\beta=1$. At $\lambda\neq{}^*\lambda$, the probabilities of errors of the first and the second kinds are no longer mutually complementary. Therefore, there could arise situations $\alpha+\beta=\gamma>1$ or $\alpha+\beta=\gamma<1$ depending on the value of $\lambda$. At $\alpha\to 1$ or $\alpha\to 0$, the probabilities of errors of the first and the second kinds become again mutually complementary because of a higher order of smallness of the difference between probability distribution densities $p(\mathbf{x}\,|\,H_i)$ and $p(\mathbf{x}\,|\,H_j)$ on the tails than in the vicinity of their modes, and therefore again condition $\alpha+\beta=\gamma=1$ is satisfied. □

**Corollary 5.2**. *In the conditions of Theorem 5.3, at* $p(H_i)=1/S$, $i=1,...,S$, $\alpha+\beta=1$ *holds true, i.e., at* $\alpha\to 0$, $\beta\to 1$ *hold true, and on the contrary, at* $\alpha\to 1$, $\beta\to 0$ *is fulfilled. In particular, at* $\alpha=0.5$, $\beta=0.5$.

***Proof***. At $p(H_i)=1/S$, $i=1,...,S$, left-hand side of (6) approaches $(S-1)$. In accordance with Corollary 5.1, the value of $\lambda$ from the right-hand side of (6), approaches $(S-1)$ from above. Therefore, the difference between regions $\Gamma_j$ tends to zero and, accordingly, the probabilities $\alpha$ and $\beta$ become mutually complementary up to one.

At $\max_{\{i,j\}} J(H_i,H_j)\to 0$, the difference between likelihood functions is caused by the difference between a priori probabilities. Therefore, in the general case, $\alpha+\beta=\gamma\neq 1$. At $p(H_i)=1/S$, $i=1,...,S$, likelihood functions are identical. Therefore $\alpha+\beta=\gamma=1$ holds true. □

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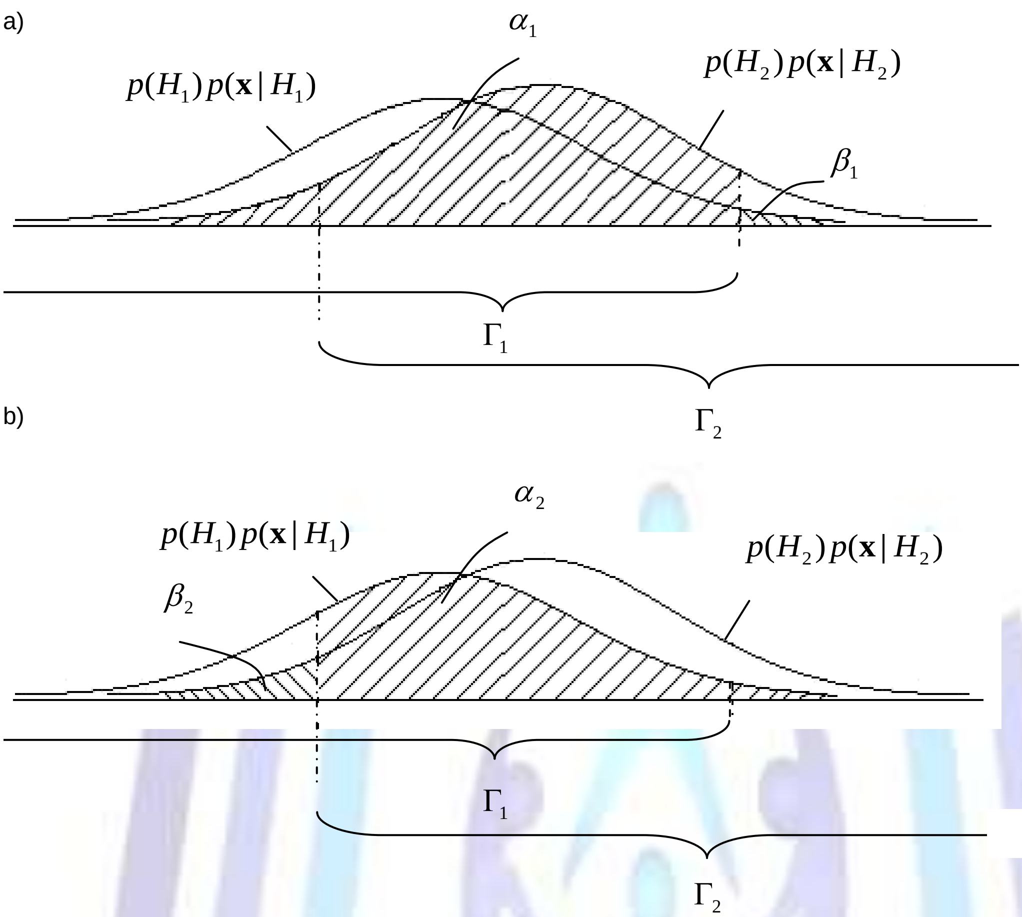


**Fig. 1. Probabilities of errors of the first and the second types in constrained Bayesian method.**

**Corollary 5.3**. *In the conditions of Theorem 5.3, at* $\alpha \to 1$, *undetermined Lagrange multiplier* $\lambda$ *in hypothesis acceptance region (3) approaches the lower limit of interval (5), i.e.* $\lambda \to \min_{\{j\}} \sum_{i=1, i\neq j}^{S} p(H_i)/p(H_j)$, *and, at* $\alpha \to 0$, *coefficient* $\lambda$ *approaches the upper limit of interval (5), i.e.* $\lambda \to \max_{\{j\}} \sum_{i=1, i\neq j}^{S} p(H_i)/p(H_j)$.

***Proof***. In accordance with the proof of Theorem 5.3, the probability of inequality (6) is equal to $(1-\alpha)$. Hence it follows that, at $\alpha \to 1$, i.e. at $(1-\alpha) \to 0$, coefficient $\lambda$ in (6) must approach its lower limit, which, in accordance with Lemma 5.2, is the value $\min_{\{j\}} \sum_{i=1, i\neq j}^{S} p(H_i)/p(H_j)$. Analogously, at $\alpha \to 0$, i.e. at $(1-\alpha) \to 1$, for guaranteeing the fulfillment of the equality with probability $(1-\alpha)$, coefficient $\lambda$ must approach its upper limit, which, in accordance of Lemma 5.2, is the value $\max_{\{j\}} \sum_{i=1, i\neq j}^{S} p(H_i)/p(H_j)$, as we wished to prove.

□

**Corollary 5.4.** *In the conditions of Theorem 5.3, at* $\alpha \to 1$, *there exist such a value* $\alpha^* \in (0;1)$ *that, if* $\alpha > \alpha^*$, $r_\delta < \alpha$ *holds true; at* $\alpha \to 0$, *there exists such a value* $\alpha_* < \alpha^*$ *(*$\alpha_* \in (0;1)$*) that, if* $\alpha < \alpha_*$, $r_\delta > \alpha$ *holds true; there exists such a value* ${}^*\alpha$ *(*$\alpha_* \leq {}^*\alpha \leq \alpha^*$*) that, at* $\alpha = {}^*\alpha$, $r_\delta = \alpha$ *holds true, and the appropriate value of Lagrange multiplier* $\lambda$ *in decision rule (3) is within interval (5).*

***Proof***. From the proof of Theorem 5.3 the validity of the given corollary is evident.

□

The results of computation of concrete examples are given in Appendix 1 for illustration of the validity of Lemmas 5.1 and 5.2, Corollaries 5.1, 5.2 and 5.3, Theorems 5.2 and 5.3.

## 5.2 Relationship between the probabilities of errors of the first and the second kinds in constrained Bayesian task and in sequential method of Bayesian type

First we consider the case when the number of hypotheses is equal to two, i.e. $S = 2$.

Hereinafter, where this creates no difficulties in understanding, for simplicity, index $m$ to the hypotheses acceptance regions is omitted and the statistics on the basis of which is made the decision is simply designated by $\mathbf{x}$ .

For any sample size $m$ , in sequential method of Bayesian type, the decision is made on the basis of hypotheses acceptance regions (3). Sequential analysis of Bayesian type ends when the matrix of repeated observation results appears in one of hypotheses acceptance regions $\Gamma_1'$ and $\Gamma_2'$ , which have the following form:

$$\Gamma_2' = \left\{ x : \frac{p(\mathbf{x} \mid H_2)}{p(\mathbf{x} \mid H_1)} > A_1 \right\} \text{ and } \Gamma_1' = \left\{ x : \frac{p(\mathbf{x} \mid H_2)}{p(\mathbf{x} \mid H_1)} < B_1 \right\}, \tag{7}$$

where $p(\mathbf{x} \mid H_2) / p(\mathbf{x} \mid H_1)$ is the likelihood function. At $\lambda > 1$, we have:

$$A_1 = \lambda \frac{p(H_1)}{p(H_2)}, \quad B_1 = \frac{1}{\lambda} \frac{p(H_1)}{p(H_2)}, \tag{8}$$

at $\lambda < 1$, we have

$$A_1 = \frac{1}{\lambda} \frac{p(H_1)}{p(H_2)}, \quad B_1 = \lambda \frac{p(H_1)}{p(H_2)},$$

and, at $\lambda = 1$, we have:

$$A_1 = B_1 = \frac{p(H_1)}{p(H_2)}.$$

By $\Gamma_1$ and $\Gamma_2$ are designated hypotheses acceptance regions in constrained Bayesian task corresponding to the statistics of observation results of size $m$ . The following average risk corresponds to this statistics:

$$r_\delta = p(H_1) \int_{\Gamma_2} p(\mathbf{x} \mid H_1) d\mathbf{x} + p(H_2) \int_{\Gamma_1} p(\mathbf{x} \mid H_2) d\mathbf{x}. \tag{9}$$

Let us suppose that $\lambda \geq 1$. Then the hypotheses acceptance regions in sequential method of Bayesian type are determined by ratios (7) and (8). Therefore expression (9) could be written down as follows:

$$r_\delta = p(H_1)[1 - \int_{\overline{\Gamma}_2} p(\mathbf{x} \mid H_1) d\mathbf{x}] + p(H_2)[1 - \int_{\overline{\Gamma}_1} p(\mathbf{x} \mid H_2) d\mathbf{x}] =$$

$$= 1 - p(H_1) \int_{\Gamma_1'} p(\mathbf{x} \mid H_1) d\mathbf{x} - p(H_2) \int_{\Gamma_2'} p(\mathbf{x} \mid H_2) d\mathbf{x} = \beta', \tag{10}$$

where $1 - \beta' = p(H_1) \int_{\Gamma_1'} p(\mathbf{x} \mid H_1) d\mathbf{x} + p(H_2) \int_{\Gamma_2'} p(\mathbf{x} \mid H_2) d\mathbf{x}$ is the average probability of correct decision, i.e. it is the average power of sequential method of Bayesian type.

At $p(H_1) = p(H_2)$ , if $p(\mathbf{x} \mid H_i)$ , $i = 1,2$ , belong to the shift family of probability distribution laws, the validity of the following equality is obvious:

$$\int_{\Gamma_1'} p(\mathbf{x} \mid H_1) d\mathbf{x} = \int_{\Gamma_2'} p(\mathbf{x} \mid H_2) d\mathbf{x} = 1 - \beta'.$$

On the basis of condition (2), we write down:

$$1 - \alpha = p(H_1) \int_{\Gamma_1} p(\mathbf{x} \mid H_1) d\mathbf{x} + p(H_2) \int_{\Gamma_2} p(\mathbf{x} \mid H_2) d\mathbf{x} =$$

$$= p(H_1)[1 - \int_{\overline{\Gamma}_1} p(\mathbf{x} \mid H_1) d\mathbf{x}] + p(H_2)[1 - \int_{\overline{\Gamma}_2} p(\mathbf{x} \mid H_2) d\mathbf{x}] =$$

$$= p(H_1)[1 - \int_{\Gamma_2'} p(\mathbf{x} \mid H_1) d\mathbf{x}] + p(H_2)[1 - \int_{\Gamma_1'} p(\mathbf{x} \mid H_2) d\mathbf{x}] =$$

$$= 1 - p(H_1) \int_{\Gamma_2'} p(\mathbf{x} \mid H_1) d\mathbf{x} - p(H_2) \int_{\Gamma_1'} p(\mathbf{x} \mid H_2) d\mathbf{x} = 1 - \alpha', \tag{11}$$

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where $\alpha' = p(H_1)\int_{\Gamma_2'} p(\mathbf{x} \mid H_1)d\mathbf{x} + p(H_2)\int_{\Gamma_1'} p(\mathbf{x} \mid H_2)d\mathbf{x}$ is the average probability of incorrect rejecting of hypotheses, i.e. the average probability of the first-kind errors for sequential method of Bayesian type.

It is also obvious that, at $p(H_1) = p(H_2)$, if $p(\mathbf{x} \mid H_i)$, $i = 1,2$, belong to the shift family of probability distribution laws, the following equality takes place:

$$\int_{\Gamma_2'} p(\mathbf{x} \mid H_1)d\mathbf{x} = \int_{\Gamma_1'} p(\mathbf{x} \mid H_2)d\mathbf{x} = \alpha' .$$

Let us consider the case $\lambda < 1$. In this case, we have: $\Gamma_1 = \Gamma_1'$ and $\Gamma_2 = \Gamma_2'$. Therefore, the following is true:

$$r_\delta = p(H_1)\int_{\Gamma_2} p(\mathbf{x} \mid H_1)d\mathbf{x} + p(H_2)\int_{\Gamma_1} p(\mathbf{x} \mid H_2)d\mathbf{x} =$$

$$= p(H_1)\int_{\Gamma_2'} p(\mathbf{x} \mid H_1)d\mathbf{x} + p(H_2)\int_{\Gamma_1'} p(\mathbf{x} \mid H_2)d\mathbf{x} = \alpha' . \quad (12)$$

In this case, condition (2) takes the form:

$$1-\alpha = p(H_1)\int_{\Gamma_1} p(\mathbf{x} \mid H_1)d\mathbf{x} + p(H_2)\int_{\Gamma_2} p(\mathbf{x} \mid H_2)d\mathbf{x} =$$

$$= p(H_1)\int_{\Gamma_1'} p(\mathbf{x} \mid H_1)d\mathbf{x} + p(H_2)\int_{\Gamma_2'} p(\mathbf{x} \mid H_2)d\mathbf{x} = 1-\beta' . \quad (13)$$

Let us consider the general case when the number of hypotheses $S$ is arbitrary. In this case, the hypotheses acceptance regions in sequential method of Bayesian type are determined as given in Item 4. Let us designate $\lambda = {}^*\lambda(\mathbf{x}) \in [\lambda_*(\mathbf{x}); \lambda^*(\mathbf{x})]$ (see the end of Item 3 and Corollary 5.4.) for which there takes place: $\Gamma_i \bigcap \Gamma_j = \varnothing$, $i, j = 1,..., S$, $i \neq j$, $\bigcup_{i=1}^{S} \Gamma_i = R^n$. Taking into account this fact, at $\lambda > {}^*\lambda(\mathbf{x})$, the following transformations are true:

$$r_\delta = \sum_{i=1}^{S} p(H_i) \sum_{j=1, j\neq i}^{S} \int_{\Gamma_j} p(\mathbf{x} \mid H_i)d\mathbf{x} = \sum_{i=1}^{S} p(H_i) \sum_{j=1, j\neq i}^{S} \left(1 - \int_{\overline{\Gamma}_j} p(\mathbf{x} \mid H_i)d\mathbf{x}\right) \leq$$

$$\leq \sum_{i=1}^{S} p(H_i) \sum_{j=1, j\neq i}^{S} \left(1 - \int_{\bigcup_{\ell=1,(\ell\neq j)}^{S} \Gamma_\ell^m / I_\ell^m} p(\mathbf{x} \mid H_i)d\mathbf{x}\right) = \sum_{i=1}^{S} p(H_i) \sum_{j=1, j\neq i}^{S} \left(1 - \sum_{\ell=1, \ell\neq j}^{S} \int_{R_{m,\ell}^n} p(\mathbf{x} \mid H_i)d\mathbf{x}\right) =$$

$$= (S-1) - \sum_{i=1}^{S} p(H_i) \sum_{j=1, j\neq i}^{S} \left( \int_{R_{m,i}^n} p(\mathbf{x} \mid H_i)d\mathbf{x} + \sum_{\ell=1, \ell\neq j, \ell\neq i}^{S} \int_{R_{m,\ell}^n} p(\mathbf{x} \mid H_i)d\mathbf{x} \right) . \quad (14)$$

Let us designate: $\alpha'_{m,i}$ is the probability of no acceptance of hypothesis $H_i$ at its validity in sequential method of Bayesian type after $m$ observations; $\beta'_{m,\ell i}$ is the probability of acceptance of hypothesis $H_\ell$ at validity of $H_i$ in sequential method of Bayesian type after $m$ observations; $\alpha_m$ is the average probability of the first kind error and $r_{\delta,m}$ is the value of the risk function in constrained Bayesian task obtained on the basis of $m$ sequential observation results.

Then, on the basis of formula (14), we write down:

$$r_{\delta,m} \leq (S-1) - \sum_{i=1}^{S} p(H_i) \sum_{j=1, j\neq i}^{S} \left( (1-\alpha'_{m,i}) + \sum_{\ell=1, \ell\neq j, \ell\neq i}^{S} \beta'_{m,\ell i} \right) =$$

$$= (S-1) - \sum_{i=1}^{S} p(H_i) \left( (S-1) - (S-1)\alpha'_{m,i} + \sum_{j=1, j\neq i}^{S} \sum_{\ell=1, \ell\neq j, \ell\neq i}^{S} \beta'_{m,\ell i} \right) =$$

$$= (S-1) \sum_{i=1}^{S} p(H_i)\alpha'_{m,i} + \sum_{i=1}^{S} p(H_i) \sum_{j=1, j\neq i}^{S} \sum_{\ell=1, \ell\neq j, \ell\neq i}^{S} \beta'_{m,\ell i} . \quad (15)$$

It is not difficult to be convinced that at $S = 2$ formulae (15) coincide with formulae (11).

Let us find the expression analogous to (11) for $1-\alpha_m$:

$$1-\alpha_m = \sum_{i=1}^{S} p(H_i)\int_{\Gamma_i} p(\mathbf{x}\,|\,H_i)d\mathbf{x} = \sum_{i=1}^{S} p(H_i)\left[1-\int_{\overline{\Gamma}_i} p(\mathbf{x}\,|\,H_i)d\mathbf{x}\right] \le$$

$$\le \sum_{i=1}^{S} p(H_i)\left[1-\int_{\bigcup_{j=1,\,j\ne i}^{S}\Gamma_j^m / I_j^m} p(\mathbf{x}\,|\,H_i)d\mathbf{x}\right] = 1-\sum_{i=1}^{S} p(H_i)\sum_{j=1,\,j\ne i}^{S}\beta'_{m,\,ji}\ . \qquad (16)$$

At $S=2$, expression (16) takes the form which completely corresponds to formula (12).

On the other hand conditions (15) and (16) can be rewritten as follows:

$$r_\delta = \sum_{i=1}^{S} p(H_i)\sum_{j=1,\,j\ne i}^{S}\int_{\Gamma_j} p(\mathbf{x}\,|\,H_i)d\mathbf{x} \ge \sum_{i=1}^{S} p(H_i)\alpha'_{m,i}\ , \qquad (17)$$

and

$$1-\alpha_m = \sum_{i=1}^{S} p(H_i)\int_{\Gamma_i} p(\mathbf{x}\,|\,H_i)d\mathbf{x} \ge \sum_{i=1}^{S} p(H_i)\int_{R_{m,i}^n} p(\mathbf{x}\,|\,H_i)d\mathbf{x} =$$

$$= \sum_{i=1}^{S} p(H_i)\left(1-\alpha'_{m,i}\right) = 1-\sum_{i=1}^{S} p(H_i)\alpha'_{m,i}\ . \qquad (18)$$

Let us introduce the designations:

$$\overline{\alpha}_m = \sum_{i=1}^{S} p(H_i)\alpha'_{m,i}\ ,$$

$$\overline{\beta}_m = \sum_{i=1}^{S} p(H_i)\sum_{j=1,\,j\ne i}^{S}\beta'_{m,\,ji}\ .$$

The quantities $\overline{\alpha}_m$ and $\overline{\beta}_m$ are the average probabilities of errors of the first and the second kinds, respectively, in sequential method of Bayesian type.

Then, on the basis of (15) and (16), we can write down:

$$r_{\delta,m} \le (S-1)\overline{\alpha}_m + (S-2)\overline{\beta}_m\,,$$

$$1-\alpha_m \le 1-\overline{\beta}_m\,.$$

On the other hand, on the basis of (17) and (18), the followings are true:

$$r_{\delta,m} \ge \overline{\alpha}_m \ \text{ and } \ \alpha_m \le \overline{\alpha}_m\,;$$

i.e. finally we have:

$$\alpha_m \le \overline{\alpha}_m \le r_{\delta,m}\,.$$

Hence, for calculation of the average probabilities of errors of the first and the second kinds in sequential method of Bayesian type, we obtain:

$$\overline{\beta}_m \le \alpha_m\,,$$

$$\max\left\{\alpha_m;\frac{r_{\delta,m}-(S-2)\overline{\beta}_m}{(S-1)}\right\} \le \overline{\alpha}_m \le r_{\delta,m}\,. \qquad (19)$$

At $\lambda \le {}^{*}\lambda(\mathbf{x})$, we have:

$$r_{\delta,m} = \sum_{i=1}^{S} p(H_i)\sum_{j=1,\,j\ne i}^{S}\int_{\Gamma_j} p(\mathbf{x}\,|\,H_i)d\mathbf{x} =$$

$$= \sum_{i=1}^{S} p(H_i)\sum_{j=1,\,j\ne i}^{S}\int_{R_{m,j}^n} p(\mathbf{x}\,|\,H_i)d\mathbf{x} = \sum_{i=1}^{S} p(H_i)\sum_{j=1,\,j\ne i}^{S}\beta'_{m,\,ji}\ . \qquad (20)$$

and

$$1-\alpha_m = \sum_{i=1}^{S} p(H_i)\int_{\Gamma_i} p(\mathbf{x}\,|\,H_i)d\mathbf{x} = \sum_{i=1}^{S} p(H_i)\int_{R_{m,i}^n} p(\mathbf{x}\,|\,H_i)d\mathbf{x} =$$

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$$= \sum_{i=1}^{S} p(H_i)\left(1-\alpha'_{m,i}\right) = 1 - \sum_{i=1}^{S} p(H_i)\alpha'_{m,i} \; . \qquad (21)$$

It is not difficult to be convinced that at $S = 2$ formulae (20) and (21) coincide with formulae (12) and (13), respectively.

On the basis of (20) and (21), for the average probabilities of errors of the first and the second kinds in sequential method of Bayesian type, we obtain:

$$r_{\delta,m} = \overline{\beta}_m \, ,$$

$$\alpha_m = \overline{\alpha}_m \, . \qquad (22)$$

In formulae (19) and (22), $r_{\delta,m}$ and $\alpha_m$ are the values of average risk and significance level of the criterion, respectively, in constrained Bayesian task as a result of its solution after obtaining the next, $m$ th observation result. The ratio between $r_{\delta,m}$ and c indicates which formulae (19) or (22) must be used for the estimation of $\overline{\alpha}_m$ and $\overline{\beta}_m$ for the sequential method of Bayesian type (see Item 5.3).

If it is necessary to know not average but all probabilities of errors of the first and the second kinds, we can act as follows. After testing hypotheses in the sequential method of Bayesian type on the basis of $m$ sequential observation results, for already determined value $\lambda$, we can calculate probabilities $\alpha'_{m,i}$ and $\beta'_{m,\ell i}$, $i,\ell = 1,...,S$, $\ell \neq i$, for example, by the Monte-Carlo method.

It is clear that, if $r_{\delta,m} \le \alpha_m$ the offered sequential method of Bayesian type rigorously surpasses any other sequential method with errors of the first and second kinds $\alpha$ and $\beta$, satisfied the following condition

$$\overline{\alpha}_m \le \min\{\alpha, \beta\} \, . \qquad (23)$$

In particular, for the number of hypotheses equal to two, if $J^*(H_1, H_2)$ is such divergence between the hypotheses that there takes place $r_{\delta,m} = \alpha_m$ then for any other hypotheses $H'_1$ and $H'_2$ for which the following $J(H'_1, H'_2) \ge J^*(H_1, H_2)$ is true the sequential method of Bayesian type rigorously surpasses the Wald method with errors of the first and the second kinds $\alpha$ and $\beta$ satisfied the condition (23).

## 5.3 Relations between the probabilities of errors of the first and the second kinds in constrained Bayesian task after obtaining m sequential observation results

In constrained Bayesian task, for any sample size $m$, after testing hypotheses, the average probabilities of errors of the first and the second kinds are calculated as follows:

$$r_{\delta,m} = \sum_{i=1}^{S} p(H_i) \sum_{j=1, j\neq i}^{S} \int_{\Gamma_j} p(\mathbf{x} \mid H_i) d\mathbf{x}, \, ,$$

$$\sum_{i=1}^{S} p(H_i) \int_{\Gamma_i} p(\mathbf{x} \mid H_i) d\mathbf{x} = 1 - \alpha_m \, .$$

As was mentioned above, at $\lambda = {}^*\lambda(\mathbf{x})$, $\Gamma_i \bigcap \Gamma_j = \varnothing$, $i, j = 1,..., S$, $i \neq j$, $\bigcup_{i=1}^{S} \Gamma_i = R^n$ holds true. Therefore, the following is true:

$$\sum_{j=1, j\neq i}^{S} \int_{\Gamma_j} p(\mathbf{x} \mid H_i) d\mathbf{x} = 1 - \int_{\Gamma_i} p(\mathbf{x} \mid H_i) d\mathbf{x} \, .$$

Then, for the value of the risk function, we obtain:

$$r_{\delta,m} = \sum_{i=1}^{S} p(H_i)\left(1 - \int_{\Gamma_i} p(\mathbf{x} \mid H_i) d\mathbf{x}\right) = 1 - \sum_{i=1}^{S} p(H_i) \int_{\Gamma_i} p(\mathbf{x} \mid H_i) d\mathbf{x} = \alpha_m \qquad (24)$$

At $\lambda > {}^*\lambda(\mathbf{x})$, the following takes place:

$$\sum_{j=1, j\neq i}^{S} \int_{\Gamma_j} p(\mathbf{x} \mid H_i) d\mathbf{x} > 1 - \int_{\Gamma_i} p(\mathbf{x} \mid H_i) d\mathbf{x} \, .$$

Therefore the following inequality is true:

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$$r_{\delta,m} > \sum_{i=1}^{S} p(H_i)\left(1-\int_{\Gamma_i} p(\mathbf{x}\,|\,H_i)d\mathbf{x}\right) = 1-\sum_{i=1}^{S} p(H_i)\int_{\Gamma_i} p(\mathbf{x}\,|\,H_i)d\mathbf{x} = \alpha_m \quad (25)$$

i.e. $r_{\delta,m} > \overline{\beta}_m$ .

At $\lambda < {}^{*}\lambda(\mathbf{x})$ , we have:

$$\sum_{j=1,\, j\neq i}^{S} \int_{\Gamma_j} p(\mathbf{x}\,|\,H_i)d\mathbf{x} < 1-\int_{\Gamma_i} p(\mathbf{x}\,|\,H_i)d\mathbf{x} .$$

Therefore the following inequality is true:

$$r_{\delta,m} < \alpha_m \, , \quad (26)$$

i.e. $r_{\delta,m} < \overline{\alpha}_m$ .

In accordance with Lemma 5.2, at infinitely decreasing divergence $J(H_i, H_j)$ , between tested hypotheses $H_i$ and $H_j$ , $i, j = 1,..., S$ ; $i \neq j$ , i.e. at $\max J(H_i, H_j) \to 0$ , the interval of definition of Lagrange multiplier reduces to finite interval (5) which, at the absence of a priori information, i.e. at $p(H_i) = 1/S$ , $i = 1,..., S$ , degenerates into point $\lambda = (S-1)$ . According to corollary 5.4, if the value of $\lambda$ changes in this interval, the above-stated ratios between $r_{\delta,m}$ and $\alpha_m$ remain valid, i.e. in interval (5), there is such a value that, if $\lambda$ is more than this value, there takes place (25) and, if $\lambda$ is less than this value, inequality (26) is fulfilled.

As in sequential method of Bayesian type after obtaining every next observation result, constrained Bayesian task is solved for all observation results having obtained by the current moment, depending on the value of $\lambda$ , ratios (24), (25) or (26) between $r_{\delta,m}$ and $\alpha_m$ , i.e. between the values of the risk function and the average probability of rejection of true hypothesis, in constrained Bayesian task solved on the basis of $m$ sequential observation results, remain true.

Using these values of $r_{\delta,m}$ and $\alpha_m$ , with the help of ratios (19) or (25), depending on the value of $\lambda$ , there the values of average probabilities of errors of the first and the second kinds in sequential method of Bayesian type are calculated.

## 6 Experimental research

For illustrating the correctness and the quality of the offered *sequential analysis method of Bayesian type* in practice, the computation results of five examples for the cases when sequentially accepted observation results are normally distributed independent random variables are given in [18]. For showing of practical applicability and usefulness of the results of Items 5.2 and 5.3, the computation results for two latest examples of the mentioned work are given below.

**Example 1** [18]**.** Tested hypotheses: $H_1 : \theta_1^1 = 1, \theta_2^1 = 1, \theta_3^1 = 1$ , $H_2 : \theta_1^2 = 4, \theta_2^2 = 4, \theta_3^2 = 4$ and $H_3 : \theta_1^3 = 8, \theta_2^3 = 8, \theta_3^3 = 8$ . A priori probabilities of hypotheses: $p(H_1) = 1/3$ , $p(H_2) = 1/3$ , $p(H_3) = 1/3$ . The significance level of the criterion in constrained Bayesian task $\alpha = 0.05$ . The parameters of sequentially incoming observation results as a three-dimensional normally distributed random vector with the mathematical expectation $\mathbf{\theta} = (4,4,4)$ and the covariance matrix

$$\mathbf{W} = \begin{pmatrix} 10 & 9 & 8 \\ 9 & 10 & 9 \\ 8 & 9 & 10 \end{pmatrix}.$$

The average probabilities of errors of the first and the second kinds in sequential method of Bayesian type at hypotheses testing are calculated by formulae (19) and are equal to: on the basis of two observations - $\overline{\beta}_m \le 0.05$ and $0.2045 \le \overline{\alpha}_m \le 0.459$ ( $\lambda = 2.42$ ); on the basis of three observations - $\overline{\beta}_m \le 0.05$ and $0.1015 \le \overline{\alpha}_m \le 0.253$ ( $\lambda = 1.98$ ) and on the basis of four observations - $\overline{\beta}_m \le 0.05$ and $0.05 \le \overline{\alpha}_m \le 0.131$ ( $\lambda = 1.51$ ), respectively.

For five, six and seven observation results by computation we have obtained: $\overline{\beta}_m \le 0.05$ , $0.05 \le \overline{\alpha}_m \le 0.06817$ ; $\overline{\beta}_m = 0.041833$ , $\overline{\alpha}_m = 0.045$ ( $r_{\delta,m} < \alpha_m$ , $\lambda < 1$ ) and $\overline{\beta}_m = 0.028967$ , $\overline{\alpha}_m = 0.034$ ( $r_{\delta,m} < \alpha_m$ , $\lambda < 1$ ), respectively.

**Example 2** [18]**.** Tested hypotheses: $H_1 : \theta_1^1 = 1, \theta_2^1 = 1$ , $H_2 : \theta_1^2 = 4, \theta_2^2 = 4$ , $H_3 : \theta_1^3 = 8, \theta_2^3 = 8$ and $H_4 : \theta_1^4 = 12, \theta_2^4 = 12$ . A priori probabilities of hypotheses: $p(H_1) = 1/4$ , $p(H_2) = 1/4$ , $p(H_3) = 1/4$ , $p(H_4) = 1/4$ . The significance level of the criterion in constrained Bayesian task is $\alpha = 0.05$ . The parameters of sequentially incoming observation results as a twodimensional normally distributed random vector with the mathematical expectation $\boldsymbol{\theta}=(4,4)$ and the covariance matrix $\mathbf{W}=\begin{pmatrix}10 & 9\\ 9 & 10\end{pmatrix}$.

The average probabilities of errors of the first and the second kinds in sequential method of Bayesian type at hypotheses testing are calculated by formulae (19) and are equal to: on the basis of two observations - $\bar{\beta}_m \le 0.05$ and $0.128 \le \bar{\alpha}_m \le 0.484$ ($\lambda = 2.455$); on the basis of three observations - $\bar{\beta}_m \le 0.05$ and $0.052 \le \bar{\alpha}_m \le 0.256$ ($\lambda = 1.925$) and on the basis of four observations - $\bar{\beta}_m \le 0.05$ and $0.05 \le \bar{\alpha}_m \le 0.126$ ($\lambda = 1.465$), respectively.

For five and six observation results by computation we have obtained: $\bar{\beta}_m \le 0.05$, $0.05 \le \bar{\alpha}_m \le 0.05915$ and $\bar{\beta}_m = 0.034$, $\bar{\alpha}_m = 0.045$ ($r_{\delta,m} < \alpha_m$, $\lambda = 0.890305 < 1$), respectively.

## 8 Conclusion

New results of investigation of *the sequential analysis method of Bayesian type* is offered in the work. In particular, some relations between the errors of the first and the second kinds depending on the divergence between the tested hypotheses are given. Also dependences of the Lagrange multiplier and risk function on the probability of incorrectly accepted hypotheses are presented. Relationship between the probabilities of errors of the first and the second kinds in constrained Bayesian task and in sequential method of Bayesian type and relations between the probabilities of errors of the first and the second kinds in constrained Bayesian task after obtaining $m$ sequential observation results are given. Computation results of the concrete examples completely confirm the rightness of theoretical researches.

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## References


[1] Armitage, P., 1950. Sequential analysis with more than two alternative hypotheses, and its relation to discriminant function analysis. Journal of the Royal Statistical Society Series B 12, 137–144.

[2] Bartroff, J., Lai, T.L., 2010. Multistage tests of multiple hypotheses. Communications in Statistics - Theory and Methods 39, 1597–1607.

[3] Baum, C.W., Veeravalli, V.V., 1994. A sequential procedure for multihypothesis testing. IEEE Transactions on Information Theory 40, 1994–2007.

[4] Betensky, R.A., 1996. An O'Brien–Fleming sequential trial for comparing three treatments. Annals of Statistics 24 (4), 1765–1791.

[5] Darkhovsky, B. (2011). Optimal Sequential Tests for Testing Two Composite and Multiple Simple Hypotheses, *Sequential Analysis* 30: 479–496.

[6] De, S., Baron M., 2012. Step-up and step-down methods for testing multiple hypotheses in sequential experiments. Journal of Statistical Planning and Inference 142, 2059–2070.

[7] De, S., Baron, M. Sequential Bonferroni methods for multiple hypothesis testing with strong control of familywise error rates I and II. Sequential Analysis, in press.

[8] Dragalin, V.P., Tartakovsky, A.G., Veeravalli, V.V., 1999. Multi hypothesis sequential probability ratio tests. Part I: asymptotic optimality. IEEE Transactions on Information Theory 45 (7), 2448–2461.

[9] Edwards, D., 1987. Extended-Paulson sequential selection. Annals of Statistics 15 (1), 449–455.

[10] Edwards, D.G., Hsu, J.C., 1983. Multiple comparisons with the best treatment. Journal of the American Statistical Association 78, 965–971.

[11] Ghosh, M., Mukhopadhyay, N. and Sen, P.K., 1997. Sequential Estimation. Wiley, New York.

[12] Glimm, E., Maurer, W., Bretz, F., 2010. Hierarchical testing of multiple endpoints in group-sequential trials. Statistics in Medicine 29, 219–228.

[13] Govindarajulu, Z., 2004. Sequential Statistics. World Scientific Publishing, Co., Singapore.

[14] Hughes, M.D., 1993. Stopping guidelines for clinical trials with multiple treatments. Statistics in Medicine 12, 901–915.

[15] Jennison, C., Johnstone, I.M., Turnbull, B.W., 1982. Asymptotically optimal procedures for sequential adaptive selection of the best of several normal means. In: Gupta, S.S., Berger, J.O. (Eds.), Statistical Decision Theory and Related Topics III, vol. 2. , Academic Press, New York, pp. 55–86.

[16] Jennison, C., Turnbull, B.W., 1993. Group sequential tests for bivariate response: interim analyses of clinical trials with both efficacy and safety endpoints. Biometrics 49, 741–752.

[17] Jennison, C., Turnbull, B.W., 2000. Group Sequential Methods with Applications to Clinical Trials. Chapman & Hall, Boca Raton, FL.

[18] Kachiashvili, K.J. (2014) The Methods of Sequential Analysis of Bayesian Type for the Multiple Testing Problem. *Sequential Analysis*, 33: 23-38. DOI: 10.1080/07474946.2013.843318

[19] Kachiashvili K.J. and Hashmi M.A. (2010) About Using Sequential Analysis Approach for Testing Many Hypotheses. *Bulletin of the Georgian Academy of Sciences*, vol. 4, no. 2, 20-25.

[20] Kachiashvili, K.J., Hashmi, M. A. & Mueed, A. (2012). Sensitivity Analysis of Classical and Conditional Bayesian Problems of Many Hypotheses Testing. *Communications in Statistics—Theory and Methods*, Volume 41, Issue 4, 591–605.

[21] Kachiashvili, G.K., Kachiashvili, K.J. & Mueed, A. (2012) Specific Features of Regions of Acceptance of Hypotheses in Conditional Bayesian Problems of Statistical Hypotheses Testing. *Sankhya, Series A*, Vol. 74 A1.

[22] Kachiashvili, K.J. & Mueed, A. (2013) Conditional Bayesian Task of Testing Many Hypotheses, *Statistics*, DOI:10.1080/02331888.2011.602681.

[23] Kendall, M. and Stuart, A. (1970). *The Advanced Theory of Statistics*. Vol. 2. - Charles Griffin & Company Limited, London. p. 221.

[24] Lai, T.L., 2000. Sequential multiple hypothesis testing and efficient fault detection-isolation in stochastic systems. IEEE Transactions on Information Theory 46 (2), 595–608.

[25] Maurer, W., Glimm, E., Bretz, F., 2011. Multiple and repeated testing of primary, coprimary, and secondary hypotheses. Statistics in Biopharmaceutical Research 3 (2), 336–352.

[26] Novikov, A., 2009. Optimal sequential multiple hypothesis tests. Kybernetika 45 (2), 309–330.

[27] Paulson, E., 1962. A sequential procedure for comparing several experimental categories with a standard or control. Annals of Mathematical Statistics 33, 438–443.

[28] Paulson, E., 1964. A sequential procedure for selecting the population with the largest mean from k normal populations. Annals of Mathematical Statistics 35, 174–180.

[29] O'Brien, P.C., 1984. Procedures for comparing samples with multiple end points. Biometrics 40, 1079–1087.

[30] O'Brien, P.C., Fleming, T.R., 1979. A multiple testing procedure for clinical trials. Biometrika 35, 549–556.

[31] Siegmund, D., 1985. Sequential Analysis: Tests and Confidence Intervals. Springer-Verlag, New York.

[32] Siegmund, D., 1993. A sequential clinical trial for comparing three treatments. Annals of Statistics 21, 464–483.

[33] Simons, G., 1967. Lower bounds for the average sample number of sequential multihypothesis tests. Annals of Mathematical Statistics 38 (5), 1343–1364.

[34] Tamhane, A.C., Mehta, C.R., Liu, L., 2010. Testing a primary and a secondary endpoint in a group sequential design. Biometrics 66, 1174–1184.

[35] Tang, D.-I., Geller, N.L., 1999. Closed testing procedures for group sequential clinical trials with multiple endpoints. Biometrics 55, 1188–1192.

[36] Wilcox, R.R., 2004. Extension of Hochberg's two-stage multiple comparison method. In: Mukhopadhyay, N., Datta, S., Chattopadhyay, S. (Eds.), Applications of Sequential Methodologies, Marcel Dekker, Inc., New York, pp. 371–380.

[37] Zacks, S., 2009. Stage-Wise Adaptive Designs. Wiley, Hoboken, NJ.

***Appendix 1.*** The results of computations for constrained Bayesian task, for the case when the number of hypotheses is equal to three, are given in Table A.1 for illustration of the validity of Lemmas 5.1 and 5.2, Corollaries 5.1, 5.2 and 5.3, Theorems 5.2 and 5.3. The values of coefficient $\lambda$ are computed depending on the change of divergence between the tested hypotheses. The following covariance matrices

$$\mathbf{W}_1=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix},\ \mathbf{W}_2=\begin{pmatrix}50 & 20\\ 20 & 50\end{pmatrix},\ \mathbf{W}_3=\begin{pmatrix}100 & 20\\ 20 & 100\end{pmatrix}\ \text{and}\ \mathbf{W}_4=\begin{pmatrix}0.1 & 0\\ 0 & 0.1\end{pmatrix}$$

are used for all computed cases.

The value of risk function (1) is designated by $r$; $\alpha$ is the value of probability in the restriction (2); $\lambda$ is the value of the Lagrange coefficient in solution (3).

**Table A.1. The computation results of hypotheses testing for Appendix 1.**

| Tested Hypotheses | | | Measurement results | Covariance matrix | A priori probabilities | | | Significance level | Risk | Lagrange multiplier | Accepted hypothesis |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $H_1$ | $H_2$ | $H_3$ | $\mathbf{x}$ | $\mathbf{W}$ | $p(H_1)$ | $p(H_2)$ | $p(H_3)$ | $\alpha$ | $r$ | $\lambda$ | $H_i$ |
| (1;1) | (1.01; 1.01) | (1.02; 1.02) | (1.1; 1.1) | $\mathbf{W}_1$ | 1/3 | 1/3 | 1/3 | 0.05 | 1.898 | 2.029078 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.00001 | 2 | 2.099144 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.6 | 0.782533 | 1.999936 | $(H_2, H_3)$ |
| | | | | | 0.25 | 0.25 | 0.5 | 0.05 | 1.84635 | 3.028105 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.00001 | 1.9996 | 3.1089551 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.76 | 0.24065 | 0.999435 | $(H_3)$ |
| | | | | $\mathbf{W}_2$ | 1/3 | 1/3 | 1/3 | 0.05 | 1.899133 | 2.006165 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.00001 | 1.999867 | 2.01568 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.67 | 0.6584 | 1.999945 | $(H_3)$ |
| | | | | | | | | 0.5 | 1.075533 | 1.999999 | $(H_2, H_3)$ |
| | | | | | 0.25 | 0.25 | 0.5 | 0.05 | 1.8496 | 3.006009 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.00001 | 2 | 3.027723 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.74 | 0.2573 | 1.000084 | $(H_3)$ |
| | | | | | | | | 0.4 | 0.802325 | 2.997474 | $(H_3)$ |
| | | | | $\mathbf{W}_3$ | 1/3 | 1/3 | 1/3 | 0.05 | 1.8982 | 2.005111 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.00001 | 1.9998 | 2.014145 | $(H_1, H_2, H_3)$ |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | 0.67 | 0.659667 | 1.999956 | $(H_3)$ |
| | | | | | | | | 0.5 | 1.0978 | 2.00 | $(H_2, H_3)$ |
| | | | | | 0.25 | 0.25 | 0.5 | 0.05 | 1.8501 | 3.004886 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.00001 | 1.99985 | 3.020386 | $(H_1, H_2, H_3)$ |
| | | | | | | | | 0.75 | 0.24875 | 1.00003 | $(H_3)$ |
| | | | | | | | | 0.4 | 0.800025 | 2.997896 | $(H_3)$ |
| (1;1) | (4;4) | (8;8) | (4.1; 4.1) | $\mathbf{W}_4$ | 1/3 | 1/3 | 1/3 | 0.05 | 0 | $2.51 \cdot e^{-19}$ | $(H_2)$ |
| | | | | | | | | 0.00001 | 0 | $2.51 \cdot e^{-19}$ | $(H_2)$ |
| | | | | | | | | 0.99 | 0 | $2.51 \cdot e^{-19}$ | $(H_2)$ |
| | | | | | 0.25 | 0.25 | 0.5 | 0.05 | 0 | $2.51 \cdot e^{-19}$ | $(H_2)$ |
| | | | | | | | | 0.00001 | 0 | $1.00 \cdot e^{-18}$ | $(H_2)$ |
| | | | | | | | | 0.99 | 0 | $2.51 \cdot e^{-19}$ | $(H_2)$ |

The correctness of Lemmas 5.1 and 5.2, Corollaries 5.1, 5.2, 5.3 and 5.4, also Theorems 5.2 and 5.3 is obvious from the results of Table A.1.

It is clear that when $p(H_i) = 1/3$, $i = 1,2,3$, we have

$$\min_{\{j\}} \sum_{i=1, i \neq j}^{S} p(H_i) / p(H_j) = \max_{\{j\}} \sum_{i=1, i \neq j}^{S} p(H_i) / p(H_j) = 2$$

and when $p(H_1) = 0.25$, $p(H_2) = 0.25$ and $p(H_3) = 0.5$ we have

$$\min_{\{j\}} \sum_{i=1, i \neq j}^{S} p(H_i) / p(H_j) = 1 \text{ and } \max_{\{j\}} \sum_{i=1, i \neq j}^{S} p(H_i) / p(H_j) = 3 .$$