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Equivalent Identities on Semirings

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Abstract: In this paper mainly we have obtained equivalent conditions on semirings, regular semirings and Idempotent semirings.

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Introduction:

M.K.Sen, Y.Q.Guo and K.P.Shum proved identities on Idempotent semirings in their paper entitled "A Class of Idempotent semirings" [1]. In this paper we obtained certain Identities on regular semirings and also we have obtained certain equivalences on semirings, regular semirings, and Idempotent semirings. We also obtained results such as $x + x \approx x$,

$$xy + yx + xy \approx xy \,,\; y + yx + y \approx y,\; xyx + xy + xyx \approx xyx,$$

$$yxy + xyx + yxy \approx yxy,\; x + xyx + x \approx x$$

Using the identities $x^2 \approx x$,

$$x + xy + x \approx x,\; x + yx + x \approx x.$$

Also by using Greens relation (i.e, $\mathfrak{L}^+$, $\mathfrak{R}^+$, $\mathfrak{D}^+$). We have obtained results for

$$a + ab + a = a \,,\; a + ba + a = a \,,\; b + ab + b = b,\; ba + b + ba = ba.$$

Besides this we have also proved the equivalent conditions for idempotent semirings like,

$$x^2 \approx x \,;\; x + xy + x \approx x \Leftrightarrow x + x \approx x \,;\; y + yx + y \approx y,$$

$$xyx + xy + xyx \approx xyx \Leftrightarrow xy + yx + xy \approx xy,$$

$$xy + x + xy \approx xy \,;\; y + xy + y \approx y \Leftrightarrow (x + y)(y + x)(x + y) \approx xy.$$

First we start with the following Preliminaries

Def 1: <u>Semiring</u> :- A semiring $(S,+,\,.)$ is an algebra with two binary operations $+$ and $.$ Such that both the additive reduct $(S,+)$ and the multiplicative reduct $(S,\,.)$ are semigroups and such that the following distributive law holds,

$$x(y + z) \approx xy + xz \,,\; (x + y)z \approx xz + yz.$$

Def 2: <u>Invertible element in a semiring</u> :- An element 'a' in a semiring 'S' is said to be invertible if there exists an element 'x' in 'S', such that

(a) $axa = a$; $xax = x$. (b) $a+x+a = a$; $x+a+x = x$.

In this 'a' and 'x' are called invertible elements in 'S'.

Theorem 1:- A semiring which satisfies the identities

$$x^2 \approx x \,,\; x + xy + x \approx x \,,\; x + yx + x \approx x \,,$$

then it satisfies $x + x \approx x$, $xy + yx + xy \approx xy$, $yx + xy + yx \approx yx$.

Proof :- Let S be a semiring which satisfies the above identities.

Then for $a,b \in S$,

$$(a+a) = (a+a)^2 = (a+a)\,(a+a) = a(a+a) + a(a+a) = a.a+a.a+a.a+a.a = a+a+a+a = a+a = a$$

and $(ab + ba + ab) = (ab + ba + ab)^2$

$$= ab\,(ab + ba + ab) + ba\,(ab + ba + ab) + ab\,(ab + ba + ab)$$

$$= (ab + aba + ab) + (bab + (bab)a + bab) + (ab + (ab)a) + ab)$$

$$= ab + b(ab) + ab = ab.$$

also $(ba + ab + ba) = (ba + ab + ba)^2$

$$= (ba + ab + ba)(\,ba + ab + ba)$$

$$= ba\,(ba + ab + ba) + ab\,(ba + ab + ba) + ba\,(ba + ab + ba)$$

$$= ba.ba + ba.ab + ba.ba + ab.ba + ab.ab + ab.ba + ba.ba + ba.ab + ba.ba$$

$= (ba + bab + ba) + (aba + ab + aba) + (ba + bab + ba)$

$= ba + aba + ba = ba.$

Remark:- From the above theorem it is observed that ‘ab’ and ‘ba’ are invertibe elements in ‘S’.

Lemma 2:- ‘If ‘S’ is a semiring which satisfies (i) $a^2 = a$; (ii) $a+ab+a = a$, $a+ba+a = a$, then ‘ab’ is invertible in ‘S’.

Proof:- By using theorem1, we have $ab + ba + ab = ab$ and $ba + ab + ba = ba$. S’ is a semiring which satisfies (i) $a^2 = a$; (ii) $a+ab+a = a$, $a+ba+a = a$,

then ‘ab’ is invertible in ‘S’.

Now, $ab.ba.ab = ab$ and $ba.ab.ba = ba$.

$ab.ba.ab = ab.ab = ab$. Also $ba.ab.ba = ba.ba = ba$.

hence ‘ab’ is invertible is ‘S’.

Def 3: **<u>Regular element in a semiring in ‘S’</u>** :- An element $a \in S$ is said to be regular if there exists $x \in S$ such that $axa = a$ and $a+x+a = a$. S’ is regular, If every element in ‘then ‘S’ is called regular semiring.

Theorem 3:- In a semiring $(S,+,.)$ if the following identities hold

(a) $x^2 \approx x$, (b) $x + xy + x \approx x$, (c) $x + yx + x \approx x$, then the following holds

(1) $x + x \approx x$, (2) $xy + yx + xy \approx xy$, (3) $yx + xy + yx \approx yx$, (4) $xyx + xy + xyx \approx xyx$

(5) $y + yx + y \approx y$, (6) $y + xy + y \approx y$, (7) $yxy + yx + yxy \approx yxy$

(8) $yxy + xyx + yxy \approx yxy$, (9) $xyx + yxy + xyx \approx xyx$, (10) $x + xyx + x \approx x$,

(11) $y + yxy + y \approx y$, (12) $x + yxy + x \approx x$, (13) $y + xyx + y \approx y$.

Proof:- proofs of (1), (2) and (3) are clear from theorem-1

(4) we have,

$(aba + ab + aba) = (aba + ab + aba)^2 = (aba + ab + aba) (aba + ab + aba)$

$= aba.aba + aba.ab + aba.aba + ab.aba + ab.ab + ab.aba + aba.aba + aba.ab + aba.aba$

$= aba + ab + aba + aba + ab + aba +aba + ab + aba$

$= aba + ab.ab + aba + aba + ab.ab + aba + aba + ab.ab + aba$

$= aba + aba + aba = aba + aba = aba.$

(5) $y + yx + y \approx y$ as $b + ba + b = (b + ba + b)(b + ba + b) = b.$

(6) $y + xy + y \approx y$ as $b + ab + b = (b + ab + b)(b + ab + b) = b.$

(7) $yxy + yx + yxy \approx yxy$

$bab + ba + bab = (bab + ba + bab)(bab + ba + bab)$

$= bab.bab + bab.ba + bab.bab + ba.bab + ba.ba + ba.bab$

$+ bab.bab + bab.ba + bab.bab$

$= bab + (bab)a + (bab) + (bab) + ba + (bab) + babab + baba + bab$

$= bab + ba +bab + bab + (bab)a + bab$

$= bab + ba + bab = bab.$

(8) $yxy + xyx + yxy \approx yxy$

$bab + aba + bab = (bab + aba + bab)(bab + aba + bab)$

$= bab.bab + bab.aba + bab.bab + aba.bab + aba.aba + aba.bab$

$+ bab.bab + bab.aba + bab.bab$

$= bab + (bab)a + (bab) + ab + (ab)a + ab + bab + (bab)a + (bab)$

$= bab + ab + bab = bab.$

(9) $xyx + yxy + xyx \approx xyx$

$aba + bab + aba = (aba + bab + aba)(aba + bab + aba)$

$= aba.aba + aba.bab + aba.aba + bab.aba + bab.bab + bab.aba$

$+ aba.aba + aba.bab + aba.aba$

$= aba + (aba)b + aba + ba + (ba)b + (ba) + aba + (aba)b + aba$

$= aba + ba + aba = aba + ba.ba + aba = aba + b(aba) + aba = aba,$

as $x + yx + x \approx x$.

(10) $x + xyx + x \approx x$ as

$a + aba + a = (a + aba + a)(a + aba + a)$

$= a + aba + a + aba + aba + a + aba + a$

$= a + a(ba) + a + aba + a + a(ba) + a$

$= a + a(ba) + a + a(ba) + a = a + a(ba) + a = a$

(11) $y + yxy + y \approx y$

$b + bab + b = (b + bab + b)(b + bab + b)$

$= b + b(ab) + b + bab + babab + bab + b + b(ab) + b$

$= b + bab + bab + b + b(ab) + b$

$= b + b(ab) + b + b(ab) + b = b + b(ab) + b = b$

(12) $x + yxy + x \approx x$ as

$a + bab + a = (a + bab + a)(a + bab + a)$

$= a + ab + a + ba + ba + ba + a + ab + a$

$= a + ba + a + ab + a = a + ab + a = a.$

(13) $y + xyx + y \approx y$

$b + aba + b = (b + aba + b)(b + aba + b)$

$= b + ba + b + ab + aba + ab + b + ba + b$

$= b + ab + aba + ab + b + ba + b$

$= b + ab + (ab)a + ab + b + ba + b$

$= b + ab + b + ba + b = b + ab + b = b$

Now we have obtained certain results on regular semirings using Green's equivalences $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{D}$

$(a,b) \in \mathfrak{L}^+ \Leftrightarrow a + b = a ; b + a = b.$

$(a,b) \in \mathfrak{R}^+ \Leftrightarrow a + b = b ; b + a = a$

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$(a,b) \in \mathfrak{D}^{+} \Leftrightarrow a + b + a = a \,;\, b + a + b = b.$

Lemma 4:- In a regular semiring 'S' if for $a,b \in S$ with $(a,ab) \in \mathcal{L}^{+}$, then $ab + a + ab = ab$,

as $(a,ab) \in \mathcal{L}^{+} \Rightarrow a + ab = a \,;\, ab + a = ab.$

Now, $ab + a = ab + a + ab = ab$, and also $a + ab + a = a$

Lemma 5:- In a regular semiring 'S' if for $a,b \in S$ with $(a,ba) \in \mathcal{L}^{+}$, then $ba + a + ba = ba$.

as $(a,ab) \in \mathcal{L}^{+} \Rightarrow a + ba = a \,;\, ba + a = ba.$

Now, $ba + a = ba + a + ba = ba$, and also $a + ba + a = a$

Lemma 6:- In a regular semiring 'S'if for $a,b \in S$ with $(b,ab) \in \mathcal{L}^{+}$, then $b + ab = b$,

$ab + b = ab.$ $ab + b + ab = ab$ and also $b + ab + b = b.$

hence $(b,ab) \in \mathcal{L}^{+} \Rightarrow ab + b + ab = ab.$

Lemma 7:- In a regular semiring 'S' if for $a,b \in S$ with $(b,ba) \in \mathcal{L}^{+}$, then $b + ab = b$,

$ba + b = ba.$ $ba + b = ba + b + ba = ba.$

Hence $ba + b + ba = ba$, and $b + ba + b = ba.$

From the above it is observed that the relation '$\mathcal{L}$' is an equivalence relation which is compatiable with multiplication. It is also observed that every element in '$\mathcal{L}$' has an inverse in 'S'

Theorem 8:- In a semiring $(S,+,\,.)$ the following are equivalent,

(1) $x+xy+x \approx x \,;\, xy + x + xy \approx xy.$

(2) D^{+} is the least distributive congruence.

$xy + yx + xy \approx xy.$

Proof :- $(1) \Rightarrow (2)$, $ab + ba + ab = aba + (aba)b + aba = aba = ab.$

Also, $ab + ba + ab = (aba)b + (aba) + (aba)b = (aba)b = ab.$

Or

$(1) \Rightarrow (2)$, as, $a + ab + a = a, \Rightarrow a \,\mathcal{L}^{+} ab.$

and $ab + a + ab = ab$ and also $a + ab = ab$ and $ab + a = a. \Rightarrow a \,\mathfrak{R}^{+} ab.$

hence $a \,\mathfrak{D}^{+} ab \Rightarrow ba \,\mathfrak{D}^{+} ab \Rightarrow ab + ba + ab = ab; ba + ab + ba = ba.$

$(2) \Rightarrow (1)$, $ab + ba + ab = ab \Rightarrow ab \,\mathfrak{D}^{+}\, ba$,

Claim :- $a + ab + a = a \,;\, ab + a + ab = ab$

Since $ab \,\mathfrak{D}^{+}\, ba \Rightarrow a.\, ab \,\mathfrak{D}^{+}\, aba \Rightarrow ab \,\mathfrak{D}^{+}\, aba \Rightarrow ab + aba + ab = ab.$

Hence $a + ab + a = a.$

Now claim :- $ab + a + ab = ab.$

Since $ab \,\mathfrak{D}^{+}\, ba$

$\Rightarrow a.\, ab \,\mathfrak{D}^{+}\, aba \Rightarrow ab \,\mathfrak{D}^{+}\, aba \Rightarrow ab + aba + ab = ab \Rightarrow abab + aba + abab = abab$

$\Rightarrow (aba)b + aba + (aba)b = (aba)b.$

hence $(2) \Rightarrow (1)$ holds.

hence (1) and (2) are equivalent.

Def 4:- A Semiring with set of Idempotents is called a **band**.

Def 5:- A commutative semiring with set of Idempotents is called a **semilattice**.

Lemma 9:- In a commutative semiring 'S', the following holds

(1) $xy + x + xy \approx x$, as

$$ab + a + ab = ab + a = a.b + a.a = a(b + a) = a + a(b + a) + a$$

(2) $yx + y + yx \approx yx$, as

$$ba + b + ba = ba + a = a + ba + aa \;\; = a + (b + a).a + a.a \;\; = a.a + (b + a).a + a.a$$

$$= a + (b + a).a + a = a.$$

Theorem 10:- For an Idempotent semiring 'S', the following are equivalent,

(1) $x^2 \approx x \,;\, x + xy + x \approx x$

(2) $x + x \approx x \,;\, y + yx + y \approx y.$

Proof:- $(1) \Rightarrow (2)$

Let (1) holds,

Now, $a + a = (a +a)^2 = (a +a)(a +a) = a + a(a +a) + a = a.$

also, $b + ba + b = b + ab + b = b + b(b + a) + b = b.$

$(2) \Rightarrow (1)$,

Now $(a +a)^2 = (a + a)(a + a) = a + a(a + a) + a = a.$

But $a + a = a \;\Rightarrow\; a^2 = a$, and also $a + ab + a = a + a(a + b) + a = a.$

Hence (1) and (2) are equivalent.

Theorem 11:- For an Idempotent semiring 'S', the following are equivalent

(1) $xyx + xy + xyx \approx xyx$

(2) $xy + yx + xy \approx xy.$

Proof:- It is clear.

Theorem 12:- For an Idempotent semiring 'S' the following are equivalent

(1) $xy + x + xy \approx xy \,;\, y + xy + y \approx y.$

(2) $(x + y)(y + x)(x + y) \approx xy.$

Proof:- $(1) \Rightarrow (2)$

$$(a +b)(b + a)(a + b) = (a.b + a.a + b.b + b.a)(a + b)$$

$$= aba + abb + aaa + aab + bba + bbb + baa + bab$$

$$= aba + ab + a + ab + ba + b + ba + ba$$

$$= ab + a + ab + b + ab$$

$$= ab + b + ab = ab$$

$(2) \Rightarrow (1)$

Now $\quad ab + a + ab = ab + a + ab + ab + a + ab$

$$= (a.b + a.a + b.b + b.a)(a + b)$$
$$= (a + b)(b + a)(a + b) = ab.$$

Theorem 13:- For an Idempotent semiring 'S' the following are equivalent,

(1) $(x + y)(y + x)(x + y) \approx xy$

(2) $xy + x + xy \approx xy \; ; \; y + xy + y \approx y.$

(3) $xy + yx + xy \approx xy$

Proof:- It is Obvious.

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