ISSN 2347-1921

Debt Management and the Developing Nationts' Economy: A Stochastic Optimal Control Analysis

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ABSTRACT

This paper explores the usefulness of a stochastic optimal control analysis to discourage the less developed nations from borrowing funds from the more developed ones to service their investments (or worst still to service their yearly budget). The lenders of these funds are only interested in evaluating whether a borrower is likely to default. So they make policies to regulate and monitor the risk of an excessive debt that significantly increases the probability of default.

The borrowing nation invests part of the loan into a cash account and a stock account in order to make more money. The net effect of the loan on the economy of the nation given that it must be repaid at a nominal interest rate compounded over a period of time is determined herein.

Amazingly, this study discovered that as the debt is serviced by the borrower, the principal amount borrowed decreases as time increase but the interest rate (though fixed) increases as time increases, thereby sending the net worth of the borrowers' economy (income) towards a big crash.

Keywords:

Stochastic optimal control; debt management; developing nations; Borrowers' economy.

Academic Discipline and Sub-Disciplines

Mathematics.

2010 Mathematics Subject Classification: 91G80, 93E20, D92.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 7, No. 3
editor@cirjam.com
www.jam.cirworld.com, www.cirworld.com

INTRODUCTION

Debt management has been on the front burner of most developing nations. Poor management of internally generated revenue of most developing nations (corruption) has always forced them to access external funds as loans (e.g. from IMF). Many developing nations who are borrowers of these funds have little or no equity in their countries and so found it difficult to sell or to refinance, because the debt exceeded the market value of the country. The creditors are only interested in evaluating whether a borrower is likely to default. So they make policies to regulate and monitor the risk of an excessive debt that significantly increases the probability of default. Consequently sound financial models are required to arrive at critical financial policies in order to safeguard these loans from being added burden to the economy of such a nation. Stein (2010) addressed the question: how should creditors, banks and bank regulators evaluate and monitor risk of an excessive debt. He applied the usefulness of the stochastic optimal control analysis in answering this question.

Mashov and Ziang (1998) designed an optimal investment strategy in a portfolio of assets subject to a multiplicative Brownian motion. They stipulated investment strategy that provides the maximal typical long-term growth rate of investor's capital. They obtained optimal fraction of capital that a nation should keep in risky assets and weights of different assets in an optimal portfolio. They found that both average return and volatility of an asset are relevant indicators for determining its optimal weight.

Hibles (2011) stipulated that government policy is typically targeted heavily on investment. While Dat, et al (2009) considered optimal investment and consumption decision of a certain constant relative risk averse (CRRA) investor who faces proportional transaction costs in a finite time horizon. They found that the problem gave rise to two free boundaries for the optimal buying and selling strategies.

Tutu and Kozat (2003) studied optimal investment in a financial market that is characterized by a finite number of assets from a signal processing perspective. They investigated how an investor should distribute capital over the various assets and determined when to reallocate the distribution of the funds over the assets to maximize the cumulative wealth over any investment period. They developed portfolio selection algorithm that maximizes the expected cumulative wealth in independent and identically distributed two-asset discrete-time markets where the market levies proportional transaction costs in buying and selling stocks. Nkeki and Nwozo (2012) considered the valuation and management of inflows of internally generated revenue and monetary allocations of a country. They obtained the present and terminal values of total wealth of the country. In this study instead, we consider the effect in the economy of developing countries who borrow funds from the developed ones either for servicing their budget or for investment. This we do by exploring the usefulness of the stochastic optimal control analysis.

Model Formulation and Its Proportional Dynamics

It is a well-known fact that many developing nations borrow funds from the more developed ones (through IMF, say), to sponsor their budget sometimes and to invest in risky as well as in risk- free assets. The essence is usually to make some profit, pay back the borrowed funds and use the remainder for the development of their country.

But the lender (developed country) evaluates what debt would maximize the expected (E) growth rate of the borrower's (developing country) net worth over the period of the loan, an horizon of length T from the present t=0. This would be the optimal debt that a prudent lender would want to offer. The lender wants to avoid borrower's bankruptcy (S = 0) by placing a very high penalty on a debt that would lead to a zero net worth, bankruptcy. The borrower has a net worth $S(t)$ equal to the value of capital $K(t)$ less debt $L(t)$ (Stein, 2010). Initially net worth $S = S(0) > 0$.

The debt ratios is maximised as;

$$W(S,T) = \max_f E\ln\left[\frac{S(T)}{S(0)}\right], \quad S = K - L > 0, f = \frac{L}{S} \tag{1}$$

Alternatively in the deterministic form we have equation (1) as:

$$E[S(T) = S(0)e^{W(S,T)}]. \tag{2}$$

The lender is very risk averse, since $S(T) = 0$ implies that W is minus infinity.

The next steps are to: explain the stochastic differential equation for net worth, relate it to the debt ratio, and specify what are the sources and characteristics of the risk and uncertainty.

We assume the borrower operates on a market of one risk-free bank with constant interest rate, r and naira holdings of i different (risky) stock which evolve as a regulated logarithmic Brownian motion, $0 \leq s < \pi$ as

$$dS_i(t) = \mu_i S_i^{(t)} dt + \sigma_i S_i^{(t)} dw_i(t) + dC_i(t) - dL_i(t) \tag{3}$$

and

$$dS_0(t) = rS_0(t)dt - (1+\alpha)dC_i(t) + (1-\lambda) - dL_i(t) \tag{4}$$

with initial values as

$$S_0(0) = 1 \text{ and } S_i(0) = S_i \text{ for } t = 0 \ldots . N.$$

Where the $dw_i(t)$ are the increments to a Wiener process. $S_i(t)$ denotes the left hand limit of the process $S_i(t)$ at time, t. By assumption, asset $S_0(t) = S_0 = 1$ is risk-free with $\sigma_0 = 0$ and $\mu_0 = r$. $C(t)$ and $L(t)$ are right continuous and non decreasing cumutative dollar purchases and sales of asset on $[t, T)$ respectively with $C(t)$=$L(t)$=0. The initial trades (purchases or sales) of asset are given by $C(t)$ and $L(t)$ and λ is the market price of risk.

For simplicity, we assume that any transaction cost incurred will be paid by additional contributions to the fund. With this exception, there are not net contributions or withdrawals from the investor's holdings implying a self-financing constraint, that is

$$\sum dC_i(t) - dL_i(t)) = 0, \forall\ t. \tag{5}$$

where Σ denote the summation operator over all assets, $i = 0, \ldots, N$.

Define a wealth process $h(t)$ as a sum additive which is assumed to be strictly positive, thus

$$h(t) = 1 - \lambda \sum_i S_i(t) + \sum S_0(t). \tag{6}$$

Again

$$\frac{dh(t)}{h(t)} = 1 - \lambda \sum \frac{dS_i(t)}{h_i(t)} + \sum \frac{S_0(t)}{h_0(t)}. \tag{7}$$

Since $h_0(t)$ and $h_i(t)$ represents the amount invested into risk free and stock respectively and as we work in self-financing constraint, we have equation (7) become

$$dh(t) = (1-\lambda)\sum\left(\frac{\mu_i S_i(t)}{h_i(t)}\right)dt + (1-\lambda)\sum\left(\frac{\sigma_i S_i(t)}{h_i(t)}\right)dw_i(t) + (1-\lambda)\sum(dC_i(t) - dL_i(t)$$
$$+\sum r\frac{S_0(t)}{h_0(t)}dt - \sum [(1+\alpha)dC_i(t) + (1-\lambda)dL_i(t)], \tag{8}$$

where $h_i(t) = \frac{S_i(t)}{h(t)}$ and $h_0(t) = \frac{S_0(t)}{h(t)}$ are the proportion of wealth held in the risky asset i and risk-free asset at time t. Note that $\Sigma h_i(t) = 1$ which implies that $h_0(t) = 1 - \Sigma h_i(t)$

Therefore equation (8) becomes in differential terms

$$dh(t) = (1-\lambda)h(t)[(r + (\mu - r))dt + \sigma dw_2(t)]\ . \tag{9}$$

For $\lambda = 0$, we have the more general SDE;

$$dh_t = a(t)h_t dt + b(t)h_t d\underline{W}(t), \tag{10}$$

where $\underline{W}(t), 0 \le t \le T$, is a one-dimensional Brownian motion on the canonical probability space (Ω, F, P) with the natural filtration $\{F_t\}_{(0 \le t < \infty)}$ and on the risk neutral measure P . The financial models in this paper assume that some proportion of the flow of funds including the borrowed amount are invested in a market that is characterized by a risk-free asset (cash account) and a risky asset (stock account). Ito's formula on (10) gives;

$$\frac{\partial v}{\partial t} + a(t)h_t\frac{\partial v}{\partial h} + \frac{1}{2}(b(t)h_t)^2\frac{\partial^2 v}{\partial h^2} - rv = 0 \tag{11}$$

Suppose h_t follows the O-U process, Ergodic Mean Reversion (EMR) described by the following differential equation (Doob, 1942):

$$dh_t = k(\mu - h_t)dt + \sigma dW_t \tag{12}$$

where $k > 0$ is a constant and the rate of mean reversion, μ is the long term drift of the process, $\sigma > 0$ is the volatility or average magnitude per square root time of random fluctuations that are modeled as Brownian motion, and W_t is a normally distributed Wiener-Levy process.

The solution to Equation (12) with the initial condition h_0 applying the Duhammel principle is

$$h_t = \mu - (\mu - h_0)e^{-kt} + \sigma\int_0^t e^{-k(t-h)}dW_h. \tag{13}$$

By Revuz and Yor, (1999),

$$\int_0^t e^{ks}dW_s = W_{T(t)}, \tag{14}$$

for any $t \geq 0$, where

$$T(t) = \frac{1-e^{-2kt}}{2k}, \quad (15)$$

so that the analytical solution of Equation (12) becomes

$$h_t = \mu - (\mu - h_0)e^{-dt} + \sigma\sqrt{\frac{1-e^{-2kt}}{2k}} \quad N\{0,1\}\ (t). \quad (16)$$

$N\{0,1\}$ denotes a Gaussian distribution with zero, and variance of unity with conditional expectation calculated as:

$$E[h_\tau | h_0] = \mu - (\mu - h_0)e^{-kt}, \quad (17)$$

and the variance calculated as

$$Var[h_t | h_0] = E[(\sigma \int_0^t \quad e^{-k(t-h)} dW_h)^2] \quad (18)$$

which by the application of Ito's isometry gives the conditional variance as:

$$Var[h_t | h_0] = \frac{\sigma^2}{2k}(1 - e^{-2kt}). \quad (19)$$

The long term variance of the process is $\lim_{t\to\infty} Var\,[h_t | h_0] = \frac{\sigma^2}{2k}$.

The solution of Equation (12) implies that S_t converges to a distribution with a mean of μ and a variance of $\frac{\sigma^2}{2k}$, where a is the speed of response, that is $\lim h_\tau \sim N(\mu, \frac{\sigma^2}{2k})$. Hence equation (19) has a Markov process with stationary Gaussian transition probability densities as

$$P(t, h) = \frac{1}{\sigma\sqrt{2\pi}} \exp\,[-\frac{(h-\mu)^2}{2\sigma^2}] \quad (20)$$

Lemma 1: Since are assumed continuous, we have

$$a(t) = \frac{\sigma}{\sqrt{2\pi}} + \frac{\mu}{2} \quad (21)$$

and

$$b(t) = (2\sigma^2 - a\sigma\Gamma 2)e^{\frac{\mu}{\sigma\sqrt{2}}}\ . \quad (22)$$

Proof: For $b(t)$ and $a(t)$ being continuous functions of t we have (by Osu, 2009)

$b(t) = \int_0^\infty \sqrt{(h-a)^2 P(h)}dh$ and $a(t) = \int_0^\infty hP(h)dh$, where $P(h)$ is as in equation (20). A little calculation gives the required. The index t denotes clock time and $a(t)$ and $b(t)$ are drift and volatility of the trading time (days, say).

Solution of Equation (11) for the worth of the developing Nations' Economy

Put $y = \ln h$, we have

$$\frac{\partial v}{\partial h} = \frac{\partial v}{h\,\partial y},\ \frac{\partial^2 v}{\partial h^2} = -\frac{\partial v}{h\,\partial y} + \frac{\partial^2 v}{h^2\,\partial y^2}\}. \quad (23a)$$

Let

$$w(y,t) = e^{rt}v(y,t), \quad (23b)$$

then equation (11) together with equation (23) becomes the two dimensional Fokker-Planck forward equation;

$$c\frac{\partial w}{\partial y} - \frac{1}{2}b^2\frac{\partial^2 w}{\partial y^2} = \frac{\partial w}{\partial t}\ , \quad (24)$$

where $c = a - \frac{b^2}{2}$, which is interpreted as the (time-independent) *subordinated market price of the risky asset.*

Case 1: $\frac{\partial w}{\partial t} \to 0$ as $t \to \infty$

Assume that diffusion has a stationary distribution, then $\frac{\partial w}{\partial t} \to 0$ as $t \to \infty$ so that equation (24) becomes

$$\frac{b^2 dw}{2dy} = c.$$

Or

$$\frac{b^2}{2} w = cy, \quad (25a)$$

which implies that

$$w = (\ln h_t)^{\frac{2c}{b^2}}. \quad (25b)$$

Using equation (23b), we have

$$v = e^{-rt} (\ln h_t)^{\frac{2c}{b^2}}. \quad (26)$$

In the developing nations, the law makers always approve the servicing of the year $T = t + \tau$ with borrowed funds. These funds normally come with interest rate r (though fixed sometimes). As the debts are serviced by the borrowers, the original amount borrowed decreases while the rate increases (figure 1 below explains this phenomenon), such that as $r \to \infty, v \to 0$.

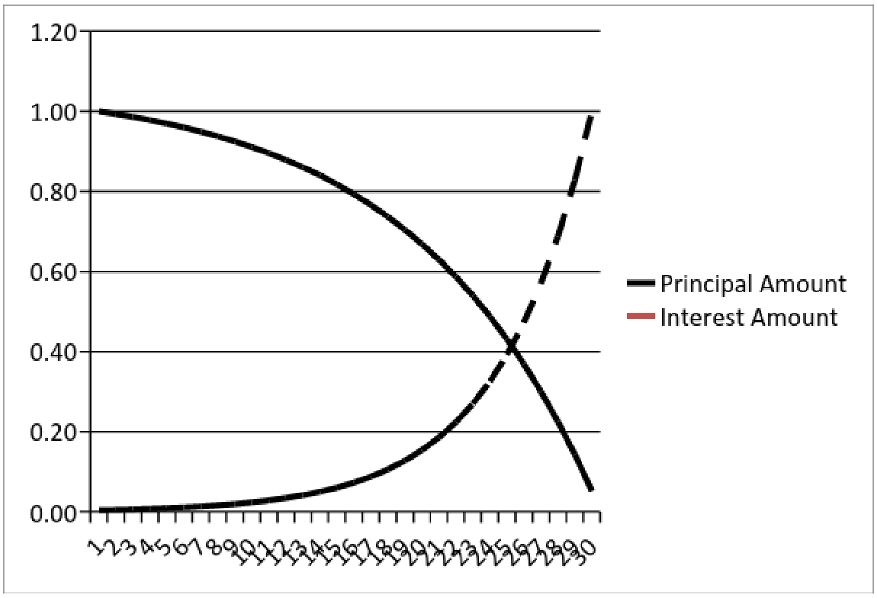

Figure 1: The plot of the interest paid over 30years versus the principal amount borrowed (in billions of Dollars). The amount is decreasing as the interest rate is increasing.

This only implies that the worth of the developing nations' economy practical becomes zero as the interest rate increases (which at some point may become even negative). At this point, the borrower (the developing country) is enslaved by the lender (the developed country). The borrower's economy crashes and she is at the mercy of the lender. But as $r \to 0$, the worth of the developing nations' economy grows exponentially positive as in figure 2 below.

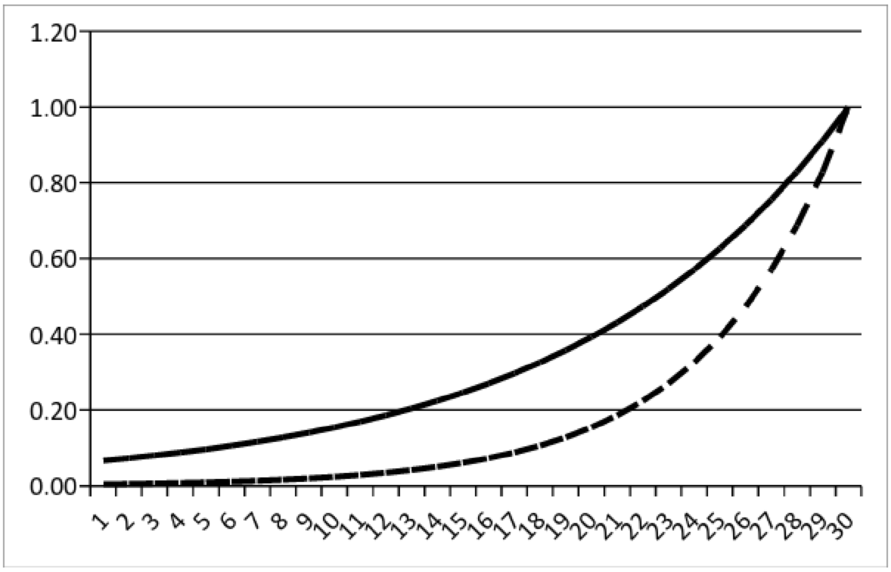

Figure 2: The exponential growth rate of worth (in billions of Dollars) with the interest rate equal to zero.

In what follows we state;

Lemma 2: For $\frac{\partial w}{\partial t} \nrightarrow 0$ as $t \to \infty$, we have (using (23a) and (24)) that;

$$v(h,t) = e^{-(r-1)t}\sinh\left(\ln h_t\right). \quad (27)$$

Proof: By Risken (1989), the general form of the Fokker-Planck equation (24) is

$$[-A(y)\frac{\partial w}{\partial y} + \frac{1}{2}B(y)\frac{\partial^2}{\partial y^2}]w = \frac{\partial w}{\partial t}, \quad (28)$$

with the initial condition

$$w(y,o) = f(y), \quad y \in R.$$

Hesam, et al (2012) had shown that for the drift coefficient $A(y) = -1$ and the diffusion coefficient $B(y) = 1$, the solution of equation (28) for $f(y) = y$ is given as;

$$w(y,t) = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} w(k,h)y^k t^h = y + t. \quad (29)$$

Therefore,

$$v = (\ln h_t + t)e^{-rt}. \quad (30)$$

For $(y) = \sinh y$, $y \in R$,$B(y,t) = e^t\cosh t$ and$A(y,t) = e^t\coth y \cosh y + e^t\sinh y - \coth y$, the solution of (28) becomes;

$$w(y,t) = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} w(k,h)y^k t^h = e^t\sinh y, \quad (31)$$

so that as required

$$v(h,t) = e^{-(r-1)t}\sinh(\ln h_t).$$

Case 2: $c = 0$

For $c = 0$ (that is the time-independent subordinated market price of the risky asset is zero), equation (24) becomes

$$\frac{\partial^2 w}{\partial y^2} = -\frac{2}{b^2}\frac{\partial w}{\partial t} \qquad t \geq 0 \; 0 < y < a \quad (32)$$

or

$$w_{hh} = -\frac{2}{b^2}w_t$$

$$w(y,0) = 0, \; w(y,b) = f(y), \; w(0,t) = w(a,t) = 0.$$

Now write

$$w = Y(y)T(t), \quad (33)$$

and substituting into equation (32) gives (with $\rho = -\frac{2}{b^2}$)

$$Y_{yy} + \lambda^2 H_{hs} = 0, T_t + \rho\lambda^2 T = 0. \quad (34)$$

The general solution for T is

$$T = e^{-\rho\lambda^2 t} \quad (35)$$

and the general solution for Y is

$$Y = A\cos\lambda y + B\sin\lambda y, \quad (36)$$

where A and B are constants; the condition $w(a,t) = 0$ implies $Y(0) = 0$ so that A=0. Similarly the condition $w(a,t) = 0$ show that $Y(a) = 0$ and hence, unless $B = 0$ (this would give $w = 0$), $\sin\lambda a = 0$.

Thus the only possible values of λ are $\frac{\mathrm{nth}}{a}$, were n is an integer. There is no known method of rejecting any values of n, so we let n range over all positive values Y is now proportional to $\sin\frac{n\pi x}{a}$. By equation (33), the general solution for w is

$$w = \sum_{n=1}^{\infty} e^{\frac{-\rho n^2\pi^2 t}{a^2}} A_n \sin\frac{n\pi y}{a} \ . \quad (37)$$

Setting $t = 0$ in equation (37) gives

$$w(y,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}. \quad (38)$$

The right-hand side of equation (38) is the Fourier sine series of w and hence

$$A_n = \frac{2}{a}\int_0^a w(x', 0)\sin\frac{n\pi y'}{a}\,dy'. \quad (39)$$

The value function is therefore given by

$$w = \int_0^a \ [w(y',0)\frac{2}{a}\sum_{n=1}^{\infty}\exp\{-\frac{\rho n^2\pi^2 t}{a^2}\}\sin\frac{n\pi x'}{a}\sin\frac{n\pi y}{a}\ ]dy' \quad (40)$$

If $w(y,0)$ is integrable, the exponential factor in equation (40) guarantees the uniform convergence of the series for all t such that $t \geq t_0 > 0$, even if $w(y,0)$ is discontinuous.

Recall that $\rho = -\frac{2}{b^2}$, so that at $y = T$, we have

$$w(y,0) = \varepsilon\exp\{2(\frac{n\pi}{ba})^2 T\}\sin\frac{n\pi y}{a} \ . \quad (41)$$

Equation (41) vanishes for $y = 0$ and $y = a$. The correct form at $y = T$ is

$$w(y,T) = \varepsilon\exp\{-2(\frac{n\pi}{ba})^2(t-T)\}\sin\frac{n\pi y}{a} \ . \quad (42)$$

By making n arbitrary large, we can make $w(y,0)$ as large as we choose for all ε, however small. This is the case of not well-posedness. Using equation (23a), we have the worth of the developing nation's economy as

$$v(y,T) = \varepsilon\exp\{-rt\}\exp\{-2(\frac{n\pi}{ba})^2(t-T)\}\sin\frac{n\pi y}{a} \ . \quad (43)$$

Estimation of the interest rate r.

The solution of the SDE (12) by Ito's formula is given as

$$h_t = h_0[b(t)\underline{W_t} + (a(t) - \frac{1}{2}b^2(t))t] \ \forall \ t \in [0,T]. \quad (44)$$

Note that the behaviour

$$\frac{1}{t}\log h_t \cong (a(t) - \frac{1}{2}b^2(t)) \ , \quad (45a)$$

as well as

ISSN 2347-1921

$$\frac{1}{t}\log E(h_t) \cong a(t). \quad (45b)$$

Therefore (see Osu and Okoroafor, 2007)

$$r = \frac{\log h_t}{\log E(h_t)} = \frac{a(t) - \frac{1}{2}b^2(t)}{a(t)}, \quad (45c)$$

where $a(t)$ and $b(t)$ are as in equations (21) and (22) respectively.

For the sake of illustration a number of simulations have been performed, making the assumption that recent historical data fit out simple model. Making use of early records of inflation, stock market prices (in form of a domestic market index with reinvested dividends), and interest rates for the some countries, we have computed the parameter values shown in the table below.

	Period	$\lvert r\rvert$	$\lvert a(t)\rvert$	$\lvert b(t)\rvert$	$\lvert c(t)\rvert$	$\lvert r - (\frac{c(t)}{b(t)})^2\rvert$
Country 1	30yrs	9.149	2.508	7.135	22.946	1.194
Country 2	30yrs	1.622	4.308	4.753	6.988	1.009
Country 3	30yrs	7.001	5.000	17.503	2.900	1.500

Conclusion

We have applied stochastic analysis to analyse the impact of the economy developing countries whose best financial policy is to embark only borrowing of fund from the developed ones. Equations (26), (29) and (43) shows that if $r \to -\infty$, the worth of the borrowers' economy experience a bubble and the lenders suffer a great default for lack of patronage. Therefore the lenders make such policies that will keep $r > 0$ and infact $r \to \infty$.

It is very important for the government of the developing nations to have a good economic policy to avoid borrowing. Some of these developing countries are endowed with natural resources. A good management of the internally generated revenue will help curb the effect of funds and hence stop enslavement. If at all they must borrow, they should avoid the interest rate policy since they have enough endowment to 'trade by barter'.

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