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Concomitants of Record Values From a General Farlie-Gumbel-Morgenstern Distribution

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ABSTRACT

In this paper, we discuss the distributions of concomitants of record values arising from a polynomial-type single parameter extension of general Farlie-Gumbel-Morgenstern bivariate distribution. We derive the single and the product moments of concomitants of record values generally for any marginal distributions. The results obtained are applied to two-parameters exponential marginal distributions. In this case, we show that the maximal positive correlation between the two variables is approximately ≈.423. Best linear unbiased estimators based on concomitants of record values of some parameters involved in the distribution are derived. Moreover, we obtain predictors of concomitants of record values by two methods. Finally a numerical illustration is presented to highlight the theoretical results obtained.

Keywords:

Farlie-Gumbel-Morgenstern family; Concomitants; Record values; Best linear unbiased estimator; Best linear unbiased predictor.

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1 INTRODUCTION

The Farlie-Gumbel-Morgenstern (FGM) family of bivariate distributions has found extensive use in practice especially in lifetime tests and in the context of reliability. Balakrishnan and Lai (2009) showed several applications for the FGM in the literature. The FGM family of distributions, was originally introduced by Morgenstern (1956) for Cauchy marginals, Gumbel (1960) investigated the same structure for exponential marginals and further generalized by Farlie (1960). Johnson and Kotz (1975) and (1977) studied the multivariate case and presented detailed analysis of probabilistic and statistical characteristics. Huang and Kotz (1984) extended the bivariate FGM distribution in their attempts to increase the dependence between the underlying variables by introducing an additional parameter. The FGM distribution is characterized by the marginal distribution functions F_X and F_Y of random variables X and Y, respectively, and the association parameter α. The cumulative distribution function (cdf) $F_{X,Y}(x,y)$ of the FGM distribution is given by

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)\{1+\alpha(1-F_X(x))(1-F_Y(y))\}.$$

The generalizations of FGM family of bivariate distributions received a great deal of attention of many researchers. For example, Huang and Kotz (1999) considered a polynomial-type single parameter extension of FGM bivariate distribution with uniform marginals. The cdf, which they considered is

$$H_\alpha(x,y) = xy\{1+\alpha(1-x^p)(1-y^p)\},\ p \ge 1,\ 0 \le x,y \le 1, \tag{1}$$

where the admissible range of α is $-(\max\{1,p\})^{-2} \le \alpha \le p^{-1}$ and the range for correlation coefficient is $-(p+2)^{-2}\min\{1,p^2\} \le \rho \le 3p(p+2)^{-2}$. The maximal positive correlation for (2) $\rho = 3/8$ is attained for $p = 2$, an improvement over the case $p = 1$ for which $\rho = 1/3$.

In the present article, we study the records and their concomitants of a general form of Huang and Kotz (1999) extension, where the joint cdf and the joint pdf of X and Y are given respectively by

$$F_p(x,y) = F_X(x)F_Y(y)\{1+\alpha(1-F_X{}^p(x))(1-F_Y{}^p(y))\},\ p \ge 1, \tag{2}$$

$$f_p(x,y) = f_X(x)f_Y(y)\{1+\alpha[(p+1)F_X{}^p(x)-1][(p+1)F_Y{}^p(y)-1]\},\ p \ge 1. \tag{3}$$

To our knowledge there are no studies concerning the records and their concomitants of the general form in (2) and (3).

Let $\{(X_{(i)},Y_{[i]}), i \ge 1\}$ be independent and identically distributed random variables from some continuous bivariate distribution. Let $\{X_{(n)}, n \ge 1\}$ be the sequence of upper record values arising from the sequence of X's. Then the Y-variable paired with the X-value which is qualified as the nth record is called the concomitant of the nth record value and is denoted by $Y_{[n]}$. In many situations the only available observations are bivariate record values, i.e., records and their concomitants, and hence we must make inferences based on records and their concomitants. Such situations often occur in life time experiments, sporting matches, weather data recording and some other experimental fields. Some properties of concomitants of record values were discussed in Houchens (1984), Arnold et al. (1998) and Ahsanullah and Nevzorov (2000). For a general review of concomitants of ordered random variables see Raqab et al. (2002). However, the concomitants in case of record values have not been extensively studied as compared with the concomitants of order statistics. This branch is relatively new in the field of ordered random variables.

If $\{X_{(n)}, n \ge 1\}$ is the sequence of upper record values then the probability density function (pdf), $g_n(.)$, of the nth upper record value can be obtained by using the following expression given by Ahsanullah (1995)

$$g_n(x) = \frac{1}{\Gamma n}\left[-\log(1-F_X(x))\right]^{n-1} f_X(x), \tag{4}$$

where $f_X(x)$ and $F_X(x)$ are the pdf and the cdf of X respectively. Then the pdf of the concomitant $Y_{[n]}$ of the nth upper record value, for $n \ge 1$, is given by

$$f_{[n]}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y\,|\,x)\,g_n(x)\,dx, \tag{5}$$

where $f_{Y|X}(y\,|\,x)$ is the conditional pdf of Y given $X = x$ of the parent bivariate distribution. Ahsanullah (1995) has given the joint distribution of mth and nth upper record values for $m < n$ as

$$g_{m,n}(x_1,x_2)=\frac{[-\log(1-F_X(x_1))]^{m-1}}{\Gamma m}\frac{[-\log(1-F_X(x_2))+\log(1-F_X(x_1))]^{n-m-1}}{\Gamma(n-m)}\frac{f_X(x_1)f_X(x_2)}{1-F_X(x_1)},x_1<x_2. \quad (6)$$

The joint pdf of concomitants of mth and nth upper record values for $m<n$ is given by

$$f_{[m,n]}(y)=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}f_{Y|X}(y_1\,|\,x_1)f_{Y|X}(y_2\,|\,x_2)g_{m,n}(x_1,x_2)dx_1dx_2, \quad (7)$$

The rest of this paper is organized as follows: In Sections 2 and 3, we derive the distribution of concomitants of record values arising from the general form in (2) and thier single and product moments. In Section 4, we investigate the results obtained in Sections 2 and 3 for the two-parameter exponential marginal distributions. Best linear unbiased estimators (BLUEs) based on concomitants of record values of some parameters involved in the distribution are derived in Section 5. In Section 6, we present two different methods for obtaining predictors of future concomitants of record values. Finally, in Section 7, numerical illustrations are presented to highlight the theoretical results obtained.

2 Concomitants of Record Values

In this section, we drive the distribution of concomitants of record values arising from the general form in (2).

Theorem 1 Let $(X_{(i)},Y_{[i]}),i=1,2,...$ be a sequence of independent observations from (2). If $Y_{[n]}$ is the concomitant of the nth record value on the X sequence of observations, then the pdf of $Y_{[n]}$ is given by

$$f_{[n]}(y)=f_Y(y)\{1+\alpha[p+(1+p)\sum_{j=1}^{\infty}\frac{\prod_{s=0}^{j-1}(p-s)(-1)^j}{j!(j+1)^n}][(1+p)F_Y{}^p(y)-1]\}. \quad (8)$$

proof The conditional pdf of Y given $X=x$ is given by

$$f(y\,|\,x)=f_Y(y)\{1+\alpha[(p+1)F_X{}^p(x)-1][(p+1)F_Y{}^p(y)-1]\}. \quad (9)$$

Using (9) and (4) in (5) we get

$$f_{[n]}(y)=\frac{1}{\Gamma n}\int_{-\infty}^{\infty}f_X(x)f_Y(y)\{1+\alpha[(p+1)F_X{}^p(x)-1][(p+1)F_Y{}^p(y)-1]\}[-\log(1-F_X(x))]^{n-1}dx.$$

Using the transformation $-\log(1-F_X(x))=t,$

$$f_{[n]}(y)=f_Y(y)\{\{1-\alpha[(1+p)F_Y{}^p(y)-1]\}\int_0^{\infty}\frac{t^{n-1}e^{-t}}{\Gamma n}dt+\alpha(1+p)[(1+p)F_Y{}^p(y)-1]\int_0^{\infty}(1-e^{-t})^p\frac{t^{n-1}e^{-t}}{\Gamma n}dt\},$$

Applying the binomial theorem, and after simplifications the proof is complete.■

Putting p=1, in (8) we get the same result of the classic FGM distribution as Houchens (1984).

Theorem 2 Let $(X_{(i)},Y_{[i]}),i=1,2,...$ be a sequence of independent observations from (2), then the joint pdf of $Y_{[m]}$ and $Y_{[n]}$ for $m<n$ is given

$$\begin{aligned}f_{[m,n]}(y)=&f_Y(y_1)f_Y(y_2)[1+\alpha\{(1+p)I_1-1\}\{(1+p)F_Y{}^p(y_1)-1\}+\alpha\{(1+p)I_2-1\}\{(1+p)F_Y{}^p(y_2)-1\}\\&+\alpha^2\{(1+p)^2I_3-(1+p)I_1-(1+p)I_2+1\}\{(1+p)F_Y{}^p(y_1)-1\}\{(1+p)F_Y{}^p(y_2)-1\}],\end{aligned} \quad (10)$$

where

$$I_1=1+\sum_{t=1}^{\infty}\frac{\prod_{j=0}^{t-1}(p-j)(-1)^t}{t!(t+1)^m}, \quad (11)$$

$$I_1 = 1 + \sum_{t=1}^{\infty} \frac{\prod_{j=0}^{t-1}(p-j)(-1)^t}{t!(t+1)^n}, \tag{12}$$

and

$$I_3 = I_1 + I_2 + \sum_{i=1}^{\infty}\left[\frac{\prod_{j=0}^{i-1}(p-j)(-1)^i}{i!(i+1)^{n-m}} \sum_{t=1}^{\infty} \frac{\prod_{j=0}^{t-1}(p-j)(-1)^t}{t!(t+i+1)^m}\right] - 1. \tag{13}$$

proof By using (6) and (9) in (7), and noticing that $\int_{-\infty}^{\infty}\int_{-\infty}^{x_2} g_{m,n}(x_1,x_2)dx_1dx_2 = 1,$ we get

$$\begin{aligned} f_{[m,n]}(y) = f_Y(y_1) f_Y(y_2) [1 + \alpha\{(1+p)J_1 - 1\}\{(1+p)F_Y{}^p(y_1) - 1\} + \alpha\{(1+p)J_2 - 1\}\{(1+p)F_Y{}^p(y_2) - 1\} \\ + \alpha^2\{(1+p)^2 J_3 - (1+p)J_1 - (1+p)J_2 + 1\}\{(1+p)F_Y{}^p(y_1) - 1\}\{(1+p)F_Y{}^p(y_2) - 1\}], \end{aligned} \tag{14}$$

where

$$J_i = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} F_X^p(x_i) g_{m,n}(x_1,x_2)dx_2dx_1,\ i = 1,2,$$

$$J_3 = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} F_X^p(x_1) F_X^p(x_2) g_{m,n}(x_1,x_2)dx_2dx_1.$$

Using (6) and applying the transformations $-\log(1-F_X(x_1)) = u,\ -\log(1-F_X(x_2)) = v,$ we get

$$J_1 = \frac{1}{\Gamma m \Gamma(n-m)} \int_0^{\infty}\int_u^{\infty} (1+e^{-u})^p u^{m-1}(v-u)^{n-m-1} e^{-v} dvdu.$$

Using the transformation $v - u = s$, and the binomial theorem, we get

$$J_1 = \frac{1}{\Gamma m \Gamma(n-m)} \int_0^{\infty}\int_0^{\infty} (1 + \sum_{t=1}^{\infty} \frac{\prod_{j=0}^{t-1}(p-j)(-1)^t e^{-tu}}{t!}) u^{m-1} s^{n-m-1} e^{-s-u} dsdu.$$

Integrating with respect to s and u, and after simplifications we obtain $J_1 = I_1$. Proceeding in a similar manner we get $J_2 = I_2$ and $J_3 = I_3$. Upon substituting the values of J_1, J_2 and J_3 the proof is complete.■

Putting $p = 1$ in (10) we get the same result as Chacko and Thomas (2006) for the classic FGM distribution.

Notice that, if p is an integer number, the pdf of the largest order statistic of a random sample of size $p+1$, $Y_{p+1:p+1}$, arising from marginal distribution of Y, will be

$$f_{p+1:p+1}(y) = (p+1) F_Y^p(y) f_Y(y),$$

Consequently the pdf of the concomitant of the nth upper record value and the joint pdf of the concomitants of the mth and nth upper record values can be written in terms of marginal pdf of Y and the pdf of the largest order statistic of a random sample of size $p+1$ as follows:

$$f_{[n]}(y) = f_Y(y) - \alpha\{1 - (1+p)\sum_{j=0}^{p} \frac{\binom{p}{j}(-1)^j}{(j+1)^n}\}\{f_{p+1:p+1}(y) - f_Y(y)\}], \tag{15}$$

and

$$f_{[m,n]}(y)=f_Y(y_1)f_Y(y_2)+\alpha\{(1+p)I_1-1\}\{f_{p+1:p+1}(y_1)-f_Y(y_1)\}f_Y(y_2)$$
$$+\alpha\{(1+p)I_2-1\}\{f_{p+1:p+1}(y_2)-f_Y(y_2)\}f_Y(y_1) \qquad (16)$$
$$+\alpha^2\{(1+p)^2I_3-(1+p)I_1-(1+p)I_2+1\}\{f_{p+1:p+1}(y_1)-f_Y(y_1)\}\{f_{p+1:p+1}(y_2)-f_Y(y_2)\},$$

where I_1, I_2 and I_3 are defined in (11)-(13) respectively.

3 The Moments of Concomitants of Record Values

From (8) the kth moment of the concomitant of the nth upper record value is given by

$$E[(Y_{[n]})^k]=\mu^{(k)}+\alpha[p+(1+p)\sum_{j=1}^{\infty}\frac{\prod_{s=0}^{j-1}(p-s)(-1)^j}{j!(j+1)^n}][(1+p)\int_{-\infty}^{\infty}y^k f_Y(y)F_Y{}^p(y)dy-\mu^{(k)}], \qquad (17)$$

where $\mu^{(k)}=E[Y^k]$.

Putting $p=1$ in (17), we get the same result as Houchens (1984) for the classic FGM distribution.

From (10), the product moment of concomitants of the mth and nth upper record values for $m<n$ is given by

$$E[Y_{[m]}Y_{[n]}]=\mu^2+\alpha\mu\{(1+p)I_1+(1+p)I_2-2\}\{(p+1)\int_{-\infty}^{\infty}yf_Y(y)F_Y^p(y)dy-\mu\}$$
$$+\alpha^2\{(1+p)^2I_3-(1+p)I_1-(1+p)I_2+1\}\{(p+1)\int_{-\infty}^{\infty}yf_Y(y)F_Y^p(y)dy-\mu\}^2, \qquad (18)$$

where $\mu=E[Y]$.

Notice that, if p is an integer number, the kth moment of the concomitant of the nth upper record value and the product moment of concomitants of the mth and nth upper record values for $m<n$ are given respectively by

$$E[(Y_{[n]})^k]=\mu^{(k)}-\alpha\{1-(1+p)\sum_{j=0}^{p}\frac{\binom{p}{j}(-1)^j}{(j+1)^n}\}\{\mu^{(k)}_{p+1:p+1}-\mu^{(k)}\},$$

$$E[Y_{[m]}Y_{[n]}]=\mu^2+\alpha\mu\{(1+p)I_1+(1+p)I_2-2\}\{\mu_{p+1:p+1}-\mu\}$$
$$+\alpha^2\{(1+p)^2I_3-(1+p)I_1-(1+p)I_2+1\}\{\mu_{p+1:p+1}-\mu\}^2,$$

where $\mu^{(k)}_{p+1:p+1}=E[Y^{(k)}_{p+1:p+1}]$ and $\mu_{p+1:p+1}=E[Y_{p+1:p+1}]$, as for $p=1$ we get the same result as Chacko and Thomas (2006) for the classic FGM distribution.

4 Exponential Marginals

In the present and the subsequent sections, we shall investigate concomitants of record values for the bivariate random variable (X,Y), having bivariate pdf given by (2) with two-parameter exponential marginals, with density functions,

$$f_X(x)=\frac{1}{\lambda_1}\exp(\frac{-(x-\mu_1)}{\lambda_1}),\ x\ge\mu_1 \text{ and } f_Y(y)=\frac{1}{\lambda_2}\exp(\frac{-(y-\mu_2)}{\lambda_2}),\ y\ge\mu_2. \qquad (19)$$

The correlation coefficient between the two variables X,Y is given by

$$\rho=\alpha\left(p+(1+p)\sum_{j=1}^{\infty}\frac{\prod_{s=0}^{j-1}(p-s)(-1)^j}{j!(j+1)^2}\right)^2=\alpha\gamma \text{ (say)} \qquad (20)$$

where $\gamma = \left(p + (1+p)\sum_{j=1}^{\infty} \frac{\prod_{s=0}^{j-1}(p-s)(-1)^{j}}{j!(j+1)^{2}} \right)^{2}$.

Notice from (20) that ρ depends only on α and p.

Since $0 \le F_X(x) \le 1$ and $0 \le F_Y(y) \le 1$, we can easily see from (2) that the admissible range of α is $-p^{-2} \le \alpha \le p^{-1}$.

Now, we shall discuss the influence of $p \ge 1$ on ρ. From (20), we find that for a specific value of p, the range of ρ is $-p^{-2}\gamma \le \rho \le \gamma p^{-1}$.

Table 1 shows the admissible values of the dependence parameter α and the correlation coefficient ρ for the exponential marginal distributions with respect to different values of $p \ge 1$. We find that the strongest positive correlation coefficient $\rho = .422872$ is attainted for $p = 6.06074$, while the negative lower bound of correlation coefficient for this value is $-.069775$ which is weaker than the negative lower bound at $p = 1$. From Figure 1, we see that the upper bound of the positive correlation coefficient $\rho = .25$ is attended for both values $p = 1$ and $p = 48.95$. However the upper bound of the positive correlation coefficient for p in the interval $(1, 48.95)$ is greater than 0.25. We see also that the upper bound of the positive correlation coefficient decreases as p tends to infinity while the admissible ranges of ρ and α shrinks as p increases.

Table 1. The admissible values of α and ρ

p	α		ρ	
	Lower bound	**Upper bound**	**Lower bound**	**Upper bound**
1	-1	1	- .25	.25
1.5	-.444444	.666667	- .205736	.308604
4	-1/16	1/4	- .102934	.411736
5.2	-.036982	.192308	-.081033	.421372
6	-1/36	1/6	-.070478	.422866
6.06047	-.027226	.165004	-.069775	.422872
7	-1/49	1/7	-.060225	.421576
10	-.01	1/10	-.0407990	.407990

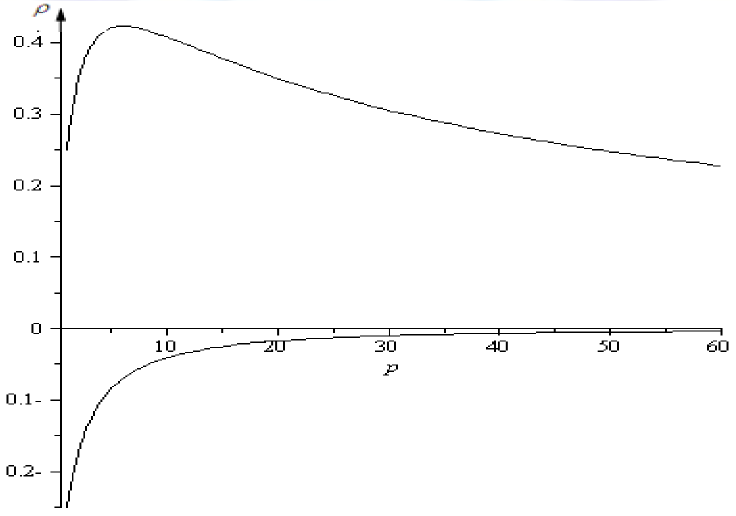

Figure 1 Bounds of correlations coefficient ρ as a function of parameter p.

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Now, Let $U=(X-\mu_1)/\lambda_1$ and $V=(Y-\mu_2)/\lambda_2$ be the standard exponential random variables. Clearly upon substitution with $F_U(u)=(1-e^{-u})$ and $F_V(v)=(1-e^{-v})$ into (8) and (10), we obtain the pdf of the concomitant of the nth upper record value and the joint pdf of the concomitants of the mth and nth upper record values with standard exponential margins, respectively.

We have,

$$\int_{-\infty}^{\infty} v^k f_V(v) F_V{}^p(v)dv = \int_0^{\infty} v^k e^{-v}(1-e^{-v})^p dv$$
$$= k!(1+\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^i}{i!(i+1)^{k+1}}), \qquad k\ge 1, \tag{21}$$

and

$$\mu^{(k)} = E[V^k] = k!. \tag{22}$$

Substituting (21) and (22) into (17), we obtain the kth moment of the concomitant of the nth upper record value

$$E[(V_{[n]})^k] = k!(1+\alpha[p+(1+p)\sum_{j=1}^{\infty}\frac{\prod_{s=0}^{j-1}(p-s)(-1)^j}{j!(j+1)^n}][p+(1+p)\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^i}{i!(i+1)^{k+1}}]) = \varepsilon_n \text{ (say)} \tag{23}$$

Thus the variance of $V_{[n]}$ is given by

$$\operatorname{var}(V_{[n]}) = \left\{1+2\alpha(p+1)[p+(1+p)\sum_{j=1}^{\infty}\frac{\prod_{s=0}^{j-1}(p-s)(-1)^j}{j!(j+1)^n}]\{\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^i}{i!(i+1)^3} - \sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^i}{i!(i+1)^2}\}\right.$$
$$\left. -\left[\alpha[p+(1+p)\sum_{j=1}^{\infty}\frac{\prod_{s=0}^{j-1}(p-s)(-1)^j}{j!(j+1)^n}][p+(1+p)\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^i}{i!(i+1)^2}]\right]^2\right\} = \rho_{n,n} \text{ (say)}. \tag{24}$$

From (18), the product moment of concomitants of the mth and nth upper record values for m<n is given by

$$E[V_{[m]}V_{[n]}] = 1+\alpha\{(1+p)I_1+(1+p)I_2-2\}\{p+(1+p)\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^i}{i!(i+1)^2}\}$$
$$+\alpha^2\{(1+p)^2 I_3-(1+p)I_1-(1+p)I_2+1\}\{p+(1+p)\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^i}{i!(i+1)^2}\}^2. \tag{25}$$

Hence, the covariance of $V_{[m]}$ and $V_{[n]}, m<n$ is given by

$$\operatorname{cov}(V_{[m]}V_{[n]}) = \left\{1+\alpha\{(1+p)I_1+(1+p)I_2-2\}\{p+(1+p)\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^i}{i!(i+1)^2}\}+\alpha^2\{(1+p)^2 I_3-(1+p)I_1-(1+p)I_2+1\}\right.$$

$$\times\{p+(1+p)\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^{i}}{i!(i+1)^{2}}\}^{2}\Bigg\}-\Bigg\{1+\alpha[p+(1+p)\sum_{j=1}^{\infty}\frac{\prod_{s=0}^{j-1}(p-s)(-1)^{j}}{j!(j+1)^{m}}][p+(1+p)\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^{i}}{i!(i+1)^{2}}]\Bigg\}$$

$$\times\Bigg\{1+\alpha[p+(1+p)\sum_{j=1}^{\infty}\frac{\prod_{s=0}^{j-1}(p-s)(-1)^{j}}{j!(j+1)^{n}}][p+(1+p)\sum_{i=1}^{\infty}\frac{\prod_{s=0}^{i-1}(p-s)(-1)^{i}}{i!(i+1)^{2}}]\Bigg\}=\rho_{m,n}\ \text{(say)}, \tag{26}$$

and I_1, I_2 and I_3 are defined in Equations (11)- (13) respectively.

Putting $p=1$ and $\mu_1=\mu_2=0$, in (24) and (26) we get the same result as Mohammed(2011), for the standard FGM distribution with exponential marginals (Gumbel's bivariate exponential distribution model II).

5 Estimation of The Location and Scale Parameters of The Exponential Margins

In this section we discuss the estimation of the location and the scale parameters μ_1, λ_1 and μ_2, λ_2 when the association parameter α is either known or unknown.

Ahsanullah (1980) derived the BLUEs of μ_1 and λ_1 based on the first n record values drawn from the marginal distribution of X as

$$\hat{\mu}_1=\frac{1}{n-1}(nX_{(1)}-X_{(n)}), \tag{27}$$

and

$$\hat{\lambda}_1=\frac{1}{n-1}(X_{(n)}-X_{(1)}). \tag{28}$$

Now we want to estimate μ_2 and λ_2 using the concomitants of record values.

5.1 Estimation of μ_2 and λ_2 When α is Known

Let $Y_{[n]}$ denote the vector of concomitants of the first n record values, that is $Y_{[n]}=(Y_{[1]},Y_{[2]},\ldots,Y_{[n]})'$, where $Y_{[i]}=\lambda_2 V_{[i]}+\mu_2,\ i=1\ldots n.$ From (23), we can write

$$E[Y_{[n]}]=\lambda_2\varepsilon_n+\mu_2 1\ , \tag{29}$$

where $\varepsilon_n=(\varepsilon_1,\ldots,\varepsilon_n)'$ denotes the column vector of expected values of the concomitant of upper record values from the standard exponential distribution and 1 is n×1 vector whose components are all1's.

The variance covariance matrix of $Y_{[n]}$ is given by

$$D[Y_{[n]}]=\lambda_2^2\Sigma$$

where $\Sigma=(\rho_{i,j})$,and $\rho_{i,j}$ are determined by (24) and (26), $i,j=1,\ldots n.$

Clearly $\varepsilon_n, \rho_{n,n}$, and $\rho_{m,n}$ are known constants provided that α , m and n are known.

Proceeding as in David and Nagaraja (2003), the BLUEs $\hat{\mu}_2$ of μ_2 and $\hat{\lambda}_2$ of λ_2 are given by

$$\hat{\mu}_2=-\varepsilon_n'\Gamma Y_{[n]}=\sum_{i=1}^{n}a_i y_{[i]}, \tag{30}$$

$$\hat{\lambda}_2=1'\,\Gamma Y_{[n]}=\sum_{i=1}^{n}b_i y_{[i]}, \tag{31}$$

where $\Gamma = \Sigma^{-1}(1\varepsilon_n^{'} - \varepsilon_n 1^{'})/\Delta$, $\Delta = (1^{'}\Sigma^{-1}1)(\varepsilon_n^{'}\Sigma^{-1}\varepsilon_n) - (1^{'}\Sigma^{-1}\varepsilon_n)^2$, and a_i, b_i, $i = 1,2,..., n$ are constants.

The variances and covariance of μ_2 and λ_2 are given respectively by

$$\mathrm{var}(\hat{\mu}_2) = \lambda_2^2 \varepsilon_n^{'} \Sigma^{-1} \varepsilon_n / \Delta, \mathrm{var}(\hat{\lambda}_2) = \lambda_2^2 1^{'} \Sigma^{-1} 1 / \Delta, \tag{32}$$

and

$$\mathrm{cov}(\hat{\mu}_2, \hat{\lambda}_2) = -\lambda_2^2 \varepsilon_n^{'} \Sigma^{-1} 1 / \Delta.$$

5.2 Estimation of μ₂ and λ₂ When α is Unknown

Following Chacko and Thomas (2006), if α is unknown, we may replace α in (30) and (31) by a rough moment type estimator. If r is the sample correlation coefficient between $X_{(i)}$ and $Y_{[i]}, i = 1,2,...$, then the rough moment type estimator $\tilde{\alpha}$ f or α is obtained by equating r with the correlation coefficient given by (20). Thus

$$\tilde{\alpha} = \begin{cases} -p^{-2} & \text{if } r \le -\gamma p^{-2} \\ p^{-1} & \text{if } r \ge \gamma p^{-1} \\ r\gamma^{-1} & \text{otherwise.} \end{cases} \tag{33}$$

6 Predictors of Concomitants of Record Values

One would wish to use past data for predicting a future observation. In this section we discuss the prediction of future concomitants of record values. Let $(X_{(i)}, Y_{[i]}), i = 1,2,...n$ represent the first observed n upper record values and their concomitants. We present two different methods for obtaining the mth predicted concomitant, $m > n$. For the first method, we obtain the best linear unbiased predictor (BLUP) $Y_{[m]}^*$ of $Y_{[m]}, m > n$, while the second method we use the conditional distribution of $Y_{[m]}$ given $X_{[m]}$ for obtaining the predictor which we call the conditional predictor $\tilde{Y}_{[m]}^*$.

6.1 The BLUP of $Y_{[m]}$

Using the generalized linear regression model, see Goldberger (1962), the BLUP $Y_{[m]}^*$ of $Y_{[m]}, m > n$ is

$$Y_{[m]}^* = \hat{\mu}_2 + \hat{\lambda}_2 \varepsilon_m + w^{'}\Sigma^{-1}(Y_{[n]} - \hat{\mu}_2 1 - \hat{\lambda}_2 \varepsilon_n), \tag{34}$$

where ε_m is the expected value of $Y_{[m]}^*$, $\hat{\mu}_2$ and $\hat{\lambda}_2$ are the BLUE of μ_2 and λ_2, respectively, $w^{'}$ is the vector of covariances of the prediction observation with the vector of observed concomitants of record values. i.e. $(\rho_{1,m},..., \rho_{n,m})$, Σ is the standard variance-covariance matrix, $Y_{[n]}$ is the vector of observed concomitants of record values, and ε_n is the vector of expected values of the concomitant of record values from the standard exponential distribution.

6.2 The Conditional Predictor of $Y_{[m]}$

Another method for obtaining a predicted value $\tilde{Y}_{[m]}^*$ of the mth concomitant $Y_{[m]}, m > n$, can be applied by using the predicted mth record value and the conditional cdf of Y given $X = x$. Ahsanullah (1980) derived the BLUP of mth record value, $X_{(m)}^*, m > n$, based on the first n record values drawn from the marginal distribution of X as

$$X_{(m)}^* = \frac{1}{n-1}\{(m-1)X_{(n)} - (m-n)X_{(1)}\}. \tag{35}$$

The conditional cdf of Y given $X = x$ is given by

$$F(y|x) = F_Y(y) + \alpha[(p+1)F_X^p(x) - 1][F_X^{p+1}(y) - F_Y(y)]. \tag{36}$$

Let $(U_{(i)}, V_{[i]})$, $i = 1,2,...n$ represent the first observed n upper record values from the standard exponential distribution and their concomitants. The cdf of V given $U = u$ is given by

$$F(v|u) = (1\text{-}e^{-v}) + \alpha[(p+1)(1\text{-}e^{-u})^p - 1][(1\text{-}e^{-v})^{p+1} - (1\text{-}e^{-v})]. \tag{37}$$

Suppose that the mth predicted record value and its concomitant is $(X^*_{(m)}, \tilde{Y}^*_{[m]})$ where $m > n$. Setting $F(v^*_{[m]} \mid u^*_{(m)}) = R$, where R is a random number, we can solve (37) in $v^*_{[m]}$ given $u^*_{(m)}$, where

$$u^*_{(m)} = (x^*_{(m)} - \hat{\mu}_1)/\hat{\lambda}_1, \tag{38}$$

and

$$v^*_{[m]} = (\tilde{y}^*_{[m]} - \hat{\mu}_2)/\hat{\lambda}_2, \tag{39}$$

Substituting with (27), (28) and (35) in (38) we find that

$$u^*_{(m)} = m.$$

Notice that the value of $v^*_{[m]}$ depends on the value of the random number R, and since $0 < R < 1$, so we can replace R by its mean (0.5). Thus substituting with $u^*_{(m)} = m$, and $R = 0.5$ in (37), we get

$$F(v^*_{[m]} \mid m) = (1\text{-}e^{-v^*_{[m]}}) + \alpha[(p+1)(1\text{-}e^{-m})^p - 1][(1\text{-}e^{-v^*_{[m]}})^{p+1} - (1\text{-}e^{-v^*_{[m]}})] = 0.5, \tag{40}$$

thus solving (40) in $v^*_{[m]}$ and substituting with its value in (39), we obtain the predicted value

$$\tilde{y}^*_{[m]} = \hat{\lambda}_2 v^*_{[m]} + \hat{\mu}_2. \tag{41}$$

Remarks:

- If α is unknown, we can replace it in (40) by its estimate given in (33).
- For improving $\tilde{y}^*_{[m]}$ and reducing the sensitivity of $v^*_{[m]}$ to R, we apply the following algorithm, using a variance reduction technique (see, Wilson (1984)),

  Algorithm 1:

  1-Generate a sequence of s paired random numbers $(R_1, 1-R_1)\ldots(R_s, 1-R_s)$.

  2-Solve (40) for $R = R_i$ to obtain $v^*_{i[m]}$ and for $R = 1-R_i$ to obtain $v'_{i[m]}$.

  3-Compute $v_{i[m]} = \dfrac{v^*_{i[m]} + v'_{i[m]}}{2}$.

  4- $v^*_{[m]} = \dfrac{1}{s}\sum_{i=1}^{s} v_{i[m]}$.

7 Numerical Illustration

We calculated the coefficients a_i and b_i in the BLUEs $\hat{\mu}_2$ and $\hat{\lambda}_2$ of μ_2 and λ_2, respectively, given by (30) and (31) for $i = 1(1)10$ and taking arbitrary values for α in the admissible range $(-.02, 0.1, 0.15)$, for $p = 6.067 \cong 6$ (the strongest positive correlation coefficient case). The results are presented in Tables 2-4.

From Tables 2&3, we can see tha $\mathrm{var}(\hat{\mu}_2)$ and $\mathrm{var}(\hat{\lambda}_2)$ decreases as the value of the association parameter α and the number of concomitants increase.

In order to illustrate numerically the estimators obtained and the predicted concomitants, we generated 9 observations of record values and their concomitants from (2) with exponential margins in (19) with $\alpha = 0.15$, $\mu_1 = 4$, $\lambda_1 = 2$, $\mu_2 = 10$, $\lambda_2 = 5$, as follows: (7.8691,12.4961), (8.5810,12.0865), (10.6856,26.3902), (14.5684,17.8104), (16.4118,20.9911), (20.6285,19.5828), (21.9594,20.1107), (23.4171,12.1424), (25.5482,20.3680).

We assume that we have only 8 or 6 observations and we require to predict the 9th concomitant value ($m = n+1$ or $m = n+3$). For calculating the BLUP in (34) or the conditional predictor in (41) we must first calculate $\hat{\mu}_2$ and $\hat{\lambda}_2$. Assuming α is known, using (30), (31) and the coefficients a_i and b_i, $i = 1,..,n$ given in Tables 2&3 for $\alpha = 0.15$, we get

$$\text{for } n = 8, \quad \hat{\mu}_2 = 9.0036 \text{ and } \hat{\lambda}_2 = 4.2225,$$
$$\text{for } n = 6, \quad \hat{\mu}_2 = 6.9433 \text{ and } \hat{\lambda}_2 = 5.6185.$$

First method (BLUP):

Calculating $\varepsilon_9, w', \Sigma^{-1}$ and ε_n and substituting with the corresponding $\hat{\mu}_2$ and $\hat{\lambda}_2$ in (34) we get

$$\text{for } n=8, \quad Y^*_{[9]} = 19.2345,$$
$$\text{for } n=6, \quad Y^*_{[9]} = 21.6544.$$

Second method (The conditional predictor):

Using (40) and (41), we get

$$\text{for } n=8, \quad \tilde{Y}^*_{[9]} = 18.4741,$$
$$\text{for } n=6, \quad \tilde{Y}^*_{[9]} = 21.1913.$$

while using Algorithm 1, with s=50 , we have

$$\text{for } n=8, \quad \tilde{Y}^*_{[9]} = 19.4517,$$
$$\text{for } n=6, \quad \tilde{Y}^*_{[9]} = 22.1582.$$

We see from the above results that the predicted values of the 9th concomitant using both methods are almost the same and near the true value of the 9th observation.

We may conclude that the conditional predictor is simple to apply and requires less calculations than the BLUP and gives satisfactory results. Moreover it can be improved by applying Algorithm 1.

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Table 2.The coefficients $a_i, i=1,...,n$ **in the BLUE** $\hat{\mu}_2=\sum_{i=1}^{n}a_i y_{[i]}$ **.**

n	α	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	$\mathrm{var}(\hat{\mu}_2)/\lambda_2^2$
2	-0.02	-1.5929	-1.5120									697.696
	0.1	-1.5929	-1.9970									44.0073
	0.15	-1.5929	-2.1991									21.3527
3	-0.02	-8.9471	-0.2502	10.1973								162.952
	0.1	3.0742	-0.1234	-1.9508								15.399
	0.15	2.3702	-0.0564	-1.3138								8.1959
4	-0.02	-5.7026	-5.7025	2.3214	6.4324							70.393
	0.1	2.3536	0.4447	-0.5937	-1.2046							8.9246
	0.15	1.0874	0.3342	-0.3776	-0.8306							5.0799
5	-0.02	-4.2864	-2.1891	0.3137	2.6616	4.5003						41.410
	0.1	2.0043	0.5820	-0.1624	-0.5844	-0.8395						6.4755
	0.15	1.6332	0.4272	-0.0851	-0.3845	-0.5909						3.8824
6	-0.02	-3.5610	- 2.1114	-0.3873	1.2262	2.4879	3.3455					29.174
	0.1	1.8119	0.6227	0.0179	-0.3143	-0.5104	-0.6278					5.2986
	0.15	1.5026	0.4534	0.0328	-0.1957	-0.3457	-0.4474					3.3094
7	-0.02	-3.1476	-2.0233	-.6899	0.5551	1.5270	2.1871	2.5915				22.97
	0.1	1.6966	0.6360	0.1078	-0.1752	-0.3387	-0.4352	-0.4912				4.6472
	0.15	1.4260	0.4611	0.0890	-0.1020	-0.2217	-0.3008	-0.3526				2.9976
8	-0.02	-2.8927	-1.9538	-.8432	0.1918	0.9986	1.5459	1.8811	2.0722			19.42
	0.1	1.6229	0.6405	0.1586	-0.094	-0.2382	-0.3219	-0.3701	-0.3971			4.250
	0.15	1.378	0.4631	0.1193	-0.0498	-0.1516	-0.2172	-0.2586	-0.2836			2.8118
9	-0.02	-2.7255	-1.9026	-0.9311	-0.0274	0.6761	1.1530	1.4448	1.6111	1.7016		17.181
	0.1	1.5732	0.6420	0.1903	-0.0434	-0.1742	-0.2494	-0.2924	-0.3164	-0.3294		3.9906
	0.15	1.3471	0.4634	0.1375	-0.0178	-0.1083	-0.1652	-0.2007	-0.2219	-0.234		2.6930
10	-0.02	-2.6102	-1.8649	-0.9868	-0.1711	0.4633	0.8928	1.1555	1.3052	1.3866	1.4296	15.679
	0.1	1.5382	0.6423	0.2115	-0.0087	-0.1306	-0.1999	-0.2393	-0.2611	-0.2730	-0.2793	3.8106
	0.15	1.3256	0.4632	0.1492	0.0031	-0.0796	-0.1306	-0.1620	-0.1806	-0.1912	-0.1969	2.6126

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Table 3. The coefficients $b_i, i=1,\ldots,n$ **in the BLUE** $\hat{\lambda}_2=\sum_{i=1}^{n} b_i y_{[i]}$.

n	α	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	$\mathrm{var}(\hat{\lambda}_2)/\lambda_2^2$
2	-0.02	1.5929	1.5929									736.01
	0.1	1.5929	1.5929									35.628
	0.15	1.5929	1.5929									15.853
3	-0.02	9.7696	0.6133	-10.3829								181.63
	0.1	-2.1861	0.3522	1.8339								10.346
	0.15	-1.4682	0.2568	1.2114								4.6667
4	-0.02	6.4117	2.4771	-2.2318	-6.6570							82.495
	0.1	-1.5386	-0.1583	0.6145	1.0824							5.1180
	0.15	-1.0352	-0.0839	0.3947	0.7245							2.2955
5	-0.02	4.9266	2.6218	-0.1264	-2.7027	-4.7193						50.621
	0.1	-1.2345	-0.2779	0.2390	0.5424	0.7309						3.2614
	0.15	-0.8328	-0.1622	0.1487	0.3493	0.4970						1.4482
6	-0.02	4.1584	2.5395	0.6160	-1.1826	-2.5881	-3.5431					36.898
	0.1	-1.0705	-0.3127	0.0853	0.3122	0.4505	0.5351					2.4064
	0.15	-0.7254	-0.1837	0.05170	0.1941	0.2955	0.3677					1.0612
7	-0.02	3.7175	2.4454	0.9387	-0.4669	-1.5635	-2.3078	-2.7636				29.846
	0.1	-0.9736	-0.3238	0.0098	0.1953	0.3061	0.3733	0.4129				1.9462
	0.15	-0.6634	-0.1900	0.0062	0.1183	0.1951	0.2490	0.2846				0.8567
8	-0.02	3.4444	2.3711	1.1029	-0.0777	-.9973	-1.6208	-2.0024	-2.2200			25.762
	0.1	-0.9122	-0.3275	-0.0326	0.1281	0.2224	0.2789	0.3120	0.3308			1.6708
	0.15	-0.6252	-0.1916	-0.0180	0.0764	0.1389	0.1819	0.2101	0.2275			0.7372
9	-0.02	3.2648	2.3160	1.1974	0.1579	-0.6509	-1.1986	-1.5337	-1.7246	-1.8284		23.183
	0.1	-0.8710	-0.3287	-0.0588	0.0857	0.1694	0.2189	0.2477	0.2640	0.2729		1.4926
	0.15	-0.6003	-0.1918	-0.0325	0.0509	0.1044	0.1404	0.1639	0.1783	0.1866		0.6617
10	-0.02	3.1406	2.2755	1.2574	0.3125	-0.4217	-0.9186	-1.2223	-1.3953	-1.4893	-1.5389	21.443
	0.1	-0.8421	-0.3291	-0.0764	0.0571	0.1333	0.1780	0.2038	0.2184	0.2263	0.2306	1.3699
	0.15	-0.5831	-0.1916	-0.0418	0.0342	0.0816	0.1130	0.1331	0.1455	0.1526	0.1565	0.6108

Table 4. $\mathrm{cov}(\hat{\mu}_2, \hat{\lambda}_2)/\lambda_2^2$.

n	2	3	4	5	6	7	8	9	10
α = -0.02	716.353	171.88	76.089	45.695	32.737	26.124	22.312	19.912	18.295
α = 0.1	39.318	12.424	6.606	4.474	3.470	2.923	2.592	2.377	2.229
α = 0.15	18.103	5.971	3.253	2.245	1.775	1.522	1.373	1.278	1.215