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Stability of the Vibrations of a Damped General Inhomogeneous Wave Equation

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ABSTRACT

In this paper, we consider the vibrations of an inhomogeneous damped wave under distributed disturbing force. The well-possedness of the system is studied. We prove that the amplitude of such vibrations is bounded under some restriction of the disturbing force. Finally, we establish the uniform exponential stabilization of the system when the disturbing force is insignificant. The results are achieved directly by means of an exponential energy decay estimate.

Keywords: Inhomogeneous wave; Energy decay estimate; Exponential stability; Bounded-input bounded-output; C_0 -semigroup; Distributed damping.

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1 Introduction

In this paper, we consider the mechanical vibrations of a clamped inhomogeneous string of length L governed by general one-dimensional wave equation

$$m(x)u_{tt}+2\alpha(x)u_t+\beta(x)u=u_{xx}+f(x,t), \quad on \quad (0,L)\times\mathbb{R}^+, \tag{1}$$

where the parameters $m(x),\ \alpha(x)$ and $\beta(x)$ respectively denote mass per unit length, coefficient of damping, the square of the natural frequency of the motion at the point x and $\mathbb{R}^+=(0,\infty).$ We assume that all the above functions are positive and continuous up to second order partial derivatives over $[0,L].$ For a general inhomogeneous string they belong to $C^2[0,L].$ The distributed force $f:(0,L)\times\mathbb{R}^+\to\mathbb{R}$ is the uncertain disturbance appearing in the model, which is continuously differentiable for all $t\geq 0.$

For a clamped string, the boundary conditions are

$$u(0,t)=0, \qquad u(L,t)=0 \quad on \quad \mathbb{R}^+ \tag{2}$$

and let initially the string is set to vibrate with

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) \quad on \quad (0,L). \tag{3}$$

The mathematical theory of stabilization of distributed parameter system is currently a topic of interest in view of application of vibrating control of various structures like string, beams, plate etc. The study of the stabilization is significant in the sense to suppress the vibrations to assure a good performance of the overall system.

The vibrations of exible structures are usually governed by nonlinear partial differential equation. Transverse vibrations of such equation is treated by Nayfeh [18] for a spatial variable beam. As the non-linear study of such structures is rather cumbersome for analytical treatment, linearized mathematical model are chosen just for simplicity and concise result. The question of energy decay estimates in context of boundary stabilization of a wave equation has earlier been studied by several authors (cf. Chen [1], [2], Gorain [4], Lagnese [11], [12], Komornik and Zuazua [8], and the reference therein). Chen [1] first established explicitly the exponential energy decay rate for the solution of wave equation by considering certain geometries of the domain. The theory of boundary stabilization of wave equation has been improved by Lagnese [12] and a faster energy decay rate is obtained by Komornik [9] constructing with a special type of feedback. There are different types of stability for the vibrations of exible structures and the most important of all these is the uniform stability. The question of uniform stabilization or point-wise stabilization of Euler-Bernoulli beams or serially connected beams has been studied by a number of authors (cf. K. Liu and Z. Liu [13], Lions [14] and references therein). Recently, Gorain [7] has established the uniform stabilization of longitudinal vibrations of inhomogeneous clamped beam. A bounded-input and bounded-output stability of a damped non-linear string is obtained by Shahruz [20], whereas Smyshlyaev et al. [21] discussed about the bounded stabilization of a 1-D wave equation with a in-domain antidumping.

Our aim in this work is to study the stability results of different types for the solutions of the mathematical problem (1) subject to the boundary (2) and initial condition (3). To achieve the results, we adopt a direct method by constructing suitable Layapunov functional associated with the energy functional of the system.

Lemma 1 *For every solution of* (1) - (3), *the total energy* $E:\mathbb{R}^+\to\mathbb{R}^+$ *is defined at time* t *by*

$$E(u(t))=\frac{1}{2}\left[\int_0^L u_x^2\,dx+\int_0^L \beta(x)u^2\,dx+\int_0^L m(x)u_t^2\,dx\right] \tag{4}$$

satisfies

$$\frac{d}{dt}E(u(t))=-2\int_0^L \alpha(x)u_t^2\,dx+\int_0^L u_t\,fdx. \tag{5}$$

Proof. We multiply (1) by $u_t.$ Integrating with respect to $x\in[0,L],$ using the boundary condition (2), the result follows.

Remark 1 *We have* $\dfrac{dE(u(t))}{dt}\neq 0,$ *it follows from* (5) *that the system is not energy conserving. On the other hand, when the uncertain disturbing force is not present, that is,* $f=f(x,t)\equiv 0$ *for all* $(x,t)\in(0,L)\times(0,\infty),$ *the system* (1) - (3) *is energy dissipating.*

Integrating(5) with respect to t over $[0,t]$, we get

$$E(u(t)) - E(u(0)) = -2\int_0^t\int_0^L [\alpha(x)(u_\tau(x,\tau))^2]dxd\tau \quad for \quad t \geq 0, \tag{6}$$

where

$$E(u(0)) = \frac{1}{2}\int_0^L [m(x)u_1{}^2 + \beta(x)u_0{}^2 + u_{0x}{}^2]dx. \tag{7}$$

In view of (6) and (7), we may conclude that $u_0 \in H_0^1(0,L)$ and $u_1 \in L^2(0,L)$, where

$$H_0^1(0,L) = \{\varphi : \varphi \in H^1(0,L) \quad and \quad \varphi(0) = 0 = \varphi(L)\}$$

is the subspace of the classical Sobolev space

$$H^1(0,L) = \{\varphi : \varphi \in L^2(0,L), \quad \frac{d\varphi}{dx} \in L^2(0,L)\}$$

of real valued function of order one. Then obviously

$$E(u(t)) \leq E(u(0)) < \infty \quad for \quad t \geq 0. \tag{8}$$

Now, we have to study bounded-input and bounded-output stability of the system in presence of uncertain input disturbance $f(x,t)$. We introduce two function spaces as specified in Gorain [6]

$$X := \{\varphi(x,t) : (0,L)\times\mathbb{R}^+ \to \mathbb{R} : \sup_{t\in\mathbb{R}^+}[\int_0^L \varphi^2 dx]^{\frac{1}{2}} < \infty\} \tag{9}$$

$$Y := \{\varphi(x,t) : (0,L)\times\mathbb{R}^+ \to \mathbb{R}^+ : \sup_{t\in\mathbb{R}^+}\sup_{x\in(0,L)} |\varphi| < \infty\} \tag{10}$$

with $\|\varphi\|_X = \sup_{t\in\mathbb{R}^+}[\int_0^L \varphi^2 dx]^{\frac{1}{2}} < \infty \quad and \quad \|\varphi\|_Y^2 = \sup_{t\in\mathbb{R}^+}\sup_{x\in(0,L)}|\varphi| < \infty$. From (9) and (10), it is clear that $Y \subset X$ as $L^\infty(0,L) \subset L^2(0,L)$.

2 Setting of the Semigroup

In this section, we obtain the existence and uniqueness of solution for the initial-boundary value problem (1)-(3). We will use the standard $L^2(0,L)$ space, the scalar product and norm are denoted by

$$\langle u,v\rangle_{L^2(0,L)} = \int_0^L u\bar{v}\, dx, \qquad \|u\|_{L^2(0,L)}^2 = \int_0^L u^2\, dx.$$

We have the Poincaré inequality

$$\|u\|_{L^2(0,L)}^2 \leq C_p \|u_x\|_{L^2(0,L)}^2, \quad for \quad all \ u \in H_0^1(0,L),$$

where C_p is the Poincaré constant.

Taking $u_t = v,$ the initial boundary problem (1)-(3) can be reduced to the following abstract initial value problem

$$\frac{d}{dt}U(t) = AU(t) + F(x,t), \qquad U(0) = U_0 \qquad for \quad all \ t > 0, \tag{11}$$

with $U(t) = (u,v)^T$ and $U_0 = (u_0,u_1)^T$, where the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m(x)}(u_{xx} - \beta(x)\,u - 2\,\alpha(x)\,v) \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ f(x,\,t) \end{pmatrix} \tag{12}$$

Now, we introduce the phase space $\mathcal{H} = H_0^1(0,L) \times L^2(0,L)$ endowed with the inner product given by

$$\langle (u,v),(u_1,v_1)\rangle_{\mathcal{H}} = \int_0^L u_x \overline{u}_{1x}\, dx + \int_0^L \beta(x) u \overline{u}_1\, dx + \int_0^L m(x) v \overline{v}_1\, dx,$$

for $U = (u,v)$, $\widetilde{U} = (u_1, v_1)$ and the norm by

$$\|(u,v)\|_{\mathcal{H}}^2 = \|u_x\|_{L^2(0,L)}^2 + \|\sqrt{\beta}u\|_{L^2(0,L)}^2 + \|\sqrt{m}v\|_{L^2(0,L)}^2.$$

We can easily show that the norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to usual norm in $\mathcal{H}$. Instead of dealing with (1), we will consider (11) in the Hilbert space H, with the domain $\mathcal{D}(\mathcal{A})$ of the operator A given by

$$D(A) = \{(u,v) \in H;\, u \in H^2(0,L) \cap H_0^1(0,L)\}. \tag{13}$$

Firstly, we show that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions on the space $\mathcal{H}$.

Proposition 1 *The operator $\mathcal{A}$ generates a C_0-semigroup $S_{\mathcal{A}}(t)$ of contractions on the space $\mathcal{H}$.*

Proof. We will show that $\mathcal{A}$ is a dissipative operator and 0 belongs to resolvent set of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$. Then our conclusion will follow using the well known Lumer-Phillips theorem (cf. [19]). We observe that if $U = (u,v) \in D(\mathcal{A})$, then

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= \int_0^L m(x)\frac{1}{m(x)}(u_{xx} - 2\alpha(x)v - \beta(x)u)\overline{v}\, dx + \int_0^L v_x \overline{u}_x\, dx \\
&\quad + \int_0^L \beta(x) v\overline{u}\, dx \\
&= \int_0^L u_{xx}\overline{v}\, dx - 2\int_0^L \alpha(x)|v|^2\, dx - \int_0^L \beta(x) u\overline{v} dx \\
&\quad + \int_0^L v_x \overline{u}_x\, dx + \int_0^L \beta(x) v\overline{u}\, dx \\
&= \int_0^L (v_x\overline{u}_x - u_x\overline{v}_x)\, dx + \int_0^L \beta(x)(v\overline{u} - u\overline{v})\, dx \\
&\quad - 2\int_0^L \alpha(x)|v|^2\, dx \\
&= 2i\,Im\int_0^L v_x\overline{u}_x\, dx + 2i\,Im\int_0^L \beta(x) v\overline{u}\, dx \\
&\quad - 2\int_0^L \alpha(x)|v|^2\, dx.
\end{aligned}$$

Taking the real part, we have

$$Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -2\int_0^L \alpha(x)|v|^2\, dx \le 0. \tag{14}$$

Thus $\mathcal{A}$ is a dissipative operator. Now, we show that $0 \in \rho(A)$. In fact, given $\mathcal{F} = (f_1, g_1) \in \mathcal{H}$, we must show that there exits a unique $U = (u,v) \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = \mathcal{F}$, that is,

$$v = f_1 \tag{15}$$

$$u_{xx} - 2\alpha(x)v - \beta(x)u = m(x)g_1. \tag{16}$$

In view of (15), we have from (16)

$$u_{xx} - \beta(x)u = m(x)g_1 + 2\alpha(x)f_1. \tag{17}$$

It is known that there is a unique $u \in H^2(0,L)$ satisfying (17). It is easy to show that $\|U\|_H \leq C\|F\|_H$.

From proposition 1 and theorem 2.4 in Pazy (cf. [19], page 107), we can state the following result (cf. [19]).

Theorem 1 *For any* $U_0 \in H$*, there exists a unique solution* $U(t) = (u,u_t)$ *of the system* (1) - (3) *satisfying*

$$u \in C([0,\infty[;H_0^1(0,L)) \cap C^1([0,\infty[;L^2(0,L)).$$

Moreover, if $U_0 \in \mathcal{D}(\mathcal{A}),$ *then*

$$u \in C([0,\infty[;H_0^1(0,L)) \cap C^1([0,\infty[;L^2(0,L)).$$

3 Stability Results

On account of uncertain disturbance force $f(x,t)$ as a input-disturbance, the system evolves from its initial state (u_0,u_1) to (u,u_t) at an instant t. The result of the bounded-output solution for the restriction of $f(x,t)$ is in the following theorems.

Theorem 2 *If* $u(x,t)$ *be the solution of the system* (1) - (3) *with* $f \in X$, *then* $u \in Y$ *for every set of initial values* $(u_0,u_1) \in H_0^1(0,L) \times L^2(0,L).$

Theorem 3 *Let* $u(x,t)$ *be the solution of the system* (1) - (3) *corresponding to the initial value* $(u_0,u_1) \in H_0^1(0,L) \times L^2(0,L)$ *then for every* $T > 0$

$$\int_0^T E(u(t))\, dt \leq \kappa E(u(0)) + \sigma \int_0^T \|f\|^2_{L^2(0,L)}\, dt, \tag{18}$$

where κ *and* σ *are positive constant given by* (55).

In an ideal case, when the uncertain disturbance force is not present in the system (1)-(3), then the energy function given by (4) is a dissipative function of time. So naturally a question arises as to whether the energy decay with time is exponentially or not and the answer of this question is found in the following theorem.

Theorem 4 *If* $u(x,t)$ *be the solution of the system* (1) - (3) *with* $f(x,t) \equiv 0$ *and* $(u_0,u_1) \in H_0^1(0,L)) \times L^2(0,L)$ *then the solution* $\to 0$ *exponentially as time* $t \to +\infty,$ *that is, the energy function satisfies*

$$E(u(t)) \leq Ae^{-\nu t} E(u(0)), \qquad for \qquad all\ t \in \mathbb{R}^+ \tag{19}$$

for some reals $\nu > 0$ *and* $A > 1.$

To prove the above theorems, we need the following inequalities and few lemmas.

I. For any real number $\alpha > 0$, we have Young's Inequality (cf. [16])

$$|f\ g| \leq \frac{1}{2}\left(\alpha |f|^2 + \frac{|g|^2}{\alpha}\right). \tag{20}$$

II. Poincaré type Scheeffer's inequality (cf. [16])

$$\int_0^L u^2\, dx \leq \frac{L^2}{\pi^2}\int_0^L u_x^2\, dx \leq \frac{L^4}{\pi^4}\int_0^L u_{xx}^2\, dx, \tag{21}$$

as $u(x,t)$ satisfies boundary condition in (2).

III. The Cauchy – Schwartz inequality for integral calculus (cf. [16])

$$\left|\int_0^L f\ g\ dx\right| \le \left[\int_0^L f^2 dx\right]^{\frac{1}{2}} \left[\int_0^L g^2\ dx\right]^{\frac{1}{2}}. \tag{22}$$

By Mean value theorem of integral calculus, there are reals $\xi_1,\ \xi_2,\ \eta_1,\ \eta_2$ and $\zeta \in [0,L]$ satisfying the following results :

$$\int_0^L m(x)u^2\ dx = m(\xi_1)\int_0^L u^2\ dx \tag{23}$$

$$\int_0^L m(x)u_t^2\ dx = m(\xi_2)\int_0^L u_t^2\ dx \tag{24}$$

$$\int_0^L \alpha(x)u^2\ dx = \alpha(\eta_1)\int_0^L u^2\ dx \tag{25}$$

$$\int_0^L \alpha(x)u_t^2\ dx = \alpha(\eta_2)\int_0^L u_t^2\ dx. \tag{26}$$

Moreover, we define

$$\mu_0 := \frac{L}{\pi}\sqrt{m(\xi_1)},\quad \mu_1 := 2\alpha(\eta_1)\frac{L^2}{\pi^2},\quad \mu_2 := \frac{\alpha(\eta_2)}{m(\xi_2)}. \tag{27}$$

It is obvious that $m(\xi_1),\ m(\xi_2),\ \alpha(\eta_1)$ and $\alpha(\eta_2)$ are all positive and they are bounded above by their corresponding upper bound over $[0,L]$.

Next, we need to established the following lemmas.

Lemma 2 *For every solution* $u = u(x,t)$ *of the system* (1) - (3), *the time derivative of the functional* $G(u(t))$ *defined by*

$$G(u(t)) := \int_0^L \left(m(x)uu_t + \alpha(x)u^2\right)dx \tag{28}$$

satisfies

$$G'(u(t)) = 2\int_0^L m(x)u_t^2\ dx + \int_0^L u\ f dx - 2E(u(t)). \tag{29}$$

Proof. Differentiate (28) with respect to t, we get

$$\frac{dG}{dt} = \int_0^L \left(m(x)uu_{tt} + m(x)u_t^2 + 2\alpha(x)uu_t\ \right)dx. \tag{30}$$

Using (1) and applying conditions in (2), we get from (30)

$$\frac{dG}{dt} = \int_0^L u\ f dx - \int_0^L \beta(x)u^2 + \int_0^L m(x)u_t^2\ dx + \int_0^L uu_x^2\ dx. \tag{31}$$

Using (4) in (31), we get

$$\frac{dG}{dt} = \int_0^L u\ f dx + 2\int_0^L m(x)u_t^2 dx - 2E(u(t)). \tag{32}$$

Hence the lemma is proved.

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Lemma 3 *The functional* $G(u(t))$ *given by* (28) *satisfies the inequality*

$$-\mu_0 E(u(t)) \le G(u(t)) \le (\mu_0+\mu_1)E(u(t)) \quad for \quad t \ge 0. \tag{33}$$

Proof. From (4), (21), (25) and (27), we get

$$\int_0^L \alpha(x)u^2\,dx = \alpha(\eta_1)\int_0^L u^2\,dx \le \alpha(\eta_1)\frac{L^2}{\pi^2}\int_0^L u_x^2\,dx$$

$$\le 2\alpha(\eta_1)\frac{L^2}{\pi^2}E(u(t)) = \mu_1 E(u(t)) \quad for \quad t \ge 0. \tag{34}$$

Also, using (4), (20), (21), (23) and (27), we get

$$\left|\int_0^L m(x)uu_t\,dx\right| = \left|\int_0^L \sqrt{m(x)}u\,\sqrt{m(x)}u_t\,dx\right|$$

$$\le \frac{1}{2}\left[\frac{\pi}{L}\frac{1}{\sqrt{m(\xi_1)}}\int_0^L m(x)u^2\,dx + \frac{L}{\pi}\sqrt{m(\xi_1)}\int_0^L m(x)u_t^2\,dx\right]$$

$$\le \frac{1}{2}\left[\frac{L}{\pi}\sqrt{m(\xi_1)}\int_0^L u_x^2 + \frac{L}{\pi}\sqrt{m(\xi_1)}\int_0^L m(x)u_t^2\,dx\right]$$

$$= \frac{1}{2}\frac{L}{\pi}\sqrt{m(\xi_1)}\int_0^L\left[m(x)u_t^2+u_x^2\right]dx$$

$$\le \frac{L}{\pi}\sqrt{m(\xi_1)}E(u(t)) = \mu_0 E(u(t)). \tag{35}$$

From (28), (34) and (35), we get,

$$-\mu_0 E(u(t)) \le G(u(t)) \le (\mu_0+\mu_1)E(u(t)) \quad for \quad t \ge 0.$$

Hence the lemma is proved.

We are now ready to prove the above theorems. For this, we proceed like Komornik [10], Gorain [4], [6]. Let us introduce an energy like Layapunov functional denoted by $V(u(t))$ and is defined by

$$V(u(t)) := E(u(t)) + \varepsilon G(u(t)) \quad for \quad t \ge 0, \tag{36}$$

where $\varepsilon > 0$ is a small real number given by (42). The lemma 3 yields for $V(u(t))$ that estimates

$$(1-\mu_0\varepsilon)E(u(t)) \le V(u(t)) \le [1+(\mu_0+\mu_1)\varepsilon]E(u(t)) \quad for \quad t \ge 0, \tag{37}$$

where we choose $\varepsilon < \dfrac{1}{\mu_0}$ such that $V(u(t)) \ge 0 \quad for \quad t \ge 0.$

From (20) and (21), we can estimate

$$\int_0^L u\,f dx \le \frac{1}{2}\left[\frac{2p\pi^2}{L^2}\int_0^L u^2 dx + \frac{L^2}{2p\pi^2}\int_0^L f^2 dx\right]$$

$$\le \frac{1}{2}\left[2p\int_0^L u_x^2\,dx + \frac{L^2}{2p\pi^2}\int_0^L f^2 dx\right]$$

$$= p\int_0^L u_x^2\,dx + \frac{L^2}{4p\pi^2}\int_0^L f^2\,dx. \tag{38}$$

Moreover, using (20) and (21), we can estimate

$$\int_0^L u_t\,f dx \le \frac{1}{2}\left[2\varepsilon\,pm(\eta_2)\int_0^L u_t^2\,dx + \frac{1}{2p\varepsilon m(\eta_2)}\int_0^L f^2 dx\right]$$

$$= \varepsilon\,p\int_0^L m(x)u_t^2\,dx + \frac{1}{4p\varepsilon m(\eta_2)}\int_0^L f^2\,dx, \tag{39}$$

where p is a real number satisfying $0 < p < 1.$ Now, taking time derivative of (36) and applying (5) and (29) with the inequalities (38) and (39), we get

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dE}{dt} + \varepsilon\frac{dG}{dt} \\
&= -2\int_0^L \alpha(x)u_t^2\,dx + \int_0^L u_t\,fdx + \varepsilon\int_0^L u\,fdx + 2\varepsilon\int_0^L m(x)u_t^2 dx - 2\varepsilon E(u(t)) \\
&\le -2\int_0^L \alpha(x)u_t^2 dx - 2\varepsilon E(u(t)) + 2\varepsilon\int_0^L m(x)u_t^2\,dx + p\varepsilon\int_0^L m(x)u_t^2 dx \\
&\quad + p\varepsilon\int_0^L u_x^2\,dx + \frac{1}{4p\varepsilon m(\eta_2)}\int_0^L f^2\,dx + \frac{L^2\varepsilon}{4p\pi^2}\int_0^L f^2 dx \\
&= -2\varepsilon E(u(t)) + p\varepsilon\int_0^L\left[m(x)u_t^2 + u_x^2\right]dx - 2(\mu_2 - \varepsilon)\int_0^L m(x)u_t^2\,dx \\
&\quad + \left(\frac{1}{4p\varepsilon m(\eta_2)} + \frac{L^2\varepsilon}{4p\pi^2}\right)\int_0^L f^2\,dx \\
&\le -2\varepsilon(1-p)E(u(t)) - 2(\mu_2 - \varepsilon)\int_0^L m(x)u_t^2\,dx + C\int_0^L f^2\,dx,
\end{aligned} \tag{40}$$

where

$$C = \frac{1}{4p}\left(\frac{1}{\varepsilon m(\xi_2)} + \frac{L^2\varepsilon}{\pi^2}\right). \tag{41}$$

Since $\varepsilon > 0$ is small, we assume that

$$0 < \varepsilon < \varepsilon_0 := min\left\{\frac{1}{\mu_0},\, \mu_2\right\}. \tag{42}$$

Hence, from (40), we get the differential inequality

$$\frac{dV}{dt} \le -2(1-p)\varepsilon E(u(t)) + C\|f\|_{L^2(0,L)}^2$$

$$\leq \frac{-2(1-p)\varepsilon}{1+(\mu_0+\mu_1)\varepsilon} V(u(t)) + C\|f\|^2_{L^2(0,L)},$$

in view of (37). Thus we have

$$\frac{dV}{dt} + \lambda V \leq C\|f\|^2_{L^2(0,L)}, \tag{43}$$

where

$$\lambda = \frac{2(1-p)\varepsilon}{1+(\mu_0+\mu_1)\varepsilon} > 0. \tag{44}$$

Multiplying (43) by $e^{\lambda t}$ and integrating over $[0,t]$ for any $t \geq 0$, we get

$$e^{\lambda t} V(u(t)) - V(u(0)) \leq \int_0^t C\|f\|^2_{L^2(0,L)} e^{\lambda \tau} d\tau.$$

Thus we have

$$V(u(t)) \leq e^{-\lambda t}\left[V(u(0)) + \int_0^t C\|f\|^2_{L^2(0,L)} e^{\lambda \tau}\, d\tau\right]. \tag{45}$$

Using (37) into (45), we get

$$E(u(t)) \leq \frac{1}{(1-\mu_0\varepsilon)}\left[(1+(\mu_0+\mu_1)\varepsilon)E(u(0))e^{-\lambda t} + C\int_0^t \|f\|^2_{L^2(0,L)} e^{-(t-\tau)\lambda} d\tau\right], \tag{46}$$

where $E(u(0))$ is given by (7).

Proof of theorem 2 Let $f \in X$ such that $\|f\|_X = \sup_{t\in\mathbb{R}^+}\|f\|_{L^2(0,L)} < \infty.$ Putting $t-\tau=\theta$ in (46), we get

$$E(u(t)) \leq \frac{1}{1-\mu_0\varepsilon}[(1+(\mu_0+\mu_1)\varepsilon)E(u(0))e^{-\lambda t} + C\int_0^t \|f\|^2_X e^{-\lambda\theta}\, d\theta]$$

$$\leq \frac{1}{1-\mu_0\varepsilon}[(1+(\mu_0+\mu_1)\varepsilon)E(u(0))e^{-\lambda t} + C\|f\|^2_X \int_0^\infty e^{-\lambda\theta}\, d\theta]$$

$$\leq \frac{1}{\lambda(1-\mu_0\varepsilon)}[2(1-p)\varepsilon E(u(0)) + C\|f\|^2_X] \quad for \quad t \in \mathbb{R}^+. \tag{47}$$

Hence

$$\sup_{t\in\mathbb{R}^+} E(u(t)) < \infty \tag{48}$$

for every set of initial value $(u_0,u_1) \in H^1_0(0,L)) \times L^2(0,L)$ and for every $f \in X$. Thus the energy of the system (1)-(3) is uniformly bounded function of time.

Again from (2), we have $u(0,t)=0$ so we can write

$$|u(x,t)| = \left|\int_0^x u_x\, dx\right| \leq \left(\int_0^L 1^2\, dx\right)^{\frac{1}{2}} \left(\int_0^L u_x^2\, dx\right)^{\frac{1}{2}} \leq L^{\frac{1}{2}}\left(\int_0^L u_x^2\, dx\right)^{\frac{1}{2}}, \tag{49}$$

for all $(x,t) \in (0,L)\times\mathbb{R}^+$. Thus, in view of (4)

$$|u(x,t)|^2 \leq L\int_0^L u_x^2\, dx \leq 2LE(u(t)) < \infty \tag{50}$$

for every $(x,t)\in(0,L)\times\mathbb{R}^{+}$. Hence

$$u\in Y, \tag{51}$$

for every set of initial values $(u_0,u_1)\in H_0^1(0,L))\times L^2(0,L)$ and for every $f\in X$.

Hence the theorem is proved.

Remark 2 Thus the above result shows that if *the output solution* u *is* Y *-bounded for every* X *-bounded input disturbance* f. *Thus the system is bounded-input bounded-output stable.*

Proof of theorem 3 Integrating (46) over $[0,T]$ for $T>0$ we get

$$\int_0^T E(u(t))\,dt \le \frac{1}{1-\mu_0\varepsilon}\left[(1+(\mu_0+\mu_1)\varepsilon)E(u(0))\int_0^T e^{-\lambda t}\,dt + C\int_0^T e^{-\lambda t}F(t)\,dt\right], \tag{52}$$

where

$$F(t)=\int_0^t \|f\|_{L^2(0,L)}^2 e^{\lambda\tau}\,d\tau. \tag{53}$$

Integrating (52) by parts, we have

$$\begin{aligned}\int_0^T E(u(t))\,dt &\le \frac{1}{1-\mu_0\varepsilon}\left[\frac{2(1-p)\varepsilon}{\lambda^2}E(u(0))(1-e^{-\lambda T})\right.\\ &\quad\left.+\frac{C}{\lambda}\left(F(0)-e^{-\lambda T}F(t)+\int_0^T e^{-\lambda t}\frac{dF}{dt}\,dt\right)\right]\\ &=\kappa E(u(0))(1-e^{-\lambda T})\\ &\quad+\sigma\left[F(0)-e^{-\lambda T}F(t)+\int_0^T e^{-\lambda t}\frac{dF}{dt}\,dt\right]\\ &\le \kappa E(u(0))+\sigma\int_0^T \|f\|_{L^2(0,L)}^2\,dt,\end{aligned} \tag{54}$$

where

$$\kappa=\frac{2(1-p)\varepsilon}{\lambda^2(1-\mu_0\varepsilon)} \qquad and \qquad \sigma=\frac{C}{\lambda(1-\mu_0\varepsilon)}. \tag{55}$$

Since $F(0)=0$ and $\dfrac{dF}{dt}=e^{\lambda t}\|f\|_{L^2(0,L)}^2$.

Hence the theorem is proved.

Remark 3 *Thus the above result shows that if* $u(x,t)$ *is a solution of the system* (1) - (3) *with* $f\in L^2(0,T,H^2(0,L))$ *then the solution* $u\in L^2(0,T,H_0^1(0,L))$. *The factor* σ *in* (55) *may be defined as the tolerance factor of this disturbing force* f *on the total energy over* $[0,T]$.

Proof of theorem 4 When the disturbing force $f(x,t)$ is not taking into our account in the equation (1), the result (5) shows that the energy $E(u(t))$ of the system (1)-(3) is a non-increasing function of time. Consequently, the terms in (38) and (39) are insignificant following (5) and (29). Thus we can get rid off the terms involving p in (40) and hence differential inequality (43) becomes

$$\frac{dV}{dt} + \nu V \le 0 \quad for \quad t \ge 0, \tag{56}$$

where

$$\nu = \frac{2\varepsilon}{1+(\mu_0+\mu_1)\varepsilon}. \tag{57}$$

Multiplying (56) by $e^{\nu t}$ and integrating over $[0,t]$ for any $t \in \mathbb{R}^+$ we get

$$V(u(t)) \le e^{-\nu t} V(u(0)). \tag{58}$$

Using (37) into (58), we get

$$E(u(t)) \le \frac{1+(\mu_0+\mu_1)\varepsilon}{1-\mu_0\varepsilon} e^{-\nu t} E(u(0)).$$

Thus

$$E(u(t)) \le A e^{-\nu t} E(u(0)),$$

where

$$A = \frac{1+(\mu_0+\mu_1)\varepsilon}{1-\mu_0\varepsilon} > 1. \tag{59}$$

Hence the theorem is proved.

Remark 4 *Thus the above result shows that the solution of the system* (1)-(3) *decay exponentially with time and* $u(x,t) \to 0$ *as* $t \to +\infty$ *for every* $(u_0,u_1) \in H_0^1(0,L)) \times L^2(0,L).$

Remark 5 *The exponential stability result* (19) *can be obtain directly by setting* $f \equiv 0$ *in* (46). *In that case, the exponential decay rate of energy would be* λ *which is less than* ν *by an amount* $\frac{2p\varepsilon}{1+(\mu_0+\mu_1)\varepsilon}$. *Thus the exponential energy decay rate* ν *given by* (57) *is a stronger one. Again, since*

$$\nu = \frac{2\varepsilon}{1+(\mu_0+\mu_1)\varepsilon},$$

we have

$$\frac{d\nu}{d\varepsilon} = \frac{2}{1+(\mu_0+\mu_1)\varepsilon} > 0.$$

Thus the exponential decay rate ν *as a function of* ε *will be maximum for the largest admissible value of* ε *satisfying the constraint* (42). *An upper bound of which is followed by* ε_0 *that depends explicitly on* μ_0 *and* μ_2, *as defined by* (27). *It signifies that the decay of energy will be slower for a longer string.*

4 Conclusion

This study deals with different types mathematical stability results of a vibrating clamped string medelled by general inhomogeneous wave equation (1). We have established the boundedness of output solution under boundedness of input disturbances. Here we also estimate the total energy of the system over any time interval with a tolerance level of input disturbance. We have also prove that the energy of the system decay exponentially with time whenever the input disturbance is not so important.

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