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Some common fixed point results of three self-mappings in cone metric spaces

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Abstract

The aim of this paper is to present coincidence point and common fixed point results for three self mappings satisfying generalized contractive conditions. The results presented in this paper generalize and extend several well-known results in the literature.

Key Words: Cone metric spaces; Coincidence point; Common fixed point.

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1. Introduction and Preliminaries

In 2007, Huang and Zhang [1] introduced the concept of cone metric spaces which is a generalization of metric spaces, by replacing the set of real numbers by on ordered Banach space and proves some fixed point theorems for some contractive maps in normal cone metric spaces. Subsequently, some other authors [2,3,4] studied fixed point results of mappings satisfying contractive type condition in cone metric spaces, however there exists non-normal cone metric spaces [5].

Recently, Stojan Radenovic [6] has obtained coincidence point results for two mappings in cone metric spaces which satisfies new contractive conditions. The same concept was further extended by M. Rangamma and K. Prudhvi [7], Malhotra et al. [8] and proved coincidence point results and common fixed point results for three self mappings. The purpose of this paper is to generalize, extend and improves the results of [7] and [8].

We recall some definitions and properties of cone metric spaces[1].

Definition 1.1[1]. Let E be a real Banach space and P be a subset of E. The set P is called a cone if :

i) P is closed, non-empty and $P \neq \{0_E\}$, here 0_E is the zero vector of E;

ii) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;

iii) $x \in P$ and $-x \in P \Rightarrow x = 0_E$.

Given a cone $P \subset E$, we define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preccurlyeq y$ but $x \neq y$, while $x \ll y$ if and only if for $y - x \in int\, P$, where $int\, P$ denotes the interior of P.

Let P be a cone in a real Banach space E, then P is called normal, if there exist a constant $K > 0$ such that for all $x, y, \in E$,

$$0_E \preccurlyeq x \preccurlyeq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.2[1]. Let X be a non-empty set, E be a real Banach space. Suppose that the mapping $d: X \times X \to E$ satisfies

(i) $0_E \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.3[1]. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

i) If for every $c \in E$ with $0 \ll c$ there is a positive integer n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$, then the sequence $\{x_n\}$ is said to be convergent and converges to x. We denote this by $lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

ii) If for every $c \in E$ with $0 \ll c$ there is a positive integer n_0 such that, $d(x_n, x_m) \ll c$ for all $n, m > n_0$, then the sequence $\{x_n\}$ is called a Cauchy sequence in X.

(X, d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

Lemma 1.1[1]. Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X.

i) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0_E$ as $n \to \infty$.

ii) If $x_n \to x, y_n \to y$, as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Remark 1.1[4]. Let P be a cone in a real Banach space E with zero vector 0_E and $a, b, c \in P$, then;

a) If $a \preccurlyeq b$ and $b \ll c$ then $a \ll c$.

b) If $a \ll b$ and $b \ll c$ then $a \ll c$.

c) If $0_E \preccurlyeq u \ll c$ for each $c \in int\, P$ then $u = 0_E$.

d) If $c \in int\, P$ and $a_n \to 0_E$ then there exist $n_0 \in N$ such that, for all $n > n_0$ we have $a_n \ll c$.

e) If $0_E \preccurlyeq a_n \preccurlyeq b_n$ for each n and $a_n \to a, b_n \to b$ then $a \preccurlyeq b$.

f) If $a \preccurlyeq \lambda a$ where $0 \leq \lambda < 1$ then $a = 0_E$.

Let E, B be two real Banach spaces, P and C normal cones in E and B respectively. Let " $\preccurlyeq$ " and " $\leq$ " be the partial orderings induced by P and C in E and B respectively. Let $\emptyset: P \to C$ be a function satisfying:

i) If $a, b \in P$ with $a \preccurlyeq b$ then $\emptyset[a] \leq k\emptyset[b]$, for some positive real k;

ii) $\emptyset[a+b] \leq \emptyset[a] + \emptyset[b]$ for all $a, b \in P$;

iii) $\emptyset$ is sequentially continuous i.e. if $a_n, a \in P$ and $lim_{n\to\infty} a_n = a$, then $lim_{n\to\infty}\emptyset[a_n] = \emptyset[a]$;

iv) If $\emptyset[a_n] \to 0_B$ then $a_n \to 0_E$, where 0_E and 0_B are the zero vectors of E and B respectively. We denote the set of all such functions by $\Phi(P, C)$ i.e. $\emptyset \in \Phi(P, C)$ if $\emptyset$ satisfies all above properties. It is clear that $\emptyset[a] = 0_B$ if and only if $a = 0_E$.

Let (X, d) be a cone metric space with normal cone P and $\emptyset \in \Phi(P, C)$. Since $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in X$, therefore

$$\emptyset[d(x,y)] \leq k\emptyset[d(x,z)] + k\emptyset[d(z,y)] \qquad \text{--------(1.1)}$$

Example 1.1[8]. Let E be any real Banach space with normal cone P and normal constant K. Define $\emptyset: P \to P$ by $\emptyset[a] = a$, for all $a \in P$. Then $\emptyset \in \Phi(P, C)$ with $E = B, P = C$ and $k = 1$.

2. Main Results

Theorem 2.1. Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose f, g, h be self maps of X satisfy the condition

$$\emptyset[d(fx, gy)] \leq a\emptyset[d(hx, hy)] + b\emptyset[d(hx, fx) + d(hy, gy)] \qquad \text{--------(2.1)}$$

for all $x, y \in X$, where $\emptyset \in \Phi(P, C)$ and a, b are nonnegative reals with $a + 2b < 1$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is complete subspace of X, then the maps f, g and h have a unique point of coincidence in X. Moreover, if (f, h) and (g, h) are weakly compatible pairs then f, g and h have a unique common fixed point.

Proof. Suppose x_o be any arbitrary point of X. Since $f(X) \cup g(X) \subseteq h(X)$, starting with x_o we define a sequence $\{y_n\}$ such that

$y_{2n} = fx_{2n} = hx_{2n+1}\ and\ y_{2n+1} = gx_{2n+1} = hx_{2n+2}$, for all $n \geq 0$. We shall prove that $\{y_n\}$ is a Cauchy sequence in X.

If $y_n = y_{n+1}$ for some n e.g. if $y_{2n} = y_{2n+1}$, then from (2.1) we obtain

$$\begin{aligned}\emptyset[d(y_{2n+2}, y_{2n+1})] &= \emptyset[d(fx_{2n+2}, gx_{2n+1})] \\ &\leq a\emptyset[d(hx_{2n+2}, hx_{2n+1})] + b\emptyset[d(hx_{2n+2}, fx_{2n+2}) + d(hx_{2n+1}, gx_{2n+1})] \\ &= a\emptyset[d(y_{2n+1}, y_{2n})] + b\emptyset[d(y_{2n+1}, y_{2n+2}) + d(y_{2n}, y_{2n+1})]\end{aligned}$$

Since $y_{2n} = y_{2n+1}$, it follows from above inequality that,

$$\emptyset[d(y_{2n+2}, y_{2n+1})] \leq b\emptyset[d(y_{2n+1}, y_{2n+2})].$$

As $b < 1$ from (f) of remark 1.1, we obtain

$\emptyset[d(y_{2n+2}, y_{2n+1})] = 0_B$ also $\emptyset \in \Phi(P, C)$ therefore we have

$d(y_{2n+2}, y_{2n+1}) = 0_E$ i.e. $y_{2n+2} = y_{2n+1}$.

Similarly we obtain that

$y_{2n} = y_{2n+1} = y_{2n+2} = ------= \vartheta$ (say).

Therefore $\{y_n\}$ is a Cauchy sequence.

Suppose $y_n \neq y_{n+1}$ for all n. Then from (2.1) it follows that

$$\begin{aligned}\emptyset[d(y_{2n}, y_{2n+1})] &= \emptyset[d(fx_{2n}, gx_{2n+1})] \\ &\leq a\emptyset[d(hx_{2n}, hx_{2n+1})] + b\emptyset[d(hx_{2n}, fx_{2n}) + d(hx_{2n+1}, gx_{2n+1})] \\ &= a\emptyset[d(y_{2n-1}, y_{2n})] + b\emptyset[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\end{aligned}$$

$$i.e\ \emptyset[d(y_{2n}, y_{2n+1})] \leq \frac{a+b}{1-b}\emptyset[d(y_{2n-1}, y_{2n})]$$

$$= \lambda\emptyset[d(y_{2n-1}, y_{2n})]$$

Where $\lambda = \frac{a+b}{1-b} < 1 (since\ a + 2b < 1\ ).$

Writing $d_n = \emptyset[d(y_n, y_{n+1})]$, we obtain

$$d_{2n} \leq \boldsymbol{\lambda} d_{2n-1} \qquad \text{----------(2.2)}$$

Again

$$\emptyset[d(y_{2n+2}, y_{2n+1})] = \emptyset[d(fx_{2n+2}, gx_{2n+1})]$$
$$\leq a\emptyset[d(hx_{2n+2}, hx_{2n+1})] + b\emptyset[d(hx_{2n+2}, fx_{2n+2}) + d(hx_{2n+1}, gx_{2n+1})]$$
$$= a\emptyset[d(y_{2n+1}, y_{2n})] + b\emptyset[d(y_{2n+1}, y_{2n+2}) + d(y_{2n}, y_{2n+1})]$$

i.e. $\emptyset[d(y_{2n+2}, y_{2n+1})] \leq \frac{a+b}{1-b}\ \emptyset\ [d(y_{2n+1}, y_{2n})]$
$$= \mu\emptyset[d(y_{2n+1}, y_{2n})]$$

Where $\mu = \frac{a+b}{1-b} < 1(since\ \ a + 2b < 1).$

Therefore $d_{2n+1} \leq \mu d_{2n}$ ---------(2.3)

From (2.2) and (2.3) we get

$$d_{2n} \leq \lambda d_{2n-1} \leq \lambda\mu d_{2n-2} \leq ----\leq \lambda^n \mu^n d_0$$

and

$$d_{2n+1} \leq \mu d_{2n} \leq \lambda\mu d_{2n-1} \leq ----\leq \lambda^n \mu^{n+1} d_0.$$

Thus

$$d_{2n} + d_{2n+1} \leq \lambda^n \mu^n (1 + \mu) d_0 \quad \text{--------(2.4)}$$

and

$$d_{2n+1} + d_{2n+2} \leq \lambda^n \mu^{n+1} (1 + \lambda) d_0 \quad \text{-------(2.5)}$$

Let $n, m \in N$, then for the sequence $\{y_n\}$ we consider $\emptyset[d(y_n, y_m)]$ in two cases.

If n is even and $m > n$, then using (1.1) and (2.4) we obtain

$$\emptyset[d(y_n, y_m)] \leq k\ \emptyset[d(y_n, y_{n+1})] + k\emptyset[d(y_{n+1}, y_{n+2})] +$$
$$-------- + k\emptyset[d(y_{m-1}, y_m)]$$
$$\leq k[d_n + d_{n+1} + d_{n+2} + d_{n+3} + ----]$$

$$\leq k[\ \lambda^{\frac{n}{2}} \mu^{\frac{n}{2}} (1 + \mu)\ d_0 + \lambda^{\frac{n+2}{2}} \mu^{\frac{n+2}{2}} (1 + \mu) d_0 + ---]$$
$$\emptyset[d(y_n, y_m)] \leq \frac{k(\lambda\mu)^{n/2}(1+\mu)}{1-\lambda\mu}\, d_0.$$

If n is odd and $m > n$, then again using (1.1) and (2.5) we obtain

$$\emptyset[d(y_n, y_m)] \leq k\emptyset[d(y_n, y_{n+1})] + k\emptyset[d(y_{n+1}, y_{n+2})] +$$
$$-------- + k\emptyset[d(y_{m-1}, y_m)]$$
$$\leq k[d_n + d_{n+1} + d_{n+2} + d_{n+3} + ----]$$
$$\leq k[\lambda^{\frac{n-1}{2}} \mu^{\frac{n-1}{2}+1} (1 + \lambda) d_0 + \lambda^{\frac{n+1}{2}} \mu^{\frac{n+1}{2}+1} (1 + \lambda) d_0 + ---]$$
$$\emptyset[d(y_n, y_m)] \leq \frac{k(\lambda\mu)^{\frac{n-1}{2}}(1+\lambda)}{1-\lambda\mu}\, d_0.$$

Since $\lambda < 1, \mu < 1$ therefore $\lambda\,\mu < 1$, so in both the cases $\emptyset[d(y_n, y_m)] \to 0_B$ as $n \to \infty$, and since $\emptyset \in \Phi(P, C)$ we have $d(y_n, y_m) \to 0_E$ as $n \to \infty$. So by lemma 1.1, $\{y_n\} = \{hx_{n-1}\}$ is a Cauchy sequence.

Since $h(X)$ is complete, there exists $\vartheta \in h(X)$ and $u \in X$ such that $lim_{n\to\infty} y_n = \vartheta$ and $\vartheta = hu$.

We shall show that u is a coincidence point of pairs (f, h) and (g, h) i.e. $fu = gu = hu$.

If $fu \neq hu$ then $0_E \prec d(fu, hu)$. Using (2.1) we obtain

$$\emptyset[d(fu, y_{2n+1})] = \emptyset[d(fu, gx_{2n+1})]$$
$$\leq a\emptyset[d(hu, hx_{2n+1})] + b\emptyset[d(hu, fu) + d(hx_{2n+1}, gx_{2n+1})]$$
$$= a\emptyset[d(hu, y_{2n})] + b\emptyset[d(hu, fu) + d(y_{2n}, y_{2n+1})]$$

Since $y_{2n} \to hu, d_{2n} \to 0_B, d(fu, y_{2n+1}) \to d(fu, hu)$ as $n \to \infty$ and $\emptyset \in \Phi\,(P, C)$, therefore letting $n \to \infty$ in above inequality and using remark 1.1 we get

$$\emptyset[d(fu, hu)] \leq b\emptyset[d(hu, fu)]$$
$$< \emptyset[d(hu, fu)] \text{ (since } b < 1\text{)},$$

a contradiction. Therefore $fu = hu$. Similarly it can be shown that $gu = hu$.

Therefore $\quad fu = gu = hu = \vartheta$ -------------(2.6)

Thus ϑ is point of coincidence of pairs (f, h) and (g, h). We shall show that it is unique.

Suppose w is another point of coincidence of these pairs i.e. $fz = gz = hz = w$ for some $z \in X$.

Then from (2.1) it follows that

$$\begin{aligned}\emptyset[d(w, \vartheta)] &= \emptyset[d(fz, gu)] \\ &\leq a\emptyset[d(hz, hu)] + b\emptyset[d(hz, fz) + d(hu, gu)] \\ &= a\emptyset[d(w, \vartheta)] + b\emptyset[d(w, w) + d(\vartheta, \vartheta)] \\ &= a\emptyset[d(w, \vartheta)].\end{aligned}$$

Since $a < 1$, by remark 1.1 we obtain

$\emptyset[d(w, \vartheta)] = 0_B$ i.e. $w = \vartheta$. Thus point of coincidence is unique.

If pairs (f, h) and (g, h) are weakly compatible, from (2.6) we have $f\vartheta = fhu = hfu = h\vartheta$ and $g\vartheta = ghu = hgu = h\vartheta$, therefore $f\vartheta = g\vartheta = h\vartheta = p$ (say). This shows that p is another point of coincidence, therefore by uniqueness, we must have $p = \vartheta$ i.e.

$$f\vartheta = g\vartheta = h\vartheta = \vartheta.$$

Thus ϑ is unique common fixed point of self maps f, g and h.

Corollary 2.1. Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose f, g, h be self maps of X satisfy the condition

$$\emptyset[d(fx, gy)] \leq \propto \emptyset[d(hx, hy)] + \beta\emptyset[d(hx, fx)]$$
$$+\gamma\emptyset[d(hy, gy)] \text{ for all } x, y \in X$$

where $\emptyset \in \Phi(P, C)$ and $\propto, \beta, \gamma$ are non negative reals with $\propto + \beta + \gamma < 1$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is complete subspace of X, then the maps f, g *and* h have a unique point of coincidence in X. Moreover, if (f, h) and (g, h) are weakly compatible pairs then f, g and h have a unique common fixed point.

Proof. The symmetric property of d and the above inequality imply that

$$\emptyset[d(fx, gy)] \leq \propto \emptyset[d(hx, hy)] + \frac{\beta+\gamma}{2}\emptyset[d(hx, fx) + d(hy, gy)]$$

By substituting $\propto = a$ and $\frac{\beta+\gamma}{2} = b$ in above inequality, we obtain the required result as given in Theorem 2.1. It is also the Theorem 2.1 of [8].

Corollary 2.2. Let (X, d) be a cone metric space and P be normal cone with normal constant K. Suppose the self maps f, g, h of X satisfy the condition

$$\emptyset[d(fx, gy)] \leq a\emptyset[d(hx, hy)] + b\emptyset[d(hx, gy) + d(hy, fx)] \text{ for all } x, y \in X,$$

where $\emptyset \in \Phi(P, C)$ and $a, b$are nonnegative reals with $a + 2b < 1$. If $f(X) \cup g(X) \subseteq h(X) and\ h(X)$ is a complete subspace of X, then the maps f, g and h have a unique point of coincidence in X. Moreover, if (f, h) and (g, h) are weakly compatible pairs then f, g and h have a unique common fixed point.

Theorem 2.2. Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose f, g, h be self maps of X satisfy the condition

$$\emptyset[d(fx, gy)] \leq a\emptyset[d(hx, hy)] + b\emptyset[d(hx, gy) + d(hy, fx)]$$
$$+c\emptyset[d(hx, fx) + d(hy, gy)] \text{ for all } x, y \in \qquad \text{--------(2.7)}$$

where $\emptyset \in \Phi(P, C)$ and a, b, c are nonnegative reals with $a + 2b + 2c < 1$. If $f(X) \cup g(X) \subseteq h(X)$ and $h(X)$ is complete subspace of X, then the maps f, g and h have a unique point of coincidence in X. Moreover, if (f, h) and (g, h) are weakly compatible pairs then f, g and h have a unique common fixed point.

Proof. Suppose x_0 be any arbitrary point of X. Since $f(X) \cup g(X) \subset h(X)$, starting with x_0 we define a sequence $\{y_n\}$ such that

$y_{2n} = fx_{2n} = hx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = hx_{2n+2}$, for all $n \geq 0$.We shall prove that $\{y_n\}$ is a Cauchy sequence in X.

If $y_n = y_{n+1}$ for some n, e.g. if $y_{2n} = y_{2n+1}$, then from (2.7) we obtain

$$\begin{aligned}
\emptyset[d(y_{2n+2}, y_{2n+1})] &= \emptyset[d(fx_{2n+2}, gx_{2n+1})] \\
&\leq a\emptyset[d(hx_{2n+2}, hx_{2n+1})] + b\emptyset[d(hx_{2n+2}, gx_{2n+1}) + d(hx_{2n+1}, fx_{2n+2})] \\
&\quad + c\emptyset[d(hx_{2n+2}, fx_{2n+2}) + d(hx_{2n+1}, gx_{2n+1})] \\
&= a\emptyset[d(y_{2n+1}, y_{2n})] + b\emptyset[d(y_{2n+1}, y_{2n+1}) + d(y_{2n}, y_{2n+2})] \\
&\quad + c\emptyset[d(y_{2n+1}, y_{2n+2}) + d(y_{2n}, y_{2n+1})]
\end{aligned}$$

Since $y_{2n} = y_{2n+1}$, it follows from above inequality that,

$$\begin{aligned}
\emptyset[d(y_{2n+2}, y_{2n+1})] &\leq b\emptyset[d(y_{2n+1}, y_{2n+2})] + c\emptyset[d(y_{2n+1}, y_{2n+2})] \\
&= (b + c)\emptyset[d(y_{2n+1}, y_{2n+2})]
\end{aligned}$$

As $b + c < 1$ from (f) of remark 1.1, we obtain

$\emptyset[d(y_{2n+2}, y_{2n+1})] = 0_B$ also $\emptyset \in \Phi\,(P, C)$ therefore we have

$d(y_{2n+2}, y_{2n+1}) = 0_E$ i.e. $y_{2n+2} = y_{2n+1}$.

Similarly we obtain that

$y_{2n} = y_{2n+1} = y_{2n+2} = ------= \vartheta$ (say).

Therefore $\{y_n\}$ is a Cauchy sequence.

Suppose $y_n \neq y_{n+1}$ for all n. Then from (2.7) it follows that

$$\begin{aligned}
\emptyset[d(y_{2n}, y_{2n+1})] &= \emptyset[d(fx_{2n}, gx_{2n+1})] \\
&\leq a\emptyset[d(hx_{2n}, hx_{2n+1})] + b\emptyset[d(hx_{2n}, gx_{2n+1}) + d(hx_{2n+1}, fx_{2n})] \\
&\quad + c\emptyset[d(hx_{2n}, fx_{2n}) + d(hx_{2n+1}, gx_{2n+1})] \\
&= a\emptyset[d(y_{2n-1}, y_{2n})] + b\emptyset[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})] \\
&\quad + c\emptyset[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]
\end{aligned}$$

$$\begin{aligned}
i.e\ \emptyset[d(y_{2n}, y_{2n+1})] &\leq \frac{a+b+c}{1-b-c}\emptyset[d(y_{2n-1}, y_{2n})] \\
&= \lambda\emptyset[d(y_{2n-1}, y_{2n})],
\end{aligned}$$

Where $\lambda = \frac{a+b+c}{1-b-c} < 1(since\ a + 2b + 2c < 1\,).$

Writing $d_n = \emptyset[d(y_n, y_{n+1})]$, we obtain

$$d_{2n} \leq \boldsymbol{\lambda} d_{2n-1} \qquad \text{----------(2.8)}$$

Again

$$\begin{aligned}
\emptyset[d(y_{2n+2}, y_{2n+1})] &= \emptyset[d(fx_{2n+2}, gx_{2n+1})] \\
&\leq a\emptyset[d(hx_{2n+2}, hx_{2n+1})] + b\emptyset[d(hx_{2n+2}, gx_{2n+1}) + d(hx_{2n+1}, fx_{2n+2})] \\
&\quad + c\emptyset[d(hx_{2n+2}, fx_{2n+2}) + d(hx_{2n+1}, gx_{2n+1})] \\
&= a\emptyset[d(y_{2n+1}, y_{2n})] + b\emptyset[d(y_{2n+1}, y_{2n+1}) + d(y_{2n}, y_{2n+2})] \\
&\quad + c\emptyset[d(y_{2n+1}, y_{2n+2}) + d(y_{2n}, y_{2n+1})]
\end{aligned}$$

$$\begin{aligned}
\text{i.e. } \emptyset[d(y_{2n+2}, y_{2n+1})] &\leq \frac{a+b+c}{1-b-c}\ \emptyset\ [d(y_{2n+1}, y_{2n})] \\
&= \mu\emptyset[d(y_{2n+1}, y_{2n})]
\end{aligned}$$

where $\mu = \frac{a+b+c}{1-b-c} < 1(since\ \ a + 2b + 2c < 1).$

Therefore $$d_{2n+1} \le \mu d_{2n} \qquad \text{---------(2.9)}$$

From (2.8) and (2.9) we get

$$d_{2n} \le \lambda d_{2n-1} \le \lambda\mu d_{2n-2} \le ----\le \lambda^n \mu^n d_0,$$

and

$$d_{2n+1} \le \mu d_{2n} \le \lambda\mu d_{2n-1} \le ----\le \lambda^n \mu^{n+1} d_0.$$

Thus

$$d_{2n} + d_{2n+1} \le \lambda^n \mu^n (1+\mu) d_0 \qquad \text{--------(2.10)}$$

$$d_{2n+1} + d_{2n+2} \le \lambda^n \mu^{n+1} (1+\lambda) d_0 \qquad \text{-------(2.11)}$$

Let $n, m \in N$, then for the sequence $\{y_n\}$ we consider $\emptyset[d(y_n, y_m)]$ in two cases.

If n is even and $m > n$, then using (1.1) and (2.10) we obtain

$$\emptyset[d(y_n, y_m)] \le k\emptyset[d(y_n, y_{n+1})] + k\emptyset[d(y_{n+1}, y_{n+2})] + \\ -------- + k\emptyset[d(y_{m-1}, y_m)] \\ \le k[d_n + d_{n+1} + d_{n+2} + d_{n+3} + ----] \\ \le k[\lambda^{\frac{n}{2}} \mu^{\frac{n}{2}} (1+\mu) d_0 + \lambda^{\frac{n+2}{2}} \mu^{\frac{n+2}{2}} (1+\mu) d_0 + ---]$$

$$\emptyset[d(y_n, y_m)] \le \frac{k(\lambda\mu)^{\frac{n}{2}}(1+\mu)}{1-\lambda\mu} d_0.$$

If n is odd and $m > n$, then again using (1.1) and (2.11) we obtain

$$\emptyset[d(y_n, y_m)] \le \frac{k(\lambda\mu)^{\frac{n-1}{2}}(1+\lambda)}{1-\lambda\mu} d_0.$$

Since $\lambda < 1, \mu < 1$ therefore $\lambda\mu < 1$, so in both the cases $\emptyset[d(y_n, y_m)] \to 0_B$ as $n \to \infty$, and since $\emptyset \in \Phi(P, C)$ we have $d(y_n, y_m) \to 0_E$ as $n \to \infty$. So by lemma 1.1, $\{y_n\} = \{hx_{n-1}\}$ is a Cauchy sequence.

Since $h(X)$ is complete, there exists $\vartheta \in h(X)$ and $u \in X$ such that $lim_{n\to\infty} y_n = \vartheta$ and $\vartheta = hu$.

We shall show that u is a coincidence point of pairs (f, h) and (g, h) i.e. $fu = gu = hu$.

If $fu \ne hu$ then $0_E < d(fu, hu)$. Using (2.7) we obtain

$$\emptyset[d(fu, y_{2n+1})] = \emptyset[d(fu, gx_{2n+1})] \\ \le a\emptyset[d(hu, hx_{2n+1})] + b\emptyset[d(hu, gx_{2n+1}) + d(hx_{2n+1}, fu)] \\ + c\emptyset[d(hu, fu) + d(hx_{2n+1}, gx_{2n+1})] \\ = a\emptyset[d(hu, y_{2n})] + b\emptyset[d(hu, y_{2n+1}) + d(y_{2n}, fu)] \\ + c\emptyset[d(hu, fu) + d(y_{2n}, y_{2n+1})]$$

Since $y_{2n} \to hu, d(fu, y_{2n+1}) \to d(fu, hu)$ as $n \to \infty$ and $\emptyset \in \Phi(P, C)$, therefore letting $n \to \infty$ in above inequality and using remark 1.1 we get

$$\emptyset[d(fu, hu)] \le (b+c)\emptyset[d(hu, fu)] \\ < \emptyset[d(hu, fu)] \text{ (since } b + c < 1),$$

a contradiction. Therefore $fu = hu$. Similarly, it can be shown that $gu = hu$. Therefore

$$fu = gu = hu = \vartheta \qquad \text{-------------(2.12)}$$

Thus ϑ is point of coincidence of pairs (f, h) and (g, h). We shall show that it is unique.

Suppose w is another point of coincidence of these pairs i.e. $fz = gz = hz = w$ for some $z \in X$.

Then from (2.7) it follows that

$$\emptyset[d(w, \vartheta)] = \emptyset[d(fz, gu)] \\ \le a\emptyset[d(hz, hu)] + b\emptyset[d(hz, gu) + d(hu, fz)] \\ + c\emptyset[d(hz, fz) + d(hu, gu)]$$

$$= a\emptyset[d(w,\vartheta)] + b\emptyset[d(w,\vartheta) + d(\vartheta,w)]$$
$$+c\emptyset[d(w,w) + d(\vartheta,\vartheta)]$$
$$= (a+2b)\emptyset[d(w,\vartheta)].$$

Since $a + 2b < 1$, by remark 1.1 we obtain

$\emptyset[d(w,\vartheta)] = 0_B$ i.e. $w = \vartheta$. Thus point of coincidence is unique.

If pairs (f,h) and (g,h) are weakly compatible, from (2.12) we have $f\vartheta = fhu = hfu = h\vartheta$ and $g\vartheta = ghu = hgu = h\vartheta$, therefore $f\vartheta = g\vartheta = h\vartheta = p$ (say). This shows that p is another point of coincidence, therefore by uniqueness, we must have $p = \vartheta$ i.e.

$$f\vartheta = g\vartheta = h\vartheta = \vartheta.$$

Thus ϑ is unique common fixed point of self maps f, g and h.

Theorem 2.3. Let (X,d) be a cone metric space and P a normal cone with normal constant K. Suppose f,g,h be self maps of X satisfy the condition.

$$\emptyset[d(fx,gy)] \le a\emptyset[d(hx,hy)] + b\emptyset[d(hx,fx) + d(hx,gy)]$$
$$+c\emptyset d[(hy,fx) + d(hy,gy)] \text{ for all } x,y \in X \qquad \text{---------(2.13)}$$

where $\emptyset \in \Phi(P,C)$ and a,b,c are non negative reals with $a + 2b + 2c < 1$. If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is complete subspace of Xthen the maps f, g *and* h have a unique point of coincidence in X. Moreover, if (f,h) and (g,h) are weakly compatible pairs then f, g and h have a unique common fixed point.

Proof. The proof of this theorem same as Theorem 2.2.

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